2020 ARML Local Problems and Solutions

Team Round (45 minutes)

T-1 The *primorial* of a natural number is the product of all primes less than or equal to that natural number. For example, the primorial of 4 is $2 \cdot 3 = 6$. Compute the number of natural numbers whose primorial is 210.

Answer: 4

Solution: The set of primes is $\{2, 3, 5, 7, 11, \ldots\}$. Notice that $210 = 2 \cdot 3 \cdot 5 \cdot 7$, and the least prime number greater than 7 is 11. Therefore 7, 8, 9, and 10 have primorials of 210. Notice that natural numbers less than 7 have primorials no greater than $2 \cdot 3 \cdot 5 = 30$, and natural numbers greater than 10 have primorials that are at least $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. Thus the answer is **4**.

T-2 Compute the number of whole numbers less than 1000 whose digit sum is 4.

Answer: 15

Solution: Suppose that leading 0's are allowed. For example, the whole number 020 has a leading 0 and represents the number 20.
If the digit sum of a whole number less than 1000 is 4, then the digits are either 4, 0, and 0, in some order; 3, 1, and 0, in some order; 2, 2, and 0, in some order; or 2, 1, and 1, in some order.
There are 3 whole numbers less than 1000 for which the digits are 4, 0, and 0, in some order: 400, 40, and 4.
There are 6 whole numbers less than 1000 for which the digits are 3, 1, and 0, in some order: 310, 301, 130, 103, 31, and 13.
There are 3 whole numbers less than 1000 for which the digits are 2, 2, and 0, in some order: 220, 202, and 22.
There are 3 whole numbers less than 1000 for which the digits are 2, 1, and 1, in some order: 211, 121, and 112.
Thus the answer is $3 + 6 + 3 + 3 = \mathbf{15}$.

Alternate Solution: Use the "stars-and-bars" approach. Allow leading 0's; then the problem is equivalent to finding solutions to $a + b + c = 4$ where a, b, and c are integers with $a, b, c \geq 0$. There are $\binom{4+2}{2} = 15$ such solutions.

T-3 The *ARML Averages* is a baseball team for which every batter has a batting average of 0.270. That is, the probability that a batter will make a hit when they bat is 0.270; otherwise, the batter makes an out. Each time at bat is considered independent of all other times at bat. Given that the *ARML Averages* make 27 outs in a baseball game, and that the game ends when the 27th out is made, compute the integer closest to the number of times at bat they should expect to have in a game.

Answer: 37

Solution: If a batter's probability of getting a hit is 0.270, then the probability of making an out is 0.730. Because the 27th out ends the game, solve $\dfrac{27}{A} = \dfrac{730}{1000}$ to obtain the number of at-bats, which is $A = \dfrac{2700}{73} = 36\dfrac{72}{73} \approx \mathbf{37}$.

T-4 Twenty fair 20-sided dice, each of whose sides are numbered $1, 2, \ldots, 20$, are rolled, and all rolls are independent. Any dice that rolled prime numbers are discarded, and the remaining dice are rolled again. Compute the expected value of the sum of the results of the rerolled dice.

Answer: 126

Solution: The probability that a die survives to the next roll is $\frac{12}{20}$, as there are eight prime numbers less than or equal to 20, namely 2, 3, 5, 7, 11, 13, 17, and 19. Therefore the expected number of dice rerolled is 12, and the expected value of each die's roll is 10.5, so the expected value of the sum is $12 \cdot 10.5 = \mathbf{126}$.

T-5 Compute the least positive integer n for which $10n + 1$ and $10n + 9$ are prime but $10n + 3$ and $10n + 7$ are composite.

Answer: 40

Solution: If $n = 3k$ for some integer k, then $10n + 9 = 3(10k + 3)$ is composite. Similarly, if $n = 3k + 2$ for some integer k, then $10n + 1 = 3(10k + 7)$ is composite. Thus $n = 3k + 1$ for some integer k. Note that $10n + 3 = 30k + 13$ is prime for $k = 0, 1, 2, 3, 5, 6, 7$ by inspection, with respective values of $10n + 3$ equal to 13, 43, 73, 103, 163, 193, and 223. Note also that $10n + 7 = 30k + 17$ is prime for $k = 4$ with $10n + 7 = 137$. Note also that $10n + 9 = 30k + 19$ is composite for $k = 8, 9, 10$ (with $10n + 9$ equaling $259 = 7 \cdot 37$, $289 = 17^2$, and $319 = 11 \cdot 29$, respectively) and $10n + 1 = 30k + 11$ is composite for $k = 11$ and 12 (with $10n + 1$ equaling $341 = 11 \cdot 31$ and $371 = 7 \cdot 53$, respectively).
Finally, if $k = 13$, then $n = 40$, and $10n + 1 = 401$ and $10n + 9 = 409$ are prime while $10n + 3 = 403 = 13 \cdot 31$ and $10n + 7 = 407 = 11 \cdot 37$ are composite, so the answer is $n = \mathbf{40}$.

