2009 ARML Local Competition
Team Round (30 minutes)

Question 1: $3+2i$ is one root of a quadratic function $f(x)=x^2+Ax+B$, where A and B are real numbers. Compute the ordered pair (A,B).

Question 2: If $f(x)$ is a line of slope -3, compute the slope of the line $f\left(f\left(f\left(x\right)\right)\right)+f\left(f\left(x\right)\right)+f\left(x\right)$.

Question 3: A six-sided die has these markings on its sides: \$12, \$12, \$16, \$16, \$26, and \$26. The following game is played. The die is rolled and the player wins whatever amount of money comes up. The player keeps rolling the die and winning money until a money amount comes up a second time, in which case the game ends. A sample game would be rolls of \$16, \$12, and \$16, in which case the player wins \$44. Compute the expected winnings by the player in this game.

Question 4: If $f\left(x\right)=\log_x\left(x+2\right)$, compute $3^{f(3)+f(9)}$.

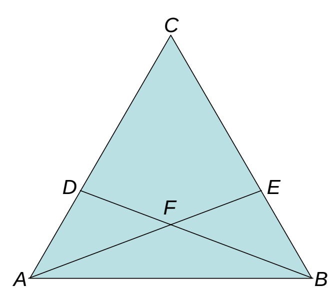
Question 5: In an equiangular hexagon *ABCDEF*, $AB=BC=3$, $CD=8$, and $EF=5$. Compute the area of *ABCDEF*.

Question 6: Compute the number of values of $\theta, 0\le\theta\le 2\pi$ for which $\left|\sin\theta\right|+\left|\cos\theta\right|=\dfrac{4}{3}$.

Question 7: You are playing a game. Your opponent has distributed five red balls into five boxes randomly. All arrangements are equally likely; that is, the left-to-right placements [0,0,5,0,0], [0,2,0,3,0], and [1,1,1,1,1] are equally likely. You place one blue ball into each box. The player with the most balls in a box wins the box (neither player wins a box with the same number of balls of each color). Whoever wins the most boxes wins the game. Compute the probability you win the game.

Question 8: When $\left(x+y+z\right)^{2009}$ is expanded and like terms are grouped, there are *k* terms with coefficients that are *not* multiples of five. Compute *k*.

Question 9: On equilateral triangle *ABC*, points *D* and *E* are on sides $\overline{AC}$ and $\overline{BC}$ such that *AD* = *BE* = 1. $\overline{BD}$ and $\overline{AE}$ meet at *F*. Given *AB* = 3, compute the area of triangle *ADF*.



