2009 ARML Local Solutions
Team Round (30 minutes)

Question 1: $3+2i$ is one root of a quadratic function $f(x)=x^2+Ax+B$, where A and B are real numbers. Compute the ordered pair (A,B).

Solution: The other root of this quadratic equation is $3-2i$. A is -1 times the sum of the roots (6), B is the product of the roots (13), so the ordered pair is $\boxed{(-6,13)}$.

Question 2: If $f(x)$ is a line of slope -3, compute the slope of the line $f\left(f\left(f(x)\right)\right)+f\left(f(x)\right)+f(x)$.

Solution: As the line is unspecified, let $f(x)=-3x$. $f(f(x))=f(-3x)=-3(-3x)=9x$ and $f\left(f(f(x))\right)=f(9x)=-3(9x)=-27x$. The sum of these three functions is $-21x$, so the slope is $\boxed{-21}$.

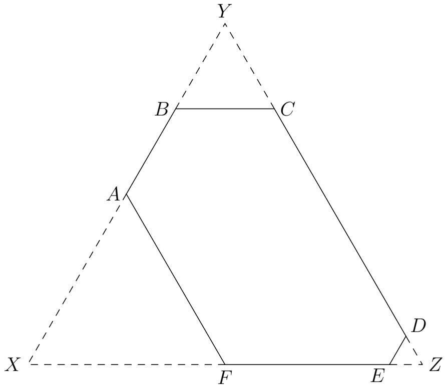
Question 3: A six-sided die has these markings on its sides: \$12, \$12, \$16, \$16, \$26, and \$26. The following game is played. The die is rolled and the player wins whatever amount of money comes up. The player keeps rolling the die and winning money until a money amount comes up a second time, in which case the game ends. A sample game would be rolls of \$16, \$12, and \$16, in which case the player wins \$44. Compute the expected winnings by the player in this game.

Solution: The expected amount of money earned per throw is \$18 (the average of the values on the faces), so the problem reduces to determining the expected number of throws. The game will end in 2 to 4 throws, with 2 throws occurring 1/3 of the time, 3 throws occurring 4/9 of the time (the probability the second throw is different than the first is 2/3, the probability the third throw is the same as either the first or second is 2/3), and 4 throws is 1 – 1/3 – 4/9 = 2/9. Thus, the expected number of throws is $2\times(1/3)+3\times(4/9)+4\times(2/9)=26/9$, so the expected winnings is $(26/9)\times\$18=\boxed{\$52}$.

Question 4: If $f(x)=\log_x(x+2)$, compute $3^{f(3)+f(9)}$.

Solution:

$$3^{f(3)+f(9)}=3^{\log_3 5+\log_9 11}=3^{\log_3 5}\times 3^{\log_9 11}=5\times\left(9^{1/2}\right)^{\log_9 11}=5\times\left(9^{\log_9 11}\right)^{1/2}=\boxed{5\sqrt{11}}.$$

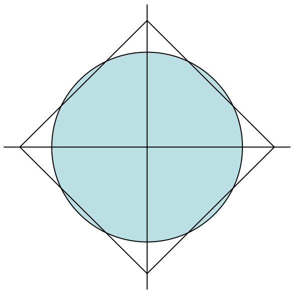


Question 5: In an equiangular hexagon *ABCDEF*, $AB=BC=3$, $CD=8$, and $EF=5$. Compute the area of *ABCDEF.*

Solution: Extend the sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ to obtain an equilateral triangle XYZ as shown. If we set $DE=u$ and $AF=v$, then we have $3+3+v=3+8+u=u+5+v$, from which we conclude that $u=1$ and $v=6$. By subtraction of areas, we get:

$$[ABCDEF]=[XYZ]-[AFX]-[BYC]-[DZE]=\frac{\sqrt{3}}{4}\left(12^2-6^2-3^2-1^2\right)=\boxed{\frac{49\sqrt{3}}{2}}.$$

Question 6: Compute the number of values of $\theta, 0\le\theta\le 2\pi$ for which $|\sin\theta|+|\cos\theta|=\frac{4}{3}$.



Solution: Points on the unit circle have coordinates $(\cos\theta,\sin\theta)$ for some value of $\theta, 0\le\theta\le 2\pi$. So, we can view these points on the plane. $|x|+|y|=k$ describes a square with corners at $(\pm k,0)$ and $(0,\pm k)$. Each edge of this square intersects the unit circle twice, provided $1<k<\sqrt{2}$, so there are $\boxed{8}$ points of intersection.

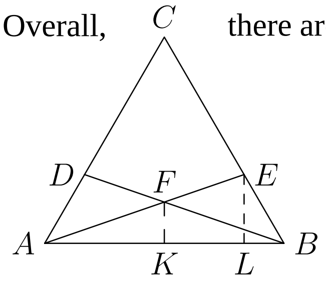
Question 7: You are playing a game. Your opponent has distributed five red balls into five boxes randomly. All arrangements are equally likely; that is, the left-to-right placements [0,0,5,0,0], [0,2,0,3,0], and [1,1,1,1,1] are equally likely. You place one blue ball into each box. The player with the most balls in a box wins the box (neither player wins a box with the same number of balls of each color). Whoever wins the most boxes wins the game. Compute the probability you win the game.

Solution: There are $\binom{5+5-1}{5-1}=126$ arrangements of the red balls, all equally likely. You win if and only if your opponent puts three or more red balls in any single box. For example: [3,1,1,0,0] means that your opponent wins the first box and you win the last two. There are five arrangements with five balls in one box, twenty arrangements with four in one box and a single ball in the other, and fifty arrangements with three in one box, and either two in one other box or a single ball in two boxes. The probability of winning is $\frac{50+20+5}{126}=\frac{75}{126}=\boxed{\frac{25}{42}}$.

Question 8: When $(x+y+z)^{2009}$ is expanded and like terms are grouped, there are *k* terms with coefficients that are *not* multiples of five. Compute *k*.

