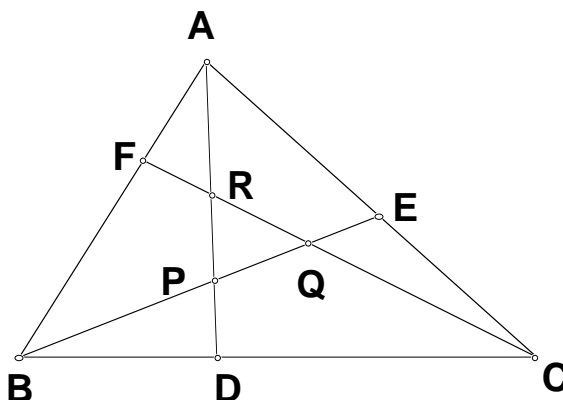


The ARML Power Contest

November 2001

Cevians

A **cevian** is a segment which joins the vertex of a triangle with a point on its opposite side (or its extension.) Throughout this problem set, unless it is stated to the contrary, the three cevians of $\triangle ABC$ will be labeled \overline{AD} , \overline{BE} , and \overline{CF} .



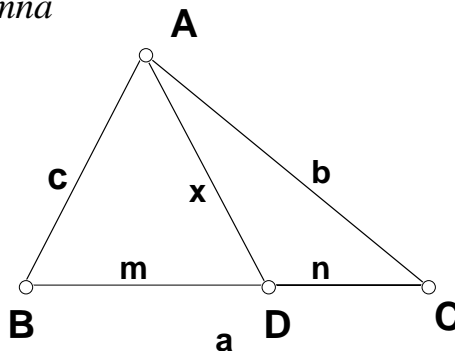
Some cevians have special names like altitudes, medians, and angle bisectors. Sometimes, \overline{AD} , \overline{BE} , and \overline{CF} are **concurrent at point P**, while other times they form an **interior cevian triangle**, called $\triangle PQR$. When concurrent, oftentimes, point P has a special name like orthocenter, centroid, incenter, or the Nagel, Gergonne, Lemoine, or Miquel point.

Triangle $\triangle DEF$ is called the **inscribed cevian triangle**. The orthic and medial triangles are special inscribed cevian triangles. The orthic triangle is formed when the cevians are altitudes while the medial triangle is formed by medians.

The following three theorems may be useful in solving this problem set:

Stewart's Formula: The length of a cevian, x , and the sides of a triangle are related by the following formula:

$$c^2n + b^2m = x^2a + mna$$



Ceva's Theorem: The three cevians, \overline{AD} , \overline{BE} , and \overline{CF} , are **concurrent** if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Routh's Theorem: If K_{ABC} is the area of ΔABC ,

then the **area of the interior cevian triangle, PQR ,**

$$K_{PQR} = \frac{(rst - 1)^2}{(st + s + 1)(tr + t + 1)(rs + r + 1)} \cdot K_{ABC}$$

and the **area of the inscribed cevian triangle DEF ,**

$$K_{DEF} = \frac{rst}{(s + 1)(t + 1)(r + 1)} \cdot K_{ABC},$$

$$\text{where } \frac{BD}{DC} = r, \frac{CE}{EA} = s, \frac{AF}{FB} = t .$$

The Problems

Part A – Cevian Length and Concurrency

1. Show that the length of the median from vertex C of ΔABC , $x = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}$.

2. Show that the length of the altitude from vertex C of ΔABC ,

$$x = \frac{\sqrt{(a + b - c)(b + c - a)(c + a - b)(a + b + c)}}{2c} .$$

3. Show that the length of the angle bisector from vertex C of ΔABC ,

$$x = \frac{\sqrt{a \cdot b \cdot (a + b - c)(a + b + c)}}{a + b} .$$

4. The altitude, median and angle bisector from vertex C of a ΔABC divide $\angle ACB$ into four equal angles. Find the degree measures of $\angle A$, $\angle B$, and $\angle ACB$.