T-6 Let n be a positive integer, and consider the list $1, 2, 2, 3, 3, 3, \ldots, n, n, \ldots, n$ where the integer k appears k times in the list for $1 \leq k \leq n$. The integer n will be called "*ARMLy*" if the median of the list is <u>not</u> an integer. The least ARMLy integer is 3. Compute the least ARMLy integer greater than 3.

Answer: 20

Solution: The condition is satisfied if there exists a positive integer m such that $1, 2, 2, 3, 3, 3, \ldots, n, n, \ldots, n$ can be split into two sequences of equal length: $1, 2, 2, 3, 3, 3, \ldots, m, m, \ldots, m$ and $m + 1, m + 1, \ldots, n, n, \ldots, n$.

Therefore $\frac{n(n+1)}{4} = \frac{m(m+1)}{2}$ or $n^2 + n = 2m^2 + 2m$. Multiplying both sides by 2 yields $2n^2 + 2n = 4m^2 + 4m$ or $n^2 + (n+1)^2 = (2m+1)^2$. The smallest Pythagorean triples satisfying the last equation are $(3, 4, 5)$, $(20, 21, 29)$, and $(119, 120, 169)$. Thus the three least integers satisfying the condition are 3, **20**, and 119.

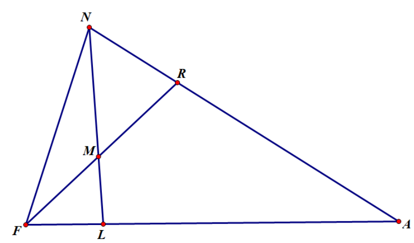
Note that these values can also be found by solving $2\binom{m}{2} = \binom{n}{2}$ using Pascal's Triangle and looking down the second column:

$$1, \underline{3}, \underline{6}, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \underline{\mathit{105}}, 120, 136, 153, 171, 190, \underline{\mathit{210}}, \ldots$$

One can rewrite $2n^2+2n = 4m^2+4m$ as $(2n+1)^2-1 = 2((2m+1)^2-1)$. If $a = 2n+1$ and $b = 2m+1$, this is equivalent to Pell's equation $a^2-2b^2 = -1$, with solutions $(1, 1), (7, 5), (41, 29), \ldots$, from which it follows that $n = 3, 20, 119, \ldots$.

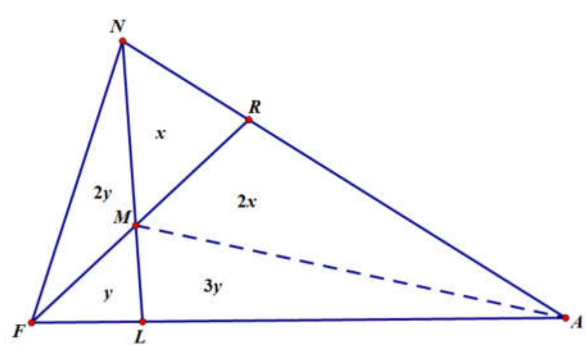
One can also look at $n(n+1) = 2m(m+1)$ and factor quickly to reach $20 \cdot 21 = 2 \cdot 10 \cdot 21 = 2 \cdot 2 \cdot 5 \cdot 3 \cdot 7 = 2 \cdot 14 \cdot 15$.

T-7 In $\triangle FAN$, point R lies on $\overline{AN}$ so that $RA = 2 \cdot RN$ and point L lies on $\overline{AF}$ so that $LA = 3 \cdot LF$. Point M is the intersection of $\overline{FR}$ and $\overline{NL}$. Given that $[ARML] = 1974$, compute $[FAN]$.



Answer: 3384

Solution: Draw $\overline{AM}$. Because $RA = 2 \cdot RN$, it follows that $[ARM] = 2 \cdot [NRM]$. Similarly, $[ALM] = 3 \cdot [FLM]$. Notice also that $[ARF] = 2 \cdot [NRF]$ so $[AMF] = 2 \cdot [NMF]$ as shown in the figure.

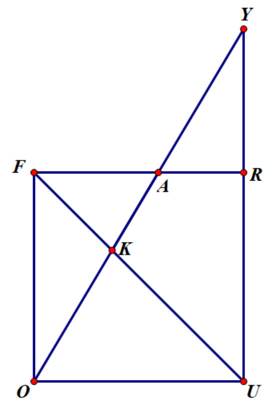


Because $[ALN] = 3 \cdot [FLN]$, it follows that $3x + 3y = 3(3y)$, so $x = 2y$. Now, $[ARML] = 2x+3y = 7y = 1974$, so $y = 282$ and $x = 564$. Thus $[FAN] = 3x+6y = 3(564)+6(282) = \mathbf{3384}$.