Solution: The coefficient of the term $x^a y^b z^c$ is $\frac{2009!}{a!b!c!}$, where we assume $a+b+c=2009$. The number of powers of five in the numerator is $\left\lfloor\frac{2009}{5}\right\rfloor+\left\lfloor\frac{2009}{5^2}\right\rfloor+\left\lfloor\frac{2009}{5^3}\right\rfloor+\left\lfloor\frac{2009}{5^4}\right\rfloor=401+80+16+3$. In order for this to equal the number of powers of five in the denominator, we need $\left\lfloor\frac{a}{5^k}\right\rfloor+\left\lfloor\frac{b}{5^k}\right\rfloor+\left\lfloor\frac{c}{5^k}\right\rfloor=\left\lfloor\frac{2009}{5^k}\right\rfloor$ for all *k*. Note that the floor function is sub-additive, so we will always have $LHS\le RHS$. Noting that $2009_{10}=31014_5$, we first distribute the three powers of 5^4 among *a*, *b*, and *c*. There are $\binom{3+3-1}{3-1}=10$ ways to place them. After doing this, we have placed 15 of the 16 powers of 5^3, so we only get to distribute one more, and there are

$\binom{1+3-1}{3-1}=3$ places for it to go. After this, all 80 powers of 5^2 have been taken care of, and we have 1 leftover power of 5 to place (3 places for it to go as well). After this, we have coefficients summing to 2005, so we can place the remaining 4 powers anywhere: there are $\binom{4+3-1}{3-1}=15$ places for these. Overall, there are $10\times3\times1\times3\times15=\boxed{1350}$ terms whose coefficients aren't multiples of 5.



Question 9: On equilateral triangle *ABC*, points *D* and *E* are on sides $\overline{AC}$ and $\overline{BC}$ such that *AD* = *BE* = 1. $\overline{BD}$ and $\overline{AE}$ meet at *F*. Given *AB* = 3, compute the area of triangle *ADF*.

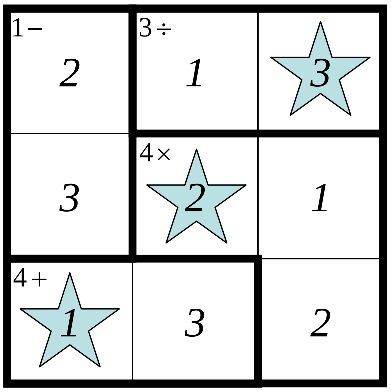
Solution: The area equals $\frac{AD\cdot AF\cdot\sin\angle DAF}{2}=\frac{AF\cdot\sin\angle CAE}{2}$. Dropping perpendiculars from *E* and *F* to $\overline{AB}$ at points *K* and *L* we get $BL=\frac{1}{2}$ and $\frac{AF}{AE}=\frac{AK}{AL}=\frac{3/2}{5/2}=\frac{3}{5}$. The area of triangle ADF is:

$$\frac{3}{10}\cdot AE\cdot\sin\angle CAE=\frac{3}{10}\cdot CE\cdot\sin\angle ACE=\frac{3}{10}\cdot2\cdot\frac{\sqrt{3}}{2}=\boxed{\frac{3\sqrt{3}}{10}}$$ using the Law of Sines on ΔACE.

Question 10: To solve a KenKen puzzle, you fill in an $n\times n$ grid with the digits $1,\dots,n$ according to the following two rules:

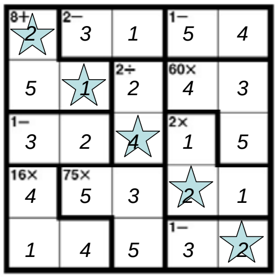
1. Each row and column contains exactly one of each digit.
2. Each bold-outlined group of cells is a cage containing digits which achieve the specified result using the specified mathematical operation: addition (+), subtraction (-), multiplication (×), and division (÷) on the digits in some order. Digits may repeat inside a cage.

A solved 3×3 KenKen appears to the right. Below is a 5×5 KenKen puzzle. On your answer sheet, enter the digits in the starred squares in the correctly solved KenKen puzzle, *in order from left to right*. In the example, you would enter 1,2,3. Digits may appear more than once in the starred squares.



1− 2	3÷ 1	☆3
3	4× ☆2	1
4+ ☆1	3	2

Solution: $\boxed{2,1,4,2,2}$ The solved puzzle is shown below. There are many ways to get to the final solution. The key thing to note is the unique arrangements of digits in the L-shaped $75\times$, $16\times$, and $2\times$ cages immediately give you the lower half of the puzzle, since the $1-$ cage in the lower right corner is uniquely determined by the other digits in the L-shaped cages.



8+ ☆2	2− 3	1	1− 5	4
5	☆1	2÷ 2	60× 4	3
1− 3	2	☆4	2× 1	5
16× 4	75× 5	3	☆2	1
1	4	5	1− 3	☆2

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Theme Round: Votes That Count (45 minutes)

"*Those who cast the votes decide nothing. Those who count the votes decide everything.*"

In this round, we will be looking at a number of different voting systems, as well as how voters choose who to vote for. In the simplest election, voters each choose a single candidate that they prefer over all of the others and the candidate with the most votes wins (this is called *plurality voting*). A slightly more complex election requires the winning candidate to receive greater than half of the votes (this is called *majority voting*), with a runoff between the two candidates receiving the most votes in the first round if no candidate has a majority (for example, the 2008 Georgia senate race).

Part 1: Preference Rankings

It is reasonable to assume that given a pair of candidates *A* and *B*, a voter either prefers one over the other ($A < B$ or $A > B$) or is indifferent between the two ($A = B$). Also, given a set of candidates, it is reasonable to assume these preference relationships are transitive ($A \le B$ and $B \le C \rightarrow A \le C$). Therefore, we can determine a preference ranking for each voter. For example, if a voter prefers candidate *A* over all others, is indifferent between candidates *B* and *C*, but prefer both of them over candidate *D*, then that voters preference ranking would be $A > B, C > D$. We can also write preference rankings vertically, with candidates on the same row being equally preferable: $\begin{array}{c} A \\ BC \\ D \end{array}$.

Question 1: Compute the number of distinct preference rankings of four candidates.

Note that $\begin{array}{c} A \\ BC \\ D \end{array}$ and $\begin{array}{c} A \\ CB \\ D \end{array}$ are *not* distinct preference rankings.

Solution: Let the *height* of a preference ranking be the number of rows in the vertical representation of a preference ranking. There are $4! = 24$ preference rankings of height 4, and one preference ranking of height 1. For height 3, there are three rows to choose where the tie should go, then $\binom{4}{2} = 6$ ways to pick the candidates in the tie, then 2 ways to pick the highest ranked remaining candidate, for a total of 36 rankings of height 3. Finally, a ranking of height two can either be two pairs or a single candidate either above or below three indifferent candidates. We know there are six arrangements of the first type and eight of the second (choose one of the four candidates then choose to put him above or below the other three). In total, there are $\boxed{75}$ preference rankings of four candidates.

A preference ranking is called *strict* if the voter always prefers one candidate over the other. For the remainder of the questions in this round, we will assume all preference rankings are strict.