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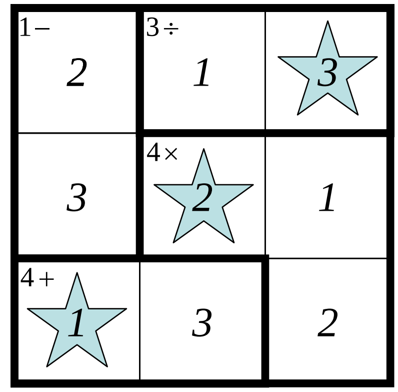
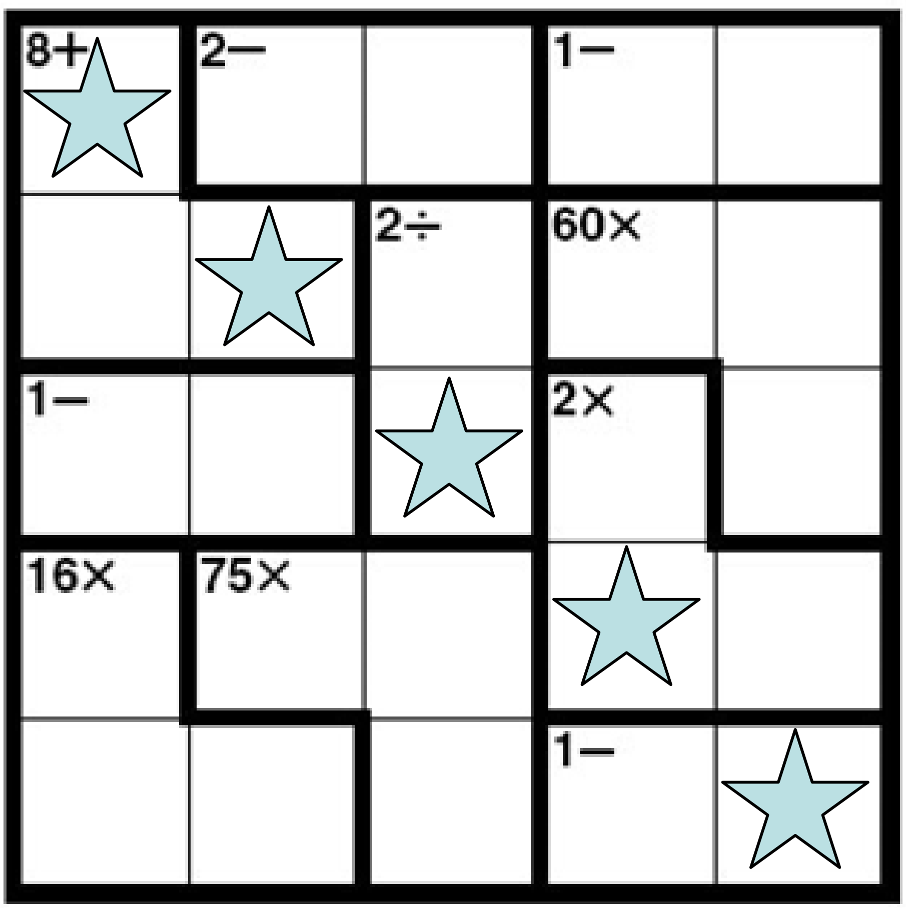
Question 10: To solve a KenKen puzzle, you fill in an $n \times n$ grid with the digits $1, \ldots, n$ according to the following two rules:

1. Each row and column contains exactly one of each digit.
2. Each bold-outlined group of cells is a cage containing digits which achieve the specified result using the specified mathematical operation: addition (+), subtraction (-), multiplication (×), and division (÷) on the digits in some order. Digits may repeat inside a cage.

A solved 3×3 KenKen appears to the right. Below is a 5×5 KenKen puzzle. On your answer sheet, enter the digits in the starred squares in the correctly solved KenKen puzzle, *in order from left to right*. In the example, you would enter 1,2,3. Digits may appear more than once in the starred squares.





2009 ARML Local Competition
Theme Round: Votes That Count (45 minutes)

"Those who cast the votes decide nothing. Those who count the votes decide everything."

In this round, we will be looking at a number of different voting systems, as well as how voters choose who to vote for. In the simplest election, voters each choose a single candidate that they prefer over all of the others and the candidate with the most votes wins (this is called *plurality voting*). A slightly more complex election requires the winning candidate to receive greater than half of the votes (this is called *majority voting*), with a runoff between the two candidates receiving the most votes in the first round if no candidate has a majority (for example, the 2008 Georgia senate race).

Part 1: Preference Rankings

It is reasonable to assume that given a pair of candidates *A* and *B*, a voter either prefers one over the other ($A < B$ or $A > B$) or is indifferent between the two ($A = B$). Also, given a set of candidates, it is reasonable to assume these preference relationships are transitive ($A \le B$ and $B \le C \to A \le C$). Therefore, we can determine a preference ranking for each voter. For example, if a voter prefers candidate *A* over all others, is indifferent between candidates *B* and *C*, but prefer both of them over candidate *D*, then that voters preference ranking would be $A > B, C > D$. We can also write preference rankings vertically, with candidates on the same row being equally preferable: $\begin{matrix} A \\ BC \\ D \end{matrix}$.

Question 1: Compute the number of distinct preference rankings of four candidates.

Note that $\begin{matrix} A \\ BC \\ D \end{matrix}$ and $\begin{matrix} A \\ CB \\ D \end{matrix}$ are *not* distinct preference rankings.

A preference ranking is called *strict* if the voter always prefers one candidate over the other. For the remainder of the questions in this round, we will assume all preference rankings are strict.

