The ARML Power Contest
February 2004

Mathematical Strings

A mathematical string is any ordered list of symbols, where a symbol can occur more than once. While strings occur in probability theory, topology, combinatorics, set theory, and other branches of mathematics, they are also very commonplace in today's world. Words, phone numbers, license plate codes, UPC symbols are all examples of strings. Using the letters from a set, such as $\{a, b, c\}$, this problem set will look at strings such as $aaaa$, $ababba$, and $bbccbccbb$. The length of a string, n, is equal to the number of symbols in the string.

The Problems

1. a) Using the set of symbols, $\{a, b, c, d, e\}$, how many strings of length n can be made? Justify your answer.

   b) Using the set of symbols, $\{a, b, c, d, e\}$, how many palindromic strings of length n can be made? (A palindromic string is the same when read forwards or backwards, like $abba$, ccc, and $bacab$.) Justify your answer.

   c) Using the set of symbols, $\{a, b, c, d, e\}$, how many strings of length n $(n \geq 2)$ can be made that contain exactly 2 $a's$?

2. Using an ordered set of symbols, $\{a_1, a_{2,}. a_3, ..., a_k\}$, where $a_1 < a_2 < a_3 <...< a_k$, many nondecreasing strings of length n can be made. (A string is considered nondecreasing if for each pair of adjacent symbols in the string, the symbol on the left side of the pair is less than or equal to the symbol on the right.) For example, using an ordered set of symbols, $\{a, b\}$, where $a < b$, there are three nondecreasing strings of length two (aa, ab, bb) and four nondecreasing strings of length three (aaa, aab, abb, bbb). In general, using an ordered set of

symbols, $\{a, b\}$, where $a < b$, the number of nondecreasing strings of length n that can be made is $n + 1$.

a) Using an ordered set of symbols, $\{a, b, c\}$, where $a < b < c$, how many nondecreasing strings of length n can be made? Justify your answer.

b) Using an ordered set of symbols, $\{a, b, c, d\}$, where $a < b < c < d$, how many nondecreasing strings of length n can be made? Justify your answer.

c) Using an ordered set of symbols, $\{a_1, a_{2,}. a_3, ..., a_k\}$, where $a_1 < a_2 < a_3 <...< a_k$, how many nondecreasing strings of length n can be made? Justify your answer.

3. The string abc has six substrings: $a, b, c, ab, bc,$ and abc. Using the symbols in the set $\{a, b\}$, $abbaa$ is a string with four substrings of length two: $ab, bb, ba,$ and aa. With a length of five, $aabba$ is an example of the longest string with no repeated substrings of length two and $aaabbbaba$ is an example of the longest string with no repeated substrings of length three.

a) Using the symbols in the set $\{a, b\}$, how long is the longest possible string with no repeated substrings of length four? Give an example of such a string.

b) Using the symbols in the set $\{a, b, c\}$, give an example of the longest possible string with no repeated substrings of length two. How long is your example?

c) Using the symbols in the set $\{a, b, c, d\}$, give an example of the longest possible string with no repeated substrings of length two. How long is your example?

d) Using the set of distinct symbols $\{a_1, a_{2,}. a_3, ..., a_k\}$, how long is the longest possible string with no repeated substrings of length m? Justify your answer.

4. In many games, such as tennis, volleyball, ping-pong, and ultimate frisbee, you must win by two points. For example, in a high school volleyball game, generally the first team to reach 15 points is the winner. However, the game can not end with a score of 15 to 14. Therefore, the score must have been tied at 14 all and "overtime" must

be played. If a represents Team A winning a point following the 14-14 tie, and b represents Team B winning a point, a string of a's and b's could represent the "overtime" portion of the game. The strings bb and $abbaaa$ could be strings representing the overtime, while $abbaa$ and $abaaaa$ could not represent an overtime.

a) List all the "overtime" strings of length 6. How many are there?

b) Determine a formula for the number of "overtime" strings of length n, where n is any positive even integer.

c) If two teams are evenly matched, what is the average or expected length of an overtime string? Show your work, justifying your answer.

5. A string consisting of two a's and three b's could represent the counting of ballots in an election where candidate A receives two votes and candidate B receives three votes. The ten different arrangements of these a's and b's ($aabbb$, $ababb$, $abbab$, $abbba$, $baabb$, $babab$, $babba$, $bbaab$, $bbaba$, $bbbaa$) could represent the ten different ways of the counting the five ballots. Notice that in only two of the ten countings, namely $bbaba$ and $bbbaa$, candidate B is always ahead of candidate A at each step in the count. If $N(A)$, the number of votes for candidate A, equals 2 and $N(B)$, the number of votes for candidate B, equals 3, the probability that candidate B is always ahead in the count is $2/10$ or $1/5$.

a) If $N(A) = 3$ and $N(B) = 4$, how many different arrangements of a's and b's are there?

b) If $N(A) = 3$ and $N(B) = 4$, make a list of the strings where candidate B is always ahead of candidate A throughout the counting of the ballots.

c) In general, if $N(A)$ represents the number of votes for candidate A and $N(B)$, the number of votes for candidate B, with $N(B) > N(A)$, what is the probability, in terms of $N(A)$ and $N(B)$, that candidate B will always be ahead of candidate A throughout the counting of the ballots? Justify your answer.