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Consider an election with four voters (w, *x*, *y*, and *z*) and three candidates (*A*, *B*, and *C*). The voters' preference rankings are $\begin{array}{cccc} w & x & y & z \\ A & A & B & B \\ B & C & C & C \\ C & B & A & A \end{array}$. Based on first place votes alone, candidates *A* and *B* are tied 2-2.

One way to break the tie is to perform a *Borda count*: For each candidate (for example, *A*), and each voter (for example, *w*), count the number of candidates that the voter *w* prefers *A* over (in this example, 2). Sum these values over all voters, and the candidate with the highest sum wins. In this case, *B* is the winner of the Borda count.

Question 2: Consider an election with *m* candidates and *n* voters where one candidate has greater than half of the first place votes, but loses a Borda count, we'll call this situation an (*m*,*n*)-*Borda upset*. Compute the ordered pair (*m*,*n*) such that an (*m*,*n*)-Borda upset exists, and $m+n$ is minimized.

	w	x	y	z	T
A	2	2	0	0	4
B	1	0	2	2	5
C	0	1	1	1	3

Solution: Clearly there cannot be a Borda upset with two candidates, so $m \ge 3$. With three candidates, and an odd number of voters, a Borda upset occurs when there are *k* voters with preference ranking $\begin{array}{c} A \\ B \\ C \end{array}$ and $k-1$ voters with preference ranking $\begin{array}{c} B \\ C \\ A \end{array}$, provided that $2k < 3k-2 \to k > 2$, so there exists a $(3,5)$-Borda upset. With four candidates, there exists a three voter Borda upset: $\begin{array}{ccc} A & A & B \\ B & B & C \\ C & C & D \\ D & D & A \end{array}$, with *A* having a majority of first place votes, but losing a Borda count to *B* 7 to 6. The answer is $\boxed{(4,3)}$.

Another voting method using preference rankings is called *instant runoff voting* (IRV). In one version of IRV, all voters submit strict preference rankings of candidates. The following process is used to determine the winner:

Step 1: Does any candidate have a majority of voters that have them as their top choice?
If yes, then that candidate is the winner. END
If not, go to step 2.
Step 2: Is there a unique candidate with the fewest first place votes?
If yes, then remove that candidate from ALL preference rankings and go to Step 1.
If no, then some other process must be used to determine who is removed from the election.

So, for example, for three candidates *A*, *B*, and *C*, if there are 10 voters with preference ranking $\begin{matrix}A\\B\\C\end{matrix}$, 8 with preference ranking $\begin{matrix}B\\A\\C\end{matrix}$, and 6 voters with a preference ranking $\begin{matrix}C\\B\\A\end{matrix}$, then in the first round of IRV, candidate *C* would be eliminated, leaving 10 voters with preference ranking $\begin{matrix}A\\B\end{matrix}$ and 14 voters with preference ranking $\begin{matrix}B\\A\end{matrix}$. Therefore, candidate *B* wins this IRV election, despite the fact that candidate *A* had more first-place votes than any other candidate when the process began. When this happens with *m* candidates and *n* voters, we call this a (*m*,*n*)*-IRV upset.* Note that if there is a tie for last place at any stage, then the IRV election does not produce a winning candidate, so there must always be a unique last place candidate at every stage for the process to conclude.

Question 3: Let n_3 and n_4 be the minimum values of *n* such that there exists a (3, *n*)-IRV upset and (4, *n*)-IRV upset, respectively. Compute the ordered pair (n_3,n_4).

Solution: With three candidates, we need a distribution of first place votes v_A, v_B, and v_C such that $v_A > v_B > v_C$ and $v_A < v_B + v_C$. Then, we have a $(3, v_A + v_B + v_C)$-IRV upset with v_A rankings of $\begin{matrix}A\\B\\C\end{matrix}$ or $\begin{matrix}A\\C\\B\end{matrix}$, v_B rankings of $\begin{matrix}B\\A\\C\end{matrix}$ or $\begin{matrix}B\\C\\A\end{matrix}$, and v_C rankings of $\begin{matrix}C\\B\\A\end{matrix}$. Then, after the first round, when candidate *C* is eliminated, there are v_A voters with preference ranking $\begin{matrix}A\\B\end{matrix}$ and $v_B + v_C$ voters with preference ranking $\begin{matrix}B\\A\end{matrix}$, an upset. The ordered triple of positive integers that satisfies the inequalities with the sum $v_A + v_B + v_C$ minimized is $(v_A, v_B, v_C) = (4,3,2)$, so $n_3 = 9$. With four candidates, consider three voters with preference ranking $\begin{matrix}A\\B\\C\\D\end{matrix}$, two with $\begin{matrix}B\\A\\C\\D\end{matrix}$, two with $\begin{matrix}C\\B\\A\\D\end{matrix}$ and one with $\begin{matrix}D\\B\\C\\A\end{matrix}$. Candidate *D* is eliminated in the first round, transferring one first place vote to candidate *B*. Candidate *C* is eliminated in the second round, transferring two first place votes to candidate *B*, who now has a majority of first place votes. It is easy to verify there are no smaller values of n_4 than 8 that lead to a $(4, n_4)$-IRV upset. The answer to the question is $\boxed{(9,8)}$.

Part 2: Winning Coalitions and Relative Power in Elections

To win the presidential election, a candidate needs a majority of the 538 electoral votes (that is, 270). Most states distribute all of their electoral votes to the candidate that gets the most votes in their statewide election. However, is California (with 55 electoral votes) really eleven times as important as, say, West Virginia (with 5 electoral votes? The answer is, naturally, it depends.

Given *n* voters (which we will denote with the set $S = \{1, \ldots, n\}$), every subset of *S* is either a winning coalition or a losing coalition. Winning coalitions are *monotonic*, that is, if *W* is a winning coalition and *V* contains *W*, then *V* is also a winning coalition.

We define a *weighted majority vote* (WMV) as follows: given an integer threshold *T* and that each voter *i* has v_i votes (v_i all integers) a subset *W* of *S* is a winning coalition of a WMV if and only if $\sum_{i \in W} v_i \geq T$. We can represent a WMV by $(T; v_1, v_2, \ldots, v_n)$. For example, the US presidential election (ignoring the fact that some states that split their electoral votes) can be represented as a WMV: $(270; 55, 34, 31, 27, \ldots, 3, 3)$.