Consider an election with four voters (w, *x*, *y*, and *z*) and three candidates (*A*, *B*, and *C*). The voters' preference rankings are $\begin{matrix} w & x & y & z \\ A & A & B & B \\ B & C & C & C \\ C & B & A & A \end{matrix}$. Based on first place votes alone, candidates *A* and *B* are tied 2-2.

One way to break the tie is to perform a *Borda count*: For each candidate (for example, *A*), and each voter (for example, *w*), count the number of candidates that the voter *w* prefers *A* over (in this example, 2). Sum these values over all voters, and the candidate with the highest sum wins. In this case, *B* is the winner of the Borda count.

	w	x	y	z	T
A	2	2	0	0	4
B	1	0	2	2	5
C	0	1	1	1	3

Question 2: Consider an election with *m* candidates and *n* voters where one candidate has greater than half of the first place votes, but loses a Borda count, we'll call this situation an (*m*,*n*)-*Borda upset*. Compute the ordered pair (*m*,*n*) such that an (*m*,*n*)-Borda upset exists, and $m+n$ is minimized.

Another voting method using preference rankings is called *instant runoff voting* (IRV). In one version of IRV, all voters submit strict preference rankings of candidates. The following process is used to determine the winner:

Step 1: Does any candidate have a majority of voters that have them as their top choice?
If yes, then that candidate is the winner. END
If not, go to step 2.

Step 2: Is there a unique candidate with the fewest first place votes?
If yes, then remove that candidate from ALL preference rankings and go to Step 1.
If no, then some other process must be used to determine who is removed from the election.

So, for example, for three candidates *A*, *B*, and *C*, if there are 10 voters with preference ranking $\begin{matrix} A \\ B \\ C \end{matrix}$, 8 with preference ranking $\begin{matrix} B \\ A \\ C \end{matrix}$, and 6 voters with a preference ranking $\begin{matrix} C \\ B \\ A \end{matrix}$, then in the first round of IRV, candidate *C* would be eliminated, leaving 10 voters with preference ranking $\begin{matrix} A \\ B \end{matrix}$ and 14 voters with preference ranking $\begin{matrix} B \\ A \end{matrix}$. Therefore, candidate *B* wins this IRV election, despite the fact that candidate *A* had more first-place votes than any other candidate when the process began. When this happens with *m* candidates and *n* voters, we call this a (*m*,*n*)-*IRV upset*. Note that if there is a tie for last place at any stage, then the IRV election does not produce a winning candidate, so there must always be a unique last place candidate at every stage for the process to conclude.

Question 3: Let n_3 and n_4 be the minimum values of *n* such that there exists a (3, *n*)-IRV upset and (4, *n*)-IRV upset, respectively. Compute the ordered pair (n_3, n_4).

Part 2: Winning Coalitions and Relative Power in Elections

To win the presidential election, a candidate needs a majority of the 538 electoral votes (that is, 270). Most states distribute all of their electoral votes to the candidate that gets the most votes in their statewide election. However, is California (with 55 electoral votes) really eleven times as important as, say, West Virginia (with 5 electoral votes? The answer is, naturally, it depends.

Given *n* voters (which we will denote with the set $S=\{1,\ldots,n\}$), every subset of *S* is either a winning coalition or a losing coalition. Winning coalitions are *monotonic*, that is, if *W* is a winning coalition and *V* contains *W*, then *V* is also a winning coalition.

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We define a *weighted majority vote* (WMV) as follows: given a threshold T and that each voter i has v_i votes, a subset W of S is a winning coalition of a WMV if and only if $\sum_{i \in W} v_i \geq T$. We can represent a WMV by $(T; v_1, v_2, \ldots, v_n)$. For example, the US presidential election (ignoring the fact that some states that split their electoral votes) can be represented as a WMV: $(270; 55, 34, 31, 27, \ldots, 3, 3)$.

Many elections or voting processes can be represented as WMVs. On some issues, the Australian government has votes involving the six state legislatures plus the federal government. Each of the six states gets one vote, while the federal government gets two. In the case of a 4-4 tie, the federal government makes the decision. It is easy to show that this voting process is equivalent to the WMV $(5;1,1,1,1,1,1,3)$, with the three votes going to the federal government: a coalition is a winning coalition if and only if it contains at least five states or the federal government and at least two states.