T-8 Triangle YOU and square $FOUR$ lie in the same plane, with R on $\overline{UY}$. Suppose that the intersection of $\overline{OY}$ and $\overline{FR}$ is point A and the intersection of $\overline{OY}$ and diagonal $\overline{UF}$ is point K. Given that $OK = 15$ and $KA = 9$, compute AY.

Answer: 16

Solution: Consider the diagram below.



Notice that $\triangle FAK \sim \triangle UOK$, so $\frac{FA}{UO} = \frac{AK}{OK} = \frac{3}{5}$. This implies $AR = \frac{2}{5}OU$. Also notice that $\triangle AFO \sim \triangle ARY$, which implies $\frac{AR}{AF} = \frac{AY}{AO}$, so $\frac{2}{3} = \frac{AY}{24}$. The answer is $AY = \mathbf{16}$.

T-9 Let a positive integer be called *diverse* if all its decimal digits are distinct. For example, 2019 is diverse, but 2020 is not diverse. Compute the arithmetic mean of all diverse integers between 1000 and 2020 inclusive.

Answer: $\frac{113088}{73}$ or $1549\frac{11}{73}$

Solution: First consider diverse integers of the form $1\underline{A}\,\underline{B}\,\underline{C}$. There are $9 \cdot 8 \cdot 7 = 504$ diverse integers of this type. Note that A can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of A, there are $8 \cdot 7 = 56$ diverse integers where A is the hundreds digit. Note also that B can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of B, there are $8 \cdot 7 = 56$ diverse integers where B is the tens digit. Note further that C can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of C, there are $8 \cdot 7 = 56$ diverse integers where C is the ones digit. Thus the sum of all diverse integers of the form $1\underline{A}\,\underline{B}\,\underline{C}$ is $504 \cdot 1000 + 56 \cdot (0+2+3+\cdots+9) \cdot 100 + 56 \cdot (0+2+3+\cdots+9) \cdot 10 + 56 \cdot (0+2+3+\cdots+9) \cdot 1 = 504000 + 56 \cdot 44 \cdot 111 = 777504$.
There are an additional 7 diverse integers 2013, 2014, ..., 2019 exceeding 1999 but not exceeding 2020, whose sum is $2016 \cdot 7 = 14112$. Therefore the answer is

$$\frac{777504 + 14112}{504 + 7} = \frac{791616}{511} = \mathbf{\frac{113088}{73}}.$$

T-10 The polynomial $f(x) = x^3 - 6x^2 + 1$ has distinct real roots p, q, r. Compute the arithmetic mean of the values of $a^2b + b^2c + c^2a$ over all permutations (a, b, c) of (p, q, r).

Answer: $\frac{3}{2}$ or $1\frac{1}{2}$ or 1.5

Solution: Let $S(a, b, c) = a^2b+b^2c+c^2a$. It suffices to compute the sum of $S(a, b, c)$ as (a, b, c) ranges through all permutations of the roots of $f(x)$. First note that $S(a, b, c) = S(c, a, b) = S(b, c, a)$ because the sum is cyclic. Similarly, $S(a, c, b) = S(b, a, c) = S(c, b, a)$, so it suffices to compute $S(a, b, c) + S(a, c, b)$. Note that $S(a, b, c) + S(a, c, b) = a^2(6 - a) + b^2(6 - b) + c^2(6 - c)$ by Vieta's formulas. Because $r^3 = 6r^2 - 1$ for $r = a, b, c$, it follows that $S(a, b, c) + S(a, c, b) = 3$. Therefore the average value of $a^2b+b^2c+c^2a$ over all permutations (a, b, c) of (p, q, r) is $\dfrac{3 \cdot 3}{6} = \mathbf{\dfrac{3}{2}}$.

T-11 The diagonals of a convex cyclic quadrilateral $ABCD$ are perpendicular and meet at point P. Suppose $\{PA, PB, PC, PD\}$ is a collection of four integers such that no element of the collection divides any other element of the collection. Compute the least possible area of $ABCD$.