Many elections or voting processes can be represented as WMVs. On some issues, the Australian government has votes involving the six state legislatures plus the federal government. Each of the six states gets one vote, while the federal government gets two. In the case of a 4-4 tie, the federal government makes the decision. It is easy to show that this voting process is equivalent to the WMV $(5; 1, 1, 1, 1, 1, 1, 3)$, with the three votes going to the federal government: a coalition is a winning coalition if and only if it contains at least five states or the federal government and at least two states.

Question 4: In the UN Security Council, there are five permanent members and ten non-permanent members. For any resolution to pass, it must have the support of *all five* of the permanent members and at least four of the ten non-permanent members. Consider the WMV $(T; P, P, P, P, P, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ where *P* denotes the integral number of votes for each of the permanent members of the council and the ones denote the votes of the non-permanent members of the council. Compute the ordered pair (*T*, *P*) such that *T* is minimized and *W* is a winning coalition in this WMV if and only if the corresponding set of council members can pass a resolution in the UN Security Council.

Solution: The smallest winning coalition is all five permanent members of the council with four non-permanent members. Therefore, $5P + 4 \geq T$. To minimize *T*, we say that $5P + 4 = T$. There are two maximal non-winning coalitions: one consists of the five permanent members as well as three non-permanent members, so $5P + 3 < T$ (this is obvious). The second consists of four permanent members and all ten of the non-permanent members, so $4P + 10 < T$. From the inequalities, we can establish that $4P + 10 < 5P + 4 \to 6 < P$, so $P = 7$ in order to minimize T and $T = 39$. The answer is $\boxed{(39,7)}$.

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To measure the relative power of a voter, we consider the *Shapley-Shubik Power Index*. It is defined as follows.

Consider all $n!$ orderings of n voters. Given an ordering $\left(p_1,\ldots,p_n\right)$ of the set $\{1,\ldots,n\}$, let $W_i=\bigcup_{j\le i}\left\{p_j\right\}$. A voter p_k is *pivotal* for this ordering if W_{k-1} is not a winning coalition, but W_k is a winning coalition. The Shapley-Shubik Power Index for a voter is the fraction of the orderings for which the voter is pivotal.

For example, consider the WMV (7;5,4,2). There are six orderings of the voters, and we will underline the pivotal voter in each case: $24\underline{5}$ $2\underline{5}4$ $42\underline{5}$ $4\underline{5}2$ $5\underline{2}4$ $5\underline{4}2$. Thus, the Shapley-Shubik Power Index of the first voter is 4/6, or 2/3, while the Shapley-Shubik Power Index of the other two voters are 1/6 each.

Question 5: Consider the WMV for the Australian federal government: $(5;1,1,1,1,1,1,3)$. Let s be the Shapley-Shubik Power Index of any one of the states and let f be the Shapley-Shubik Power Index of the federal government. Compute the ordered pair (s,f).

Solution: To be the pivotal voter, the federal government's three votes needs to be in either the third, fourth, or fifth position. There are $7!=840$ orderings of the voters, of which $3\times 6!=360$ have the federal government in the third, fourth, or fifth position, so the Shapley-Shubik Power Index of the federal government is $\frac{360}{840}=\frac{3}{7}$. The sum of the power indices across all voters is one, and all of the states are equally powerful, so $6s+\frac{3}{7}=1$. This simplifies to $6s=\frac{4}{7}\to s=\frac{4}{42}=\frac{2}{21}$. The answer is $\boxed{\left(\frac{2}{21},\frac{3}{7}\right)}$.

There are other ways to define winning coalitions. In the US legislature, a bill is sent to the president for signature ("winning") if and only if it has a coalition of at least 218 of the 435 members of the House and at least 51 of the 101 members of the Senate (including the Vice President in the Senate count, the Vice President is not required to be a part of a coalition for it to be a winning coalition) supporting it.[1]

Question 6: Compute the Shapley-Shubik Power Index of the Vice President.

Solution: We first claim that the sum of the power indices over all of the members of the House is equal to the sum of the power indices over all of the members of the Senate, namely ½. For every ordering of the members of the House and Senate where a House member is pivotal, the reverse ordering of the House and Senate members has a Senate member as pivotal, and vice versa. Since all Senate members have the same Shapley-Shubik Power Index, the power index of the vice president is $\frac{1/2}{101}=\boxed{\frac{1}{202}}$.

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[1] Technically, the Vice President does not vote, except in the case of ties. It is equivalent, however, to let the vice president vote all the time, since his vote only matters when the other 100 senators are tied 50-50.

If $W \subseteq S = \{1, 2, \ldots, n\}$ is a winning coalition, $i \in W$ is called a *swing voter* for W if $W - \{i\}$ is a losing coalition. We can define another power index called the *Banzhaf-Penrose Power Index* as follows. Let s_i be the number of winning coalitions for which voter i is a swing voter. The Banzhaf-Penrose Power Index of voter i is

$$B_i = \frac{s_i}{\sum_{j=1}^{n} s_j}.$$

Going back to our earlier WMV example, (7;5,4,2), there are three winning coalitions, {5,4}, {5,2}, and {5,4,2}. Both voters are swing voters in the first two coalitions, but only the 5 in the third coalition is a swing voter. Accordingly, there are five swing voters in total, so the Banzhaf-Penrose Power Indices for the three voters are 3/5, 1/5, and 1/5, respectively.

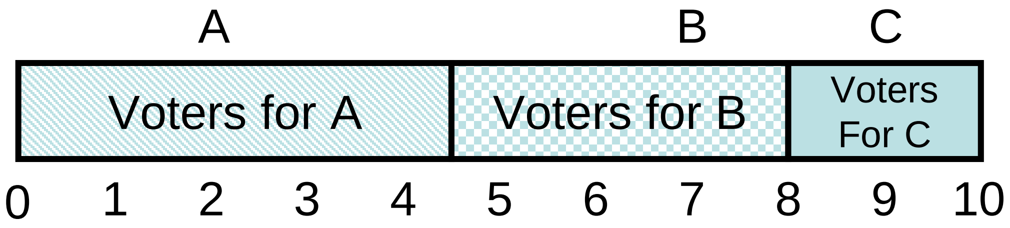
Question 7: Consider the WMV (6;1,2,3,4). Compute the ordered 4-tuple (B_1, B_2, B_3, B_4), that is, the Banzhaf-Penrose Power Index for each voter.

Solution: We list the winning coalitions and underline all swing voters: $\underline{24}$, $\underline{34}$, $\underline{123}$, $1\underline{24}$, $1\underline{34}$, $23\underline{4}$, and 1234. There are $2+2+3+2+2+1=12$ swing voters in total, so $(B_1, B_2, B_3, B_4) = \left(\frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{5}{12}\right) = \boxed{\left(\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{5}{12}\right)}$.