5. In ΔABC , the median \overline{AD} , the altitude \overline{BE} , and the angle bisector \overline{CF} are concurrent.

$$\text{Show that } \frac{a + b}{a - b} = \frac{b^2}{a^2 - c^2} .$$

6. In $\triangle ABC$, point D is on \overline{BC} and half way around the triangle from vertex A , i.e., $AC + CD = AB + BD$, and E is on \overline{AC} and half way around the triangle from vertex B , and F is on \overline{AB} and half way around the triangle from C . Prove that the cevians \overline{AD} , \overline{BE} , and \overline{CF} are concurrent.
7. Prove that the orthocenter of $\triangle ABC$ is the incenter of its orthic triangle $\triangle DEF$.

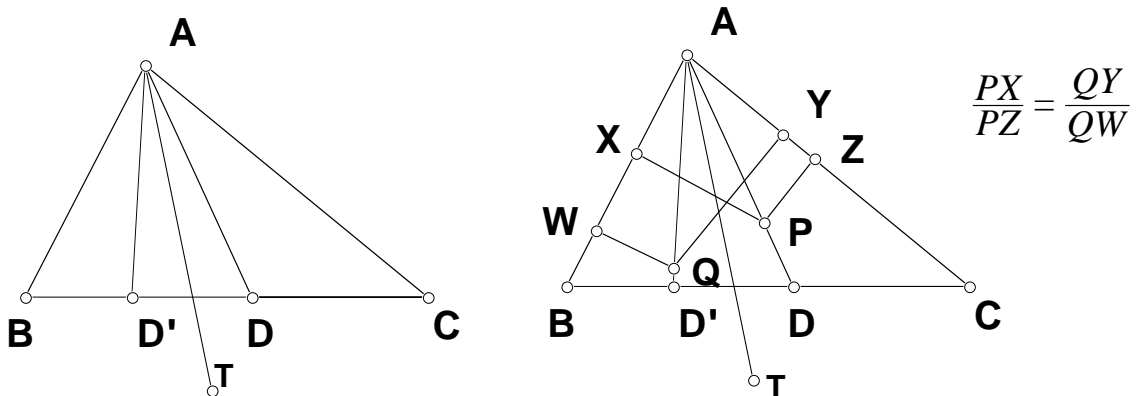
Part B – Areas of Cevian Triangles

1. The sides of $\triangle ABC$ have lengths 14, 25, and $\sqrt{149}$. The three cevians divide each side in the ratio 3:5 and form an interior cevian triangle $\triangle PQR$. Find the area of $\triangle PQR$.
2. The cevians of a triangle partition the sides into segments whose ratios are 1:1, 4:1, and 7:1, consecutively. If the area of the inscribed cevian triangle is 28, what is the area of the interior cevian triangle?
3. The cevians of a triangle partition the sides into segments whose ratios are each $1:n$. Find a simplified rational function which expresses $\frac{K_{PQR}}{K_{ABC}}$ in terms of n .

Part C – Symmedians

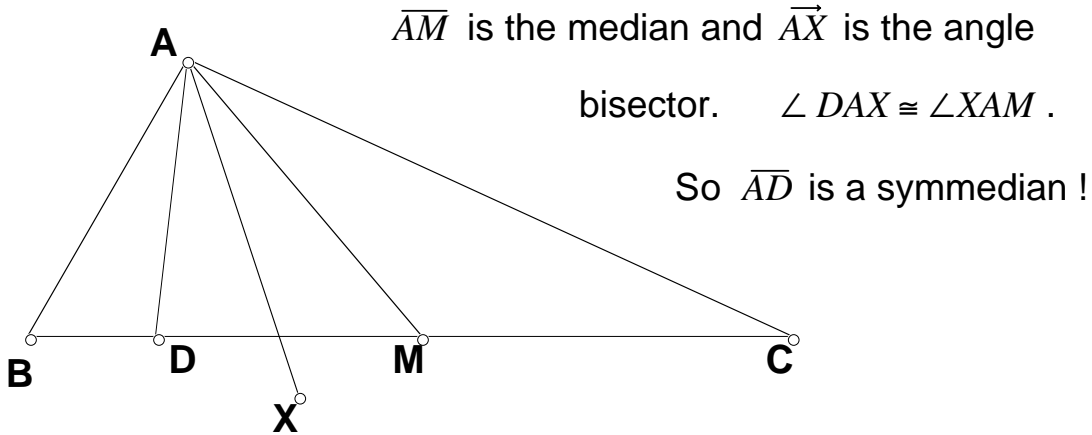
Definition: If two cevians from the same vertex of a triangle are symmetrical with respect to the angle bisector of that angle, i.e. $\angle D'AT \cong \angle DAT$ in figure below, they are said to be **isogonal** and form a **pair of isogonal conjugates**.