Answer: 152

Solution: Consider the following lemma: If positive integers a, b, c, and d satisfy $ac = bd$, then there exist positive integers x, y, z, and t such that $a = xy$, $c = zt$, $b = xt$, and $d = yz$. The proof of the lemma proceeds as follows. Let $x = \gcd(a, b)$ and $z = \gcd(c, d)$. Then $a = xy_1$, $b = xt_1$, $c = zt_2$, and $d = zy_2$ where y_1, y_2, t_1, and t_2 are positive integers. Note that $\gcd(y_1, t_1) = \gcd(y_2, t_2) = 1$. Because $ac = bd$, it follows that $xy_1zt_2 = xt_1zy_2 \to y_1t_2 = t_1y_2$. Because y_1t_2 is a multiple of t_1 and because y_1 is relatively prime to t_1, it follows that t_2 is a multiple of t_1. Similarly, t_1 is a multiple of t_2. This implies that $t_2 = t_1 = t$ and $y_2 = y_1 = y$. This completes the proof of the lemma.
By Power of a Point, it follows that $PA \cdot PC = PB \cdot PD$, so by the lemma, there exist positive integers x, y, z, t such that $PA = xy$, $PC = zt$, $PB = xt$, $PD = yz$. Note that $PA \nmid PB \iff y \nmid t$, and similarly, it follows that $t \nmid y$, $x \nmid z$, $z \nmid x$. In particular, none of x, y, z, t can be 1.

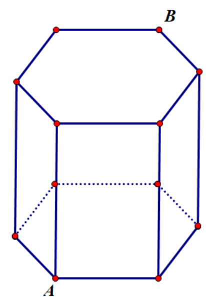
It must be that $\max(x, y, z, t) \geq 5$. Otherwise, two of x, y, z, t must be equal, say $x = y$, so $\{PA, PB, PC, PD\} = \{x^2, xt, zt, xz\}$. Then none of x, z, t divides one another, so it is impossible for them to be contained in $\{2, 3, 4\}$. (Note that $x = y$ is the only case that needs to be considered. If $x = z$, then $x \mid z$, which is a contradiction. If $x = t$, the results are similar to the case where $x = y$.)

Thus assume, without loss of generality, that $t \geq 5$. Because x and z are distinct, assume $x \geq 2$ and $z \geq 3$. Also, assume $y \geq 2$. Note that $[ABCD] = \frac{AC \cdot BD \cdot \sin(\angle APB)}{2} = \frac{AC \cdot BD}{2} = \frac{(PA+PC)(PB+PD)}{2}$. Then

$$[ABCD] = \frac{1}{2}(xy + zt)(xt + yz) \geq \frac{1}{2}(2 \cdot 2 + 3 \cdot 5)(2 \cdot 5 + 2 \cdot 3) = \mathbf{152}.$$

This is indeed achievable: one ordered quadruple satisfying these conditions is $(x, y, z, t) = (2, 2, 3, 5)$, and this implies $PA = 4$, $PB = 10$, $PC = 15$, $PD = 6$.

T-12 An ant is at vertex A of the regular right hexagonal prism shown below. The side length of the hexagon is 1 and the height of the prism is 2.

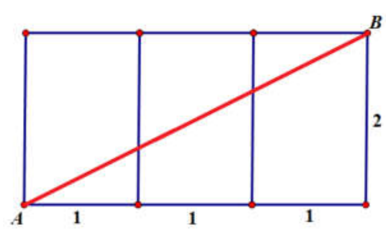


Compute the shortest distance the ant can walk on the surface of the prism, passing through the interiors of at most three faces and no more than one hexagonal face, to get from point A to point B.

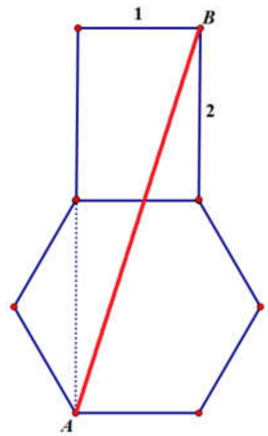
Answer: $\sqrt{13}$

Solution: There are many different ways that the ant could travel along the surface of the prism from point A to point B. However, it appears that only three do so in ways that result in short straight-line distances from A to B across the unfolded net of the prism: a path across 3 rectangles, a path across one hexagon and two rectangles, and a path across one hexagon and one rectangle. Calculate each distance and determine which is the shortest.

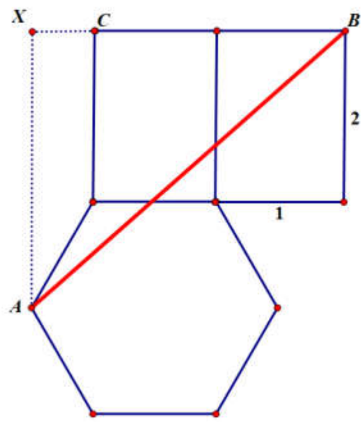
First consider the shortest path that crosses three rectangles. The ant would travel a path of length $\sqrt{3^2+2^2}=\sqrt{13}$, as shown below.