Question 4: In the UN Security Council, there are five permanent members and ten non-permanent members. For any resolution to pass, it must have the support of *all five* of the permanent members and at least four of the ten non-permanent members. Consider the WMV $(T; P, P, P, P, P, 1,1,1,1,1,1,1,1,1,1)$ where P denotes the integral number of votes for each of the permanent members of the council and the ones denote the votes of the non-permanent members of the council. Compute the ordered pair (T, P) such that T is minimized and W is a winning coalition in this WMV if and only if the corresponding set of council members can pass a resolution in the UN Security Council.

To measure the relative power of a voter, we consider the *Shapley-Shubik Power Index*. It is defined as follows.

Consider all $n!$ orderings of n voters. Given an ordering $(p_1, \ldots, p_n)$ of the set $\{1, \ldots, n\}$, let $W_i = \bigcup_{j \leq i} \{p_j\}$. A voter p_k is *pivotal* for this ordering if W_{k-1} is not a winning coalition, but W_k is a winning coalition. The Shapley-Shubik Power Index for a voter is the fraction of the orderings for which the voter is pivotal.

For example, consider the WMV (7;5,4,2). There are six orderings of the voters, and we will underline the pivotal voter in each case: $24\underline{5}$ $2\underline{5}4$ $42\underline{5}$ $4\underline{5}2$ $5\underline{2}4$ $5\underline{4}2$. Thus, the Shapley-Shubik Power Index of the first voter is 4/6, or 2/3, while the Shapley-Shubik Power Index of the other two voters are 1/6 each.

Question 5: Consider the WMV for the Australian federal government: $(5;1,1,1,1,1,1,3)$. Let s be the Shapley-Shubik Power Index of any one of the states and let f be the Shapley-Shubik Power Index of the federal government. Compute the ordered pair (s, f).

There are other ways to define winning coalitions. In the US legislature, a bill is sent to the president for signature ("winning") if and only if it has a coalition of at least 218 of the 435 members of the House and at least 51 of the 101 members of the Senate (including the Vice President in the Senate count, the Vice President is not required to be a part of a coalition for it to be a winning coalition) supporting it.[1]

[1] Technically, the Vice President does not vote, except in the case of ties. It is equivalent, however, to let the vice president vote all the time, since his vote only matters when the other 100 senators are tied 50-50.

Question 6: Compute the Shapley-Shubik Power Index of the Vice President.

If $W \subseteq S = \{1, 2, \ldots, n\}$ is a winning coalition, $i \in W$ is called a *swing voter* for W if $W - \{i\}$ is a losing coalition. We can define another power index called the *Banzhaf-Penrose Power Index* as follows. Let s_i be the number of winning coalitions for which voter i is a swing voter. The Banzhaf-Penrose Power Index of voter i is

$$B_i = \frac{s_i}{\sum_{j=1}^{n} s_j}.$$

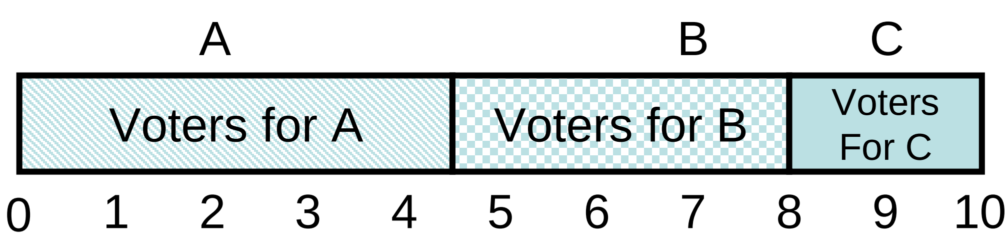
Going back to our earlier WMV example, (7;5,4,2), there are three winning coalitions, {5,4}, {5,2}, and {5,4,2}. Both voters are swing voters in the first two coalitions, but only the 5 in the third coalition is a swing voter. Accordingly, there are five swing voters in total, so the Banzhaf-Penrose Power Indices for the three voters are 3/5, 1/5, and 1/5, respectively.