Part 3: Models of Voter Preference

There are many ways that voters come up with their preference rankings for candidates. Some people are single-issue voters. That is, their preference ranking depends on how close a candidate's view on a single issue is to their own. Say a candidate's view on an issue could be represented as a real number on a 0 to 10 scale, and there were three candidates (*A*,*B*,*C*) whose views on an issue are 2, 7, and 9. Then, a single-issue voter whose view on an issue was 5 would have a preference ranking of $B > A > C$ since the distance between the voter's view on the issue and those of the candidates are 3, 2, and 4, respectively.

If all voters were single issue voters and voters' views on the issue were evenly distributed throughout the 0-10 scale, then candidate *A* would win a plurality election, as voters with views between 0 and 4.5 (45%) would vote for candidate *A*, those with views between 4.5 and 8 (35%) would vote for candidate *B*, and those with views between 8 and 10 (20%) would vote for candidate *C*. We will call this a *single-issue election*.



Question 8: In a single-issue election, candidates *A* and *B* have randomly picked their views on the issue (all views are equally likely and the views of candidates *A* and *B* are independent). Candidate *C*, knowing the views

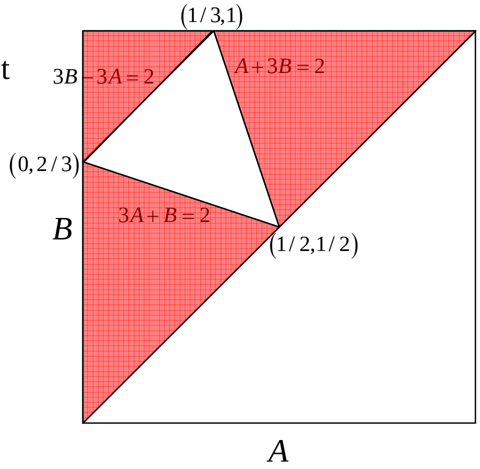
of candidates *A* and *B*, can choose his view to maximize the fraction of the votes he receives. Compute the probability that it is impossible for candidate *C* to pick a view that results in candidate *C* winning the election.

Solution: We can simplify calculations by setting the range of views on the single-issue election to be from 0 to 1. Then the probability that it is impossible for candidate *C* to pick a winning view is equal to the area of the set of points (A, B) such that it is impossible for candidate *C* to pick a winning view, given the other two views are *A* and *B*. It is easier to find the set of points where *C* can pick a winning view. Assume without loss of generality that $0 \le A \le B \le 1$. There are three cases to consider:

Cases	Strategy for *C*	Votes for *A*	Votes for *B*	Votes for *C*
$C < A$	Pick just less than *A*	$\frac{B-A}{2}$	$\frac{B-A}{2}+(1-B)$	A
$A < C < B$	See below	$A+\frac{C-A}{2}$	$\frac{B-C}{2}+(1-B)$	$\frac{B-A}{2}$
$B < C$	Pick just more than *B*	$A+\frac{B-A}{2}$	$\frac{B-A}{2}$	$(1-B)$

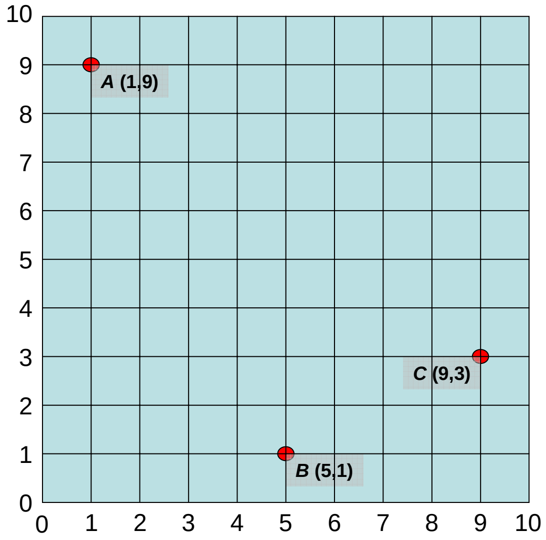
In the first case, candidate *C* wins if $\frac{B-A}{2}+(1-B) < A$ or $2 < 3A+B$. In the third case, candidate *C* wins if $A+\frac{B-A}{2} < 1-B$ or $2 > A+3B$. In the second case, candidate *C* wins if there is a number between *A* and *B* such that $A+\frac{C-A}{2} < \frac{B-A}{2}$ and $\frac{B-C}{2}+(1-B) < \frac{B-A}{2}$, or $C < B-2A$ and $C > A+2(1-B)$. Since we know that $B-2A \le 1$ and $0 \le A+2(1-B)$, there is a winning pick for *C* provided $A+2(1-B) < B-2A$ or $2 < 3B-3A$.

Plotting these three regions on the *AB*-plane, we see the region where *C* cannot win (the clear section). The triangle has a base and height of $\frac{\sqrt{2}}{3}$, and area $\frac{1}{9}$. By symmetry, there is an identical triangle on the other side of the line $A = B$, so the answer is $\boxed{\frac{2}{9}}$.



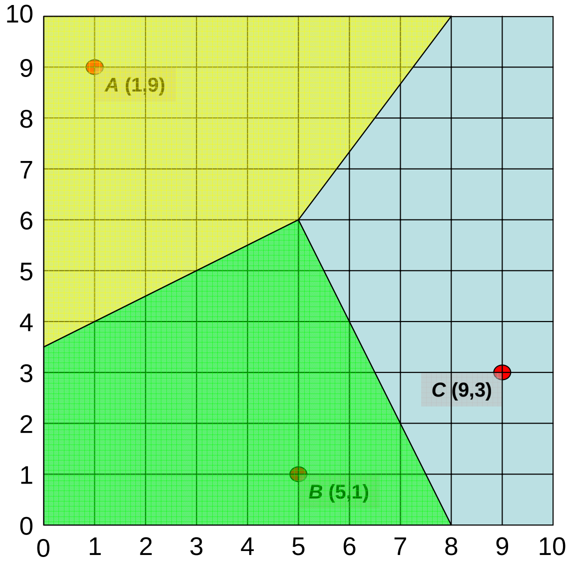
We would hope that voters are a bit more nuanced than this and perhaps base their preferences on *two* issues! Assume there are three candidates *A*, *B*, and *C* with views on two issues (again, on a 0-10 scale) as given in the graph below:

2009 ARML Local Solutions



A voter sets their preference ranking depending on the (Cartesian) distance between their views on these two topics and those of the candidate. If voters' views on both topics were independent and evenly distributed throughout the 0-10 scales, then the fraction of the votes received by a candidate would be equal to the fraction of the square $[0,10]\times[0,10]$ that is closest to the point corresponding to the candidate's views.