Next consider paths that cross one hexagon and one rectangle. One such path is shown below. Other paths pass through a rectangle and then a hexagon before ending at B. By symmetry, there is no need to consider them separately. Because the hexagon has side length 1, by the Law of Cosines, the dotted segment has length $\sqrt{1^2+1^2-2\cdot 1\cdot 1\cdot \cos(120°)}=\sqrt{3}$. Thus the ant would travel a path of length $\sqrt{1^2+(2+\sqrt{3})^2}=\sqrt{8+4\sqrt{3}}$.



Finally consider paths that cross through one hexagon and two rectangles. One such path is shown below, and there is no need to consider other such paths because of symmetry. Again, because the hexagon has side length 1, the ant would travel a path of length $\sqrt{(2+\frac{\sqrt{3}}{2})^2+(2+\frac{1}{2})^2} = \sqrt{11+2\sqrt{3}}$.



Compare the radicands to establish which distance is the least. Note that $11+2\sqrt{3} > 11+2(1.5) = 13$, so $\sqrt{13} < \sqrt{11+2\sqrt{3}}$. Note also that $8+4\sqrt{3} > 8+4(1.5) = 14 > 13$, so $\sqrt{13} < \sqrt{8+4\sqrt{3}}$. It can be seen that the path across three rectangles is the shortest of the three. Thus the minimum distance is $\mathbf{\sqrt{13}}$.

T-13 Compute the greatest value of n that satisfies $n + 20\lfloor\sqrt{n}\rfloor - 20\lfloor\sqrt[3]{n}\rfloor = 1000$.

Answer: 660

Solution: First, suppose that $\lfloor\sqrt{n}\rfloor = \lfloor\sqrt[3]{n}\rfloor = k$, where $k \geq 3$. The equation $\lfloor\sqrt{n}\rfloor = k$ implies the inequality

(1) $$(k+1)^2 > n \geq k^2$$

and the equation $\lfloor \sqrt[3]{n} \rfloor = k$ implies the inequality

(2) $$(k+1)^3 > n \geq k^3.$$

But by induction, it follows that

(3) $$k^3 > (k+1)^2 \text{ for } k \geq 3.$$

Hence, by combining (1), (2), and (3), it follows that

$$n \geq k^3 > (k+1)^2 > n,$$

which is impossible. Hence $\lfloor \sqrt{n} \rfloor - \lfloor \sqrt[3]{n} \rfloor \geq 1$ for $n \geq 3^2 = 9$. Note further that it is impossible to have $\lfloor \sqrt[3]{n} \rfloor \geq 10$, because otherwise,

$$n + 20\lfloor \sqrt{n} \rfloor - 20\lfloor \sqrt[3]{n} \rfloor \geq 10^3 + 20(1) > 1000.$$

Suppose that $\lfloor \sqrt[3]{n} \rfloor \geq 9$. Then by (2), it follows that $10^3 - 1 \geq n \geq 9^3$, which is equivalent to $729 \leq n \leq 999$. By (1), it follows that $\sqrt{n} \geq \sqrt{729} = 27$. Hence

$$n + 20\lfloor \sqrt{n} \rfloor - 20\lfloor \sqrt[3]{n} \rfloor \geq 729 + 20(27 - 9) = 1089 > 1000.$$

Thus $\lfloor \sqrt[3]{n} \rfloor \leq 8$. Next, suppose that $\lfloor \sqrt[3]{n} \rfloor = 8$. The previously established inequalities imply that

$$728 \geq n \geq 512 \quad \text{and thus} \quad 26 \geq \lfloor \sqrt{n} \rfloor \geq 22.$$

If $\lfloor \sqrt{n} \rfloor = 26$, then $n = 20(50 - \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor) = 20(50 - 26 + 8) = 640$, but this leads to a contradiction because $\lfloor \sqrt{640} \rfloor = 25$. However, if $\lfloor \sqrt{n} \rfloor = 25$, then $n = 20(50 - \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor) = 20(50 - 25 + 8) = 660$, and indeed, $\lfloor \sqrt{660} \rfloor = 25$. Because decreasing values of n have been considered up until this point, it follows that $n = \mathbf{660}$ is the greatest solution to the given equation. Moreover, by continuing the preceding analysis, it can be established that $n = 660$ is the *unique* solution to the given equation.

T-14 The *tribonacci sequence* $T_0, T_1, T_2, \ldots$ is defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 3$. Compute the number of non-empty subsets of $\{T_0, T_1, \ldots, T_{20}\}$ whose elements sum to a number in the tribonacci sequence.