The Isogonal Theorem: If \overline{AD} and $\overline{AD'}$ are cevians with point P on \overline{AD} and point Q on $\overline{AD'}$, then \overline{AD} and $\overline{AD'}$ are isogonal if and only if the ratio of distances from P to \overline{AB} and \overline{AC} are inversely proportional to the ratio of the distances from point Q to \overline{AB} and \overline{AC} . (\overline{AT} bisects angle BAC .)



1. Prove, if the cevians \overline{AD} , \overline{BE} , and \overline{CF} are concurrent, then so are their isogonal conjugates, $\overline{AD'}$, $\overline{BE'}$, and $\overline{CF'}$.

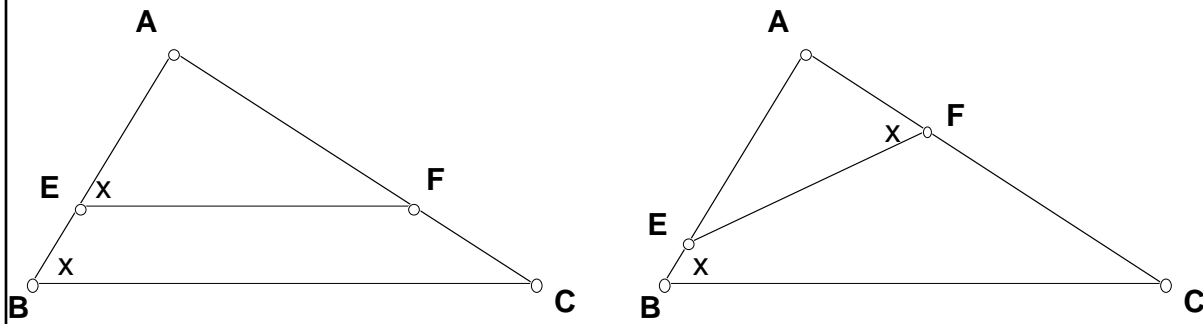
Definition: A **symmedian** is a cevian which is isogonal to a median. Since medians are concurrent at the centroid, from problem 1 above, the symmedians of a triangle must also be concurrent. (Their point of concurrency is called the **Lemoine point** of the triangle.)



2. Prove P is a point on symmedian \overline{AD} if and only if the ratio of the distances from P to \overline{AB} and \overline{AC} is proportional to $\frac{AB}{AC}$.

3. Prove if \overline{AD} is a symmedian, then $\frac{BD}{DC} = \frac{AB^2}{AC^2}$

Definition: In $\triangle ABC$ let E be a point on \overline{AB} and F a point on \overline{AC} .
 If $\triangle AEF \cong \triangle ABC$ then \overline{EF} is a **parallel of side \overline{BC}** of $\triangle ABC$ and
 if $\triangle AFE \cong \triangle ABC$ then \overline{EF} is an **antiparallel of side \overline{BC}** of $\triangle ABC$.



4. In $\triangle ABC$ we know the median \overline{AM} bisects every parallel of side \overline{BC} . Prove the symmedian \overline{AD} bisects every antiparallel of side \overline{BC} .