Question 9: Under the above assumption, if f_i is the fraction of the vote received by candidate *i*, compute the ordered triple (f_A, f_B, f_C). It is acceptable to give the answer as an ordered triple of fractions or percentages.



Solution: The set of points equidistant from two candidates views forms the perpendicular bisector of the segment between the two views. Voters on either side of the bisector favor the respective candidate. When we show all three perpendicular bisectors on the graph, we get the partition of the voter space as seen to the right. Adding up areas of rectangles and triangles, we get the voter distribution $(32.25\%, 32.75\%, 35\%)$ or $\boxed{\left(\frac{129}{400}, \frac{131}{400}, \frac{7}{20}\right)}$.

Question 10: A voter in this election with views (x_v, y_v) is indifferent between all three candidates. Compute the ordered pair (x_v, y_v).

Solution: Look at the graphic and note the bisectors all meet at $\boxed{(5,6)}$. Alternately, you can find the center of the circle that goes through the three candidates' views using the method of your choice and get the same answer.

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Individual Round (5 pairs, 10 minutes per pair)

Question 1: Given $A \wedge B = \dfrac{A+B}{AB}$, compute $(2 \wedge 6) \wedge (3 \wedge 4)$.

Solution: $(2 \wedge 6) \wedge (3 \wedge 4) = \dfrac{2+6}{2\times 6} \wedge \dfrac{3+4}{3\times 4} = \dfrac{8}{12} \wedge \dfrac{7}{12} = \dfrac{\frac{8}{12}+\frac{7}{12}}{\frac{8}{12}\times\frac{7}{12}} = \dfrac{\frac{15}{12}}{\frac{56}{144}} = \dfrac{15\times 12}{56} = \boxed{\dfrac{45}{14}}$.

Question 2: Compute the smallest value of n for which the mean and median of the set $\{13, 21, 26, 28, 41, n\}$ are equal.

Solution: The mean of the set is $\dfrac{129+n}{6}$; the middle two elements of the set are either 21 and 26 (if $n \le 21$), n and 26 (if $21 \le n \le 28$), or 26 and 28 (if $n \ge 28$). The median in these cases are $\dfrac{47}{2}$, $\dfrac{n+26}{2}$, and 27, respectively. Starting with the smallest possibility (the first one) and solving for n, we get $\dfrac{129+n}{6} = \dfrac{47}{2} \rightarrow \boxed{n=12}$.

Question 3: Compute the number of subsets S of the set $\{1,2,3,4,5,6,7,8,9\}$ such that the smallest element of S is equal to the size of S.

Solution: Condition off of k, the smallest element of S, with $1 \le k \le 5$. Then there are $k-1$ elements left to pick from $\{k+1,\ldots,9\}$ giving us $\binom{8}{0}+\binom{7}{1}+\binom{6}{2}+\binom{5}{3}+\binom{4}{4} = 1+7+15+10+1 = \boxed{34}$.

Question 4: $A_1A_2\ldots A_{2009}$ is a regular 2009-gon of area K. If $A_1A_{1005} = 6$**,** compute $\lfloor K \rfloor$.

Solution: Consider the circle circumscribing the 2009-gon. Note that $A_1A_{1004} = 6$ as well, and $\angle A_{1004}A_1A_{1005} = \dfrac{\pi}{2009} = \alpha$. The diameter of the circle circumscribing the 2009-gon is $\dfrac{6}{\cos\left(\frac{\alpha}{2}\right)}$, so its radius is $r = \dfrac{3}{\cos\left(\frac{\alpha}{2}\right)} \approx 3$. The area of the polygon is $\dfrac{2009r^2}{2}\sin\left(\dfrac{2\pi}{2009}\right) \approx \dfrac{2009r^2}{2} \times \dfrac{2\pi}{2009} = \pi r^2 \approx 28.27$, so the answer is $\boxed{28}$**.** We did play a bit fast and loose with the explanation here, but basically, the area of the 2009-gon is so close to the area of its circumscribed circle, the effect on the area would not be noted in the first or second decimal place.

Question 5: *x*, *y*, and *z* are real numbers randomly chosen between 0 and 100. Compute the probability that $\lceil x+y+z \rceil = \lceil x \rceil + \lceil y \rceil + \lceil z \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to *x*.

Solution: Write each number as $x = x_I + x_R$ where x_I is an integer and $0 < x_R \le 1$, the integer part and the remainder. Then $\lceil x \rceil = \lceil x_I + x_R \rceil = x_I + \lceil x_R \rceil = x_I + 1$ and $\lceil x \rceil + \lceil y \rceil + \lceil z \rceil = x_I + y_I + z_I + 3$. $\lceil x+y+z \rceil = \lceil (x_I + y_I + z_I) + (x_R + y_R + z_R) \rceil = (x_I + y_I + z_I) + \lceil (x_R + y_R + z_R) \rceil$ Therefore, the two sides of the equation are equal if and only if the sum of the remainders is greater than two. The probability that the sum of three numbers randomly drawn from between 0 and 1 is greater than two is $\boxed{\frac{1}{6}}$. To see this, consider the volume of the intersection of the set $x + y + z \ge 2$ and the unit cube $(0,1] \times (0,1] \times (0,1]$ in three-dimensional space.

Question 6: An integer is called "rotatable" if it only uses the digits 0, 1, 6, 8, or 9. Compute the sum of all three-digit rotatable numbers (note that the first digit cannot be a zero).

Solution: There are 100 rotatable numbers, since there are four choices for the first digit and five choices apiece for the latter digits. For each of 1,6,8, and 9, there are 25 three-digit rotatable numbers that begin with that digit. The contribution to the sum of the hundreds place alone is $25(100 + 600 + 800 + 900) = 60000$. Similarly, for each of the digits 0,1,6,8, and 9, there are 20 three-digit rotatable numbers that have that digit in the tens place and the ones place, contributing $20(00 + 10 + 60 + 80 + 90) = 4800$ and $20(0 + 1 + 6 + 8 + 9) = 480$ to the sum, respectively. The total is $\boxed{65280}$.

Question 7: Compute the number of ways to place the integers 1 through 7 in the blanks below so that the chain of inequalities is satisfied.