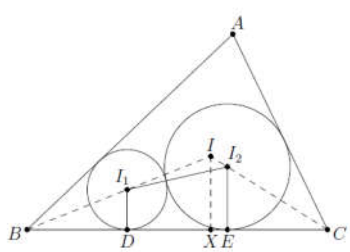
Answer: 181

Solution: Note that every $\{T_i\}$ for $0 \leq i \leq 20$ is a valid subset and if S is a valid subset with $0 \notin S$, then $S \cup \{0\}$ is also a valid subset. Therefore it suffices to examine the valid subsets of $X = \{T_1, T_2, \ldots, T_{20}\}$ with more than one element. Note that it can be shown by induction that $\sum_{j=1}^{k} T_j < T_{k+2}$ for all natural numbers k. Let B_k be the set of desired subsets, with more than one element, where its largest element is T_k. Furthermore, let $n_k = |B_k|$ and suppose $S \in B_k$. Then the sum of elements in S is strictly between T_k and T_{k+2} by the main inequality, so it must be equal to T_{k+1}. Setting up for a recursive method of computing n_k, consider a set $Y \in B_k$ where $k \geq 3$. Furthermore, let $S_0 = T_{k+1}$ be the sum of elements in Y and let $Y_0 = \{T_1, T_2 \ldots, T_k\}$. Therefore it suffices to find the subsets of $Y_1 = Y_0 \setminus \{T_k\}$ with sum $S_1 = T_{k+1} - T_k = T_{k-1} + T_{k-2}$. If $T_{k-1} \notin Y$, then $S_1 \leq T_{k-2} + \left(\sum_{j=1}^{k-3} T_j\right) < T_{k-2} + T_{k-1}$, which is a contradiction. If $T_{k-2} \in Y$, then $Y = \{T_k, T_{k-1}, T_{k-2}\}$ is a valid subset, and so there cannot be any more elements in Y. Otherwise, it suffices to find the subsets of $Y_2 = Y_1 \setminus \{T_{k-1}, T_{k-2}\}$ with sum $S_2 = S_1 - T_{k-1} = T_{k-2}$. By definition, there are n_{k-3} such sets so this yields $n_k = 1 + n_{k-3}$. Computing the first few values, note that $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$, so $n_k = \left\lfloor \frac{k+1}{3} \right\rfloor$ for all $k \in \mathbb{N}$. Therefore the number of valid subsets of X with more than one element is $\sum_{j=1}^{20} n_j = 3(1 + 2 + \cdots + 6) + 7 = 70$. Because 0 can be appended to each of these subsets, this creates 140 valid subsets. However, $\{T_i\}$ for $0 \le i \le 20$ and $\{0, T_i\}$ for $1 \le i \le 20$ are also valid subsets. Therefore there are $140 + 21 + 20 = \mathbf{181}$ non-empty subsets of $\{T_0, T_1, \ldots, T_{20}\}$ whose elements sum to a number in the tribonacci sequence.

T-15 Two externally tangent circles are drawn inside triangle ABC so that one is tangent to side $\overline{AB}$, the other is tangent to side $\overline{AC}$, and both are tangent to side $\overline{BC}$. Suppose that the product of the two circles' radii is 9. Given that $AB + BC = 35$ and $AC = 13$, compute the sum of the two circles' radii.

Answer: $\frac{54\sqrt{77}}{77}$

Solution: Let the circles be Γ_1 and Γ_2 with radii r_1 and r_2, respectively. Let Γ_1, Γ_2, and the incircle of triangle ABC have centers I_1, I_2, and I, respectively, and touch $\overline{BC}$ at D, E, and X, respectively. By definition, $DI_1 = r_1$, $EI_2 = r_2$, and let $XI = r$. Then $\triangle BDI_1$ and $\triangle BXI$ are similar right triangles, so $\frac{BD}{r_1} = \frac{BX}{r} = \frac{s-b}{r}$, where s is the semiperimeter of $\triangle ABC$. It follows that $BD = \frac{r_1(s-b)}{r}$ and similarly, $CE = \frac{r_2(s-c)}{r}$. Note that $s-a$, $s-b$, and $s-c$ are all positive.



From right trapezoid DI_1I_2E, it follows that $DE^2 + (r_2 - r_1)^2 = (r_2 + r_1)^2$ by the Pythagorean Theorem. Simplifying this gives $DE = 2\sqrt{r_1 r_2}$, so

$$a = BD + DE + EC = \frac{r_1(s-b)}{r} + \frac{r_2(s-c)}{r} + 2\sqrt{r_1 r_2}.$$

Applying the AM-GM Inequality on $r_1(s-b)$ and $r_2(s-c)$ gives $\frac{1}{2}(r_1(s-b) + r_2(s-c)) \ge \sqrt{r_1 r_2 (s-b)(s-c)}$, so

$$a \ge \frac{2\sqrt{r_1 r_2 (s-b)(s-c)}}{r} + 2\sqrt{r_1 r_2} = 2\sqrt{r_1 r_2}\left(\sqrt{\frac{s}{s-a}} + 1\right)$$

because $r = \frac{[ABC]}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$ by Heron's Formula.