Question 7: Consider the WMV (6;1,2,3,4). Compute the ordered 4-tuple (B_1, B_2, B_3, B_4), that is, the Banzhaf-Penrose Power Index for each voter.

Part 3: Models of Voter Preference

There are many ways that voters come up with their preference rankings for candidates. Some people are single-issue voters. That is, their preference ranking depends on how close a candidate's view on a single issue is to their own. Say a candidate's view on an issue could be represented as a real number on a 0 to 10 scale, and there were three candidates (*A*,*B*,*C*) whose views on an issue are 2, 7, and 9. Then, a single-issue voter whose view on an issue was 5 would have a preference ranking of $B > A > C$ since the distance between the voter's view on the issue and those of the candidates are 3, 2, and 4, respectively.

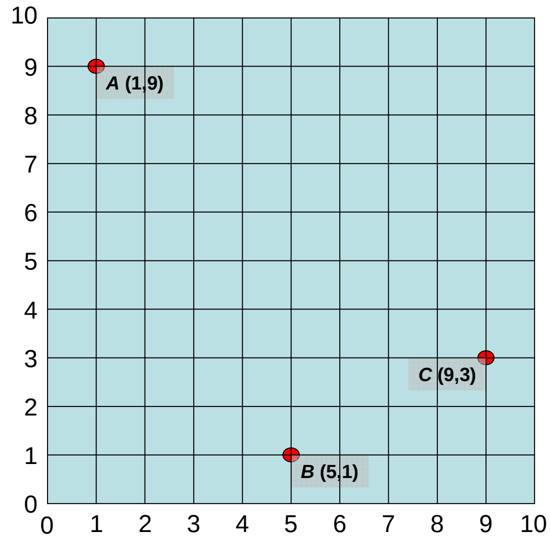
If all voters were single issue voters and voters' views on the issue were evenly distributed throughout the 0-10 scale, then candidate *A* would win a plurality election, as voters with views between 0 and 4.5 (45%) would vote for candidate *A*, those with views between 4.5 and 8 (35%) would vote for candidate *B*, and those with views between 8 and 10 (20%) would vote for candidate *C*. We will call this a *single-issue election*.



Question 8: In a single-issue election, candidates *A* and *B* have randomly picked their views on the issue (all views are equally likely and the views of candidates *A* and *B* are independent). Candidate *C*, knowing the views of candidates *A* and *B*, can choose his view to maximize the fraction of the votes he receives. Compute the probability that it is impossible for candidate *C* to pick a view that results in candidate *C* winning the election.

We would hope that voters are a bit more nuanced than this and perhaps base their preferences on *two* issues!

Assume there are three candidates *A*, *B*, and *C* with views on two issues (again, on a 0-10 scale) as given in the graph below:



A voter sets their preference ranking depending on the (Cartesian) distance between their views on these two topics and those of the candidate. If voters' views on both topics were independent and evenly distributed throughout the 0-10 scales, then the fraction of the votes received by a candidate would be equal to the fraction of the square $[0,10]\times[0,10]$ that is closest to the point corresponding to the candidate's views.

Question 9: Under the above assumption, if f_i is the fraction of the vote received by candidate *i*, compute the ordered triple (f_A, f_B, f_C). It is acceptable to give the answer as an ordered triple of fractions or percentages.

Question 10: A voter in this election with views (x_v, y_v) is indifferent between all three candidates. Compute the ordered pair (x_v, y_v).

2009 ARML Local Competition
Individual Round (5 pairs, 10 minutes per pair)

Question 1: Given $A \wedge B = \dfrac{A+B}{AB}$, compute $(2 \wedge 6) \wedge (3 \wedge 4)$.

Question 2: Compute the smallest value of *n* for which the mean and median of the set $\{13, 21, 26, 28, 41, n\}$ are equal.

Question 3: Compute the number of subsets *S* of the set $\{1,2,3,4,5,6,7,8,9\}$ such that the smallest element of *S* is equal to the size of *S*.

Question 4: $A_1A_2 \ldots A_{2009}$ is a regular 2009-gon of area *K*. If $A_1A_{1005} = 6$, compute $\lfloor K \rfloor$.