_ < _ < _ > _ < _ > _ > _

(Example: $1 < 2 < 5 > 4 < 7 > 6 > 3$)

Solution: The 7 must go in the 3rd or 5th blank. Presume it goes in the 3rd blank. All of the placements with the 7 in the 5th blank can be obtained by flipping the ones for which it goes in the 3rd, so we will double our answer at the end. There are $\binom{6}{2} = 15$ ways to choose the two numbers to place in the first two blanks, and once chosen, they must be entered in increasing order. Of the 4 remaining numbers, the largest must go in the 5th blank. There are 3 integers left to place. We have 3 choices of which one to place in the 4th blank, and the remaining two must be placed in decreasing order in the last two blanks. Thus, counting the reversed placements, there are $15 \cdot 3 \cdot 2 = \boxed{90}$ valid placements.

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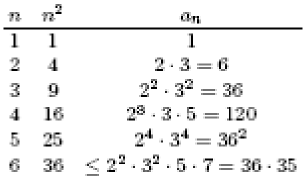
Question 8: The period of a sequence $s_0, s_1, \ldots$ is the smallest positive integer *k* such that $s_n = s_{n+k}$ for all $n \geq 0$. Compute the period of the sequence defined by $s_0 = \tan 4^\circ, s_n = \dfrac{2s_{n-1}}{1-s_{n-1}{}^2}$ for $n > 0$.

Solution: The closed form of the sequence is $s_n = \tan\left(2^n \times 4^\circ\right)$, so we need to find the smallest $n > 0$ for which $\tan 4^\circ = \tan\left(2^n \times 4^\circ\right)$. Since the tangent function is periodic with period 180°, it is equivalent to find the smallest positive *n* for which 180 divides $4\left(2^n - 1\right)$. Dividing out the 4, we seek the smallest positive *n* such that 45 divides $2^n - 1$.

We could brute-force this, but we note that $45 = 5 \times 9$, and $2^n - 1$ is a multiple of five when 4 divides *n* (this can be shown with Fermat's Little Theorem or listing the first few values of $2^n - 1$). Similarly, $2^n - 1$ is a multiple of nine when 6 divides *n*. Thus, the period is $lcm\{4,6\} = \boxed{12}$.

Question 9: For a positive integer *n*, define a_n to be the smallest positive integer with exactly n^2 positive factors. Compute the smallest *n* for which $a_n > a_{n+1}$.

Solution: Recall that if $p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$ is the factorization of *n* into powers of distinct primes, then *n* has $(e_1+1)(e_2+1)\cdots(e_k+1)$ positive factors. We start computing the first few values of a_n:



n	n^2	a_n
1	1	1
2	4	$2 \cdot 3 = 6$
3	9	$2^2 \cdot 3^2 = 36$
4	16	$2^3 \cdot 3 \cdot 5 = 120$
5	25	$2^4 \cdot 3^4 = 36^2$
6	36	$\leq 2^2 \cdot 3^2 \cdot 5 \cdot 7 = 36 \cdot 35$

Which is good enough to show that the smallest *n* for which $a_n > a_{n+1}$ is $\boxed{5}$.

Question 10: θ is an acute angle for which $\sin\theta$, $\sin 2\theta$, and $\sin 4\theta$ form a strictly increasing arithmetic sequence. Compute $\cos^3\theta - \cos\theta$.

Solution: We have $\sin 4x - \sin 2x = \sin 2x - \sin x$. We will use double-angle identities to rewrite both sides of this equality:

$$\sin 4x - \sin 2x = \sin 2x - \sin x$$

$$\sin 4x - 2\sin 2x = -\sin x$$

$$4\cos 2x \cos x \sin x - 4\cos x \sin x = -\sin x$$

$$4\left(2\cos^2 x - 1\right)\cos x - 4\cos x = -1$$

$$8\cos^3 x - 8\cos x = -1$$

$$\cos^3 x - \cos x = \boxed{-\frac{1}{8}} \text{ or } \boxed{-0.125}$$

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Relay Round (6, 8, and 10 minutes per round)

R1-1: $(47)_a = (74)_b$, where $(x)_c$ denotes the number represented by x written in base-c. Compute the smallest value for $a+b$ where a and b are both positive integers.

Solution: $4a+7=7b+4 \rightarrow 4a+3=7b$. We need to find integral values of a and b, both greater than 7, such that this equality holds. b must be odd, and $(a,b)=(15,9)$ satisfies the equation, so $a+b=\boxed{24}$.

R1-2: Let $T = TNYWR$. Let y be the product of the digits of T and let z be the sum of the digits of T. The roots of the quadratic function $f(x)=x^2+Ax+B$ have arithmetic mean y and geometric mean z. Compute the ordered pair (A,B).

Solution: $-A$ is the sum of the roots, which is twice the arithmetic mean of the digits (or $2y$). B is the product of the roots of $f(x)$, which is the square of the geometric mean (or z^2). As $y=2\times4=8$ and $z=2+4=6$, the ordered pair $(A,B)=\left(-2\times8,6^2\right)=\boxed{(-16,36)}$.

R2-1: Let $f(x)=x^2-4^{\lceil \log_{10} x \rceil}$. Compute $f\left(f(7)\right)$.

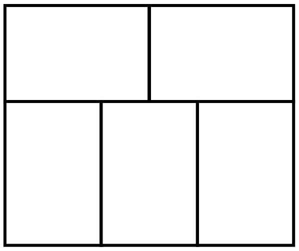
Solution: $f\left(f(7)\right)=f\left(7^2-4^{\lceil \log_{10} 7 \rceil}\right)=f\left(7^2-4^1\right)=f(45)=45^2-4^{\lceil \log_{10} 45 \rceil}=45^2-4^2=2025-16=\boxed{2009}$.

R2-2: Let $T = TNYWR$. $a_1,a_2,\ldots,a_7$ is an arithmetic sequence of increasing positive integers whose terms sum to T. Compute the smallest possible value for a_2.

Solution: If $a_2=a_1+d$, then $T=a_1+\cdots+a_7=7a_1+21d=7(a_1+3d)$. As $T=2009$, $a_1+3d=287$. To minimize a_2, you minimize a_1. The smallest positive value for a_1 is 2, so $d=95$, and $a_2=2+95=\boxed{97}$.

R2-3: Let $T = TNYWR$. Compute the side length of the largest square that can be completely covered without overlap by at most T 2×3 tiles. Tiles may not be broken, but may be rotated, and not all T tiles must be used.

Solution: Since the area of the tile is six, the area of the square must be divisible by six, which also means that the side length of the square must be divisible by six. If the square has side length $L=6k$, the area of the square is $36k^2$, requiring $6k^2$ 2×3 tiles to cover. Since $T=97$, the largest value of k such that $6k^2\le97$ is $k=4$, so $L=\boxed{24}$.