Applying the AM-GM Inequality to $\frac{4(s-a)}{s}$, $\frac{1}{2}\sqrt{\frac{s}{s-a}}$, and $\frac{1}{2}\sqrt{\frac{s}{s-a}}$, implies $\frac{1}{3}\left(\frac{4(s-a)}{s} + \sqrt{\frac{s}{s-a}}\right) \ge 1$, so

$$a \ge 2\sqrt{r_1 r_2}\left(4 - \frac{4(s-a)}{s}\right) = a \cdot \frac{8\sqrt{r_1 r_2}}{s}.$$

Because $r_1r_2 = 9$ and $s = 24$, the right-hand side is just a, so equality must hold in each inequality. Thus $r_1(s-b) = r_2(s-c)$ and $\frac{4(s-a)}{s} = \frac{1}{2}\sqrt{\frac{s}{s-a}}$. The latter implies $\frac{s-a}{s} = \frac{1}{4}$, so $a = \frac{3s}{4} = 18$. With $b = 13$, it follows that $c = 17$. Then $11r_1 = 7r_2$, but $r_1r_2 = 9$, so $r_1 = 3\sqrt{\frac{7}{11}}$ and $r_2 = 3\sqrt{\frac{11}{7}}$. Thus the answer is

$$r_1 + r_2 = 3\left(\sqrt{\frac{7}{11}} + \sqrt{\frac{11}{7}}\right) = \frac{3}{77}\left(7\sqrt{77} + 11\sqrt{77}\right) = \frac{\mathbf{54\sqrt{77}}}{\mathbf{77}}.$$

Collaborative Relay Round (3, 4, 5 minutes)

R1-A Some of the prime factors of 16005 are single-digit prime numbers. Compute the sum of these single-digit primes.

Answer: 8

Solution: Because $16005 = 3 \cdot 5 \cdot 11 \cdot 97$, the answer is $3 + 5 = \mathbf{8}$.

R1-B *Let T be the number you will receive.* Compute the number of distinct noncongruent triangles whose vertices are vertices of a regular T-gon with side length 1. *Note that in this problem, consider two triangles T_1 and T_2 noncongruent if and only if the following is true: for any of the 6 ways to label all vertices of T_1 as A_1, B_1, and C_1 and for any of the 6 ways to label all vertices of T_2 as A_2, B_2, and C_2, the triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are noncongruent.*

Answer: 5

Solution: Knowing that $T = 8$, consider how many distinct noncongruent triangles can be formed whose vertices are vertices of regular octagon $ARMLYGON$ with side length 1. Note that $AM^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 135° = 2 + \sqrt{2}$ by the Law of Cosines. Note also that if two triangles are congruent, their longest sides are congruent, so if two triangles T_1 and T_2 are such that their longest sides are not congruent, then T_1 is noncongruent to T_2. Because $AM^2 = 2 + \sqrt{2} > 1^2$, it follows that $\overline{AM}$ is the longest side of $\triangle ARM$. Note that $m\angle RAM = m\angle RMA = 22.5°$, so $m\angle AML = 135° - 22.5° = 112.5° > 90°$, so $\angle M$ is the largest angle of $\triangle AML$ and thus $\overline{AL}$ is the longest side of $\triangle AML$, so $AL > AM$. Note also that $\angle ALY$ is a right angle, so its hypotenuse $\overline{AY}$ is the longest side of $\triangle ALY$ and thus $AY > AL$. Therefore $AM < AL < AY$. Without loss of generality, consider first all triangles for which one side is $\overline{AR}$. There are three distinct noncongruent triangles for which one side is $\overline{AR}$, namely $\triangle ARM$, $\triangle ARL$, and $\triangle ARY$, and these three are pairwise noncongruent because their longest sides are pairwise noncongruent. By symmetry, $\triangle ARM \cong \triangle RAN$, $\triangle ARL \cong \triangle RAO$, and $\triangle ARY \cong \triangle RAG$, so there are only three distinct noncongruent triangles for which one side is $\overline{AR}$.
Now, consider triangles for which the shortest side is $\overline{AM}$ (again without loss of generality). The third vertex of such a triangle must be O or Y or G. Because $AO = AM = MY = \sqrt{2+\sqrt{2}}$ and because $AY = MO$ by symmetry, it follows that $\triangle AMY \cong \triangle MAO$. Note that $AM < AL < AY$ and $AG = AL$. Note also that $\overline{AY}$ is the longest side of $\triangle AMY$ and $\overline{AG}$ is the longest side of $\triangle AMG$, so $\triangle AMY$ is noncongruent to $\triangle AMG$, and both of these triangles are noncongruent to any of $\triangle ARM$, $\triangle ARL$, and $\triangle ARY$ because $AM > AR$. Thus there are two distinct noncongruent triangles of this type.
Now, consider triangles for which the shortest side is $\overline{AL}$. The other side of such a triangle must be a segment with one endpoint at A and at least as long as $\overline{AL}$. There are two such segments, namely $\overline{AG}$ and $\overline{AY}$. In $\triangle ALG$, note that $LG = AM < AL$. In $\triangle ALY$, note that $LY = AR < AL$. This results in a contradiction in both cases, and so there cannot be any triangle whose shortest side is $\overline{AL}$. Note also that $\overline{AY}$ is the longest segment that can be drawn with vertex A and so it cannot be the shortest side of any triangle one of whose vertices is A. Thus there are **5** distinct noncongruent triangles that can be created by choosing three of the vertices of a regular octagon with side length 1.