Question 5: *x*, *y*, and *z* are real numbers randomly chosen between 0 and 100. Compute the probability that $\lceil x+y+z \rceil = \lceil x \rceil + \lceil y \rceil + \lceil z \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to *x*.

Question 6: An integer is called "rotatable" if it only uses the digits 0, 1, 6, 8, or 9. Compute the sum of all three-digit rotatable numbers (note that the first digit cannot be a zero).

Question 7: Compute the number of ways to place the integers 1 through 7 in the blanks below so that the chain of inequalities is satisfied.

$$\_ < \_ < \_ > \_ < \_ > \_ > \_$$

$$(\text{Example: } 1 < 2 < 5 > 4 < 7 > 6 > 3)$$

Question 8: The period of a sequence $s_0, s_1, \ldots$ is the smallest positive integer *k* such that $s_n = s_{n+k}$ for all $n \geq 0$.

Compute the period of the sequence defined by $s_0 = \tan 4^\circ, s_n = \dfrac{2s_{n-1}}{1-s_{n-1}{}^2}$ for $n > 0$.

Question 9: For a positive integer *n*, define a_n to be the smallest positive integer with exactly n^2 positive factors. Compute the smallest *n* for which $a_n > a_{n+1}$.

Question 10: θ is an acute angle for which $\sin\theta$, $\sin 2\theta$, and $\sin 4\theta$ form a strictly increasing arithmetic sequence. Compute $\cos^3\theta - \cos\theta$.

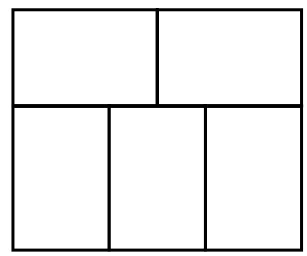
2009 ARML Local Competition

Relay Round (6, 8, and 10 minutes per round)

R1-1: $(47)_a = (74)_b$, where $(x)_c$ denotes the number represented by x written in base-c. Compute the smallest value for $a+b$ where a and b are both positive integers.

R1-2: Let $T = TNYWR$. Let y *be the product* of the digits of T and let z *be the sum* of the digits of T. The roots of the quadratic function $f(x) = x^2 + Ax + B$ have arithmetic mean y and geometric mean z. Compute the ordered pair (A,B).

R2-1: Let $f(x) = x^2 - 4^{\lceil \log_{10} x \rceil}$. Compute $f(f(7))$.



A covering of a 6x5 rectangle by 3x2 and 2x3 tiles

R2-2: Let $T = TNYWR$. $a_1, a_2, \ldots, a_7$ is an arithmetic sequence of increasing positive integers whose terms sum to T. Compute the smallest possible value for a_2.

R2-3: Let $T = TNYWR$. Compute the side length of the largest square that can be completely covered without overlap by at most T 2×3 tiles. Tiles may not be broken, but may be rotated, and not all T tiles must be used.

R3-1: M and N are distinct positive numbers such that $\log_M N = \log_N M$. Compute MN.

R3-2: Let $T = TNYWR$. The area bounded by the x-axis, the lines $x = T$, $x = 4$, and $y = mx$ is 15. Compute the largest value of m.

R3-3: Let $T = TNYWR$. An equilateral triangle and regular hexagon have equal perimeters. If the area of the triangle is T, compute the area of the hexagon.

R3-4: Let $T = TNYWR$. Let $A \wedge B = \dfrac{A+B}{AB}$ and $A \vee B = \dfrac{AB}{A+B}$. If $k \vee (k \wedge k) = \dfrac{k}{T^2}$, compute the greatest value of k.

R3-5: Let $T = TNYWR$. The sum of the two three-digit numbers $\underline{2}\underline{T}\underline{3}$ and $\underline{A}\underline{6}\underline{B}$ is a multiple of eleven. Compute the smallest possible value of $A + B$.

R3-6: Let $T = TNYWR$. $A_1A_2A_3\ldots A_T$ and $A_1A_2A_{T+1}\ldots A_{2T-2}$ are distinct regular T-gons in the plane. Let S be the set of all real numbers that are distances between distinct vertices. Compute the number of elements in S. (In the example to the left, there are two distinct non-zero distances, A_1A_2 and A_3A_4.)