A covering of a 6x5 rectangle by 3x2 and 2x3 tiles

2009 ARML Local Solutions

R3-1: M and N are distinct positive numbers such that $\log_M N = \log_N M$. Compute MN.

Solution: Let $k = \log_M N = \log_N M$. Then $M^k = N$ and $N^k = M$. Combining the two equalities gives $(MN)^k = MN \to (MN)^{k-1} = 1 \to MN = \boxed{1}$.

R3-2: Let $T = TNYWR$. The area bounded by the x-axis, the lines $x = T$, $x = 4$, and $y = mx$ is 15. Compute the largest value of m.

Solution: The region bounded by the lines (assuming $T > 0$) is a trapezoid with area $|4-T|\frac{(4+T)m}{2}$. As $T = 1$, we get $|4-1|\frac{(4+1)m}{2} = 15 \to \frac{15m}{2} = 15 \to m = \boxed{2}$.

R3-3: Let $T = TNYWR$. An equilateral triangle and regular hexagon have equal perimeters. If the area of the triangle is T, compute the area of the hexagon.

Solution: The hexagon has sides with lengths equal to half that of the triangle. H be the side length of the hexagon. The area of the triangle is $\frac{(2H)^2\sqrt{3}}{4} = H^2\sqrt{3}$. The area of the hexagon is $6\frac{H^2\sqrt{3}}{4} = \frac{3}{2}H^2\sqrt{3} = \frac{3}{2}T = \boxed{3}$.

R3-4: Let $T = TNYWR$. Let $A \wedge B = \frac{A+B}{AB}$ and $A \vee B = \frac{AB}{A+B}$. If $k \vee (k \wedge k) = \frac{k}{T^2}$, compute the greatest value of k.

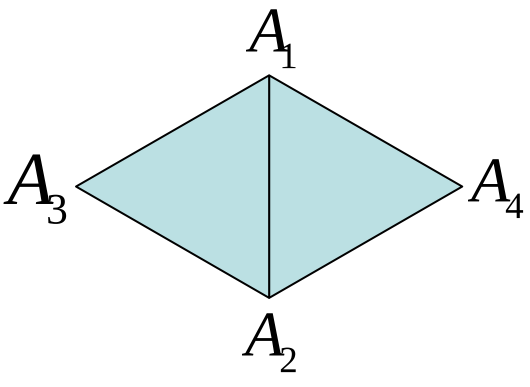
Solution: $k \vee (k \wedge k) = k \vee \left(\frac{k+k}{k \times k}\right) = k \vee \frac{2}{k} = \frac{k \times \frac{2}{k}}{k + \frac{2}{k}} = \frac{2}{k + \frac{2}{k}} = \frac{2k}{k^2+2} = \frac{k}{T^2} \to T^2 = \frac{k^2+2}{2}$. As $T = 3 \to k = \pm 4$.

The greatest value of k is $\boxed{4}$.

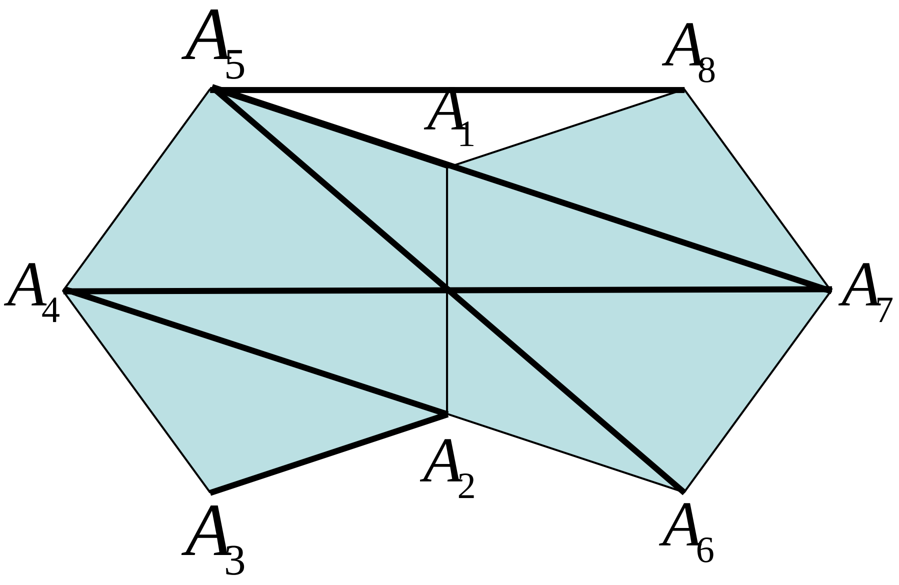
R3-5: Let $T = TNYWR$. The sum of the two three-digit numbers $\underline{2T3}$ and $\underline{A6B}$ is a multiple of eleven. Compute the smallest possible value of $A + B$.

Solution: $\underline{2T3} \equiv 3 - T + 2 \pmod{11}$ and $\underline{A6B} \equiv B - 6 + A \pmod{11}$, so $\underline{2T3} + \underline{A6B} \equiv 3 - T + 2 + B - 6 + A \equiv A + B - (T+1) \pmod{11}$. As $T = 4$, then $A + B \equiv 5 \pmod{11}$. The smallest possible value of $A + B$ is $\boxed{5}$.

R3-6: Let $T = TNYWR$. $A_1A_2A_3\dots A_T$ and $A_1A_2A_{T+1}\dots A_{2T-2}$ are distinct regular T-gons in the plane. Let S be the set of all real numbers that are distances between distinct vertices. Compute the number of elements in S. (In the example to the left, there are two distinct non-zero distances, A_1A_2 and A_3A_4.)



Solution: There are $\left\lfloor \frac{T}{2} \right\rfloor$ distinct distances between points inside a regular T-gon. There are $\left\lfloor \frac{T-2}{2} \right\rfloor (T-2)$ distinct distances between vertices in the separate T-gons when T is even, $\left\lfloor \frac{T-2}{2} \right\rfloor (T-2)+1$ if T is odd. As $T = 5$, there are two intra-pentagon distances (A_2A_3 and A_2A_4), and four inter-pentagon distances (A_4A_7, A_5A_6, A_5A_7, and A_5A_8), the distances in the two sets are distinct, so the total is $\boxed{6}$.



Final note: The authors note that the sets of intra- and inter-polygon distances do not always contain distinct elements. We welcome any proofs or comments regarding this problem for other values of T.