R2-A A bus in Springfield has a maximum capacity of 40 passengers. The bus begins its route with N passengers. At the first stop, no one gets on the bus and 4 passengers get off the bus, at which time the number of passengers on the bus is divisible by 7. At the second stop, no one gets off the bus and 2 passengers get on the bus, at which time the number of passengers on the bus is divisible by 5. Compute N.

Answer: 32

Solution: The problem implies that $N-4$ is a multiple of 7, or that $N-4 = 7k \to N = 7k+4$ for some non-negative integer k. Also, $N-2$ is a multiple of 5, which implies $N = 5m+2 \to 7k+2 = 5m$ for some non-negative integer m. Because the maximum capacity is 40, $k \le 5$. In the cases where $k = 0$, 1, 2, 3, or 5, m is not an integer, and these are contradictions. In the case where $k = 4$, it follows that $m = 6$, and so $N = 7 \cdot 4 + 4 = 5 \cdot 6 + 2 = \mathbf{32}$.

R2-B Let T be the number you will receive. Let $a_1 = -T$, $a_2 = 20.20$, and $a_n = a_{n-1} + a_{n-2}$ for each $n \ge 3$. Compute the greatest integer N for which a_N is negative.

Answer: 5

Solution: Let $k = 20.20$. Then the sequence is $-T, k, -T+k, -T+2k, -2T+3k, \ldots$. In general, if F_i is the ith term in the Fibonacci sequence which begins $F_1 = F_2 = 1$, then it follows by induction that $a_n = -F_{n-2}T + F_{n-1}k$ for $n \ge 3$. The problem asks for the greatest integer N for which a_N is negative, which is equivalent to asking for the greatest integer N for which $-F_{n-2}T + F_{n-1}k < 0 \leftrightarrow \frac{F_{N-1}}{F_{N-2}} < \frac{T}{k}$. Of interest, the quotient of consecutive Fibonacci terms converges to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, with consecutive quotients closer in absolute distance to φ and with $\frac{F_n}{F_{n-1}} < \varphi$ for even $n \ge 2$ and $\frac{F_n}{F_{n-1}} > \varphi$ for odd $n \ge 3$. With $T = 32$, $\frac{32}{20.20} < \frac{8}{5} = \frac{F_6}{F_5} < \varphi$, it follows by induction that $a_N > 0$ (because $\frac{F_{N-1}}{F_{N-2}} > \varphi$) for all even N and for all odd values of $N \ge 7$. Confirming for $N = 5$ that $\frac{F_4}{F_3} = \frac{3}{2} < \frac{32}{20.20}$, the greatest N for which a_N is negative is $N = \mathbf{5}$.

Note that if one were to wait until receiving T to begin, the sequence would be

$$-32, 20.20, -11.8, 8.4, -3.4, 5, 1.6, \ldots.$$

Because 5 and 1.6 are both positive, every term of the sequence after these will be positive, so the greatest N for which a_N will be negative is 5.

R3-A Compute the greatest possible value of $x+y$ where x and y are positive integers and $11x+21y = 2020$.

Answer: 180

Solution: To maximize $x+y$ subject to the constraint that $11x+21y = 2020$, begin by maximizing x and minimizing y. Notice that $21y \equiv 2020 \pmod{11}$, which implies $21y \equiv 7 \pmod{11} \to -y \equiv 7 \pmod{11}$, and thus the least positive value of y is $11 - 7 = 4$. This implies $x = \frac{2020-21\cdot4}{11} = 176$. Thus the greatest $x+y$ is $176+4 = \mathbf{180}$. Note that the slope of the line $11x+21y = 2020$ is $-\frac{11}{21}$, so the next lattice point will have $y = 4+11 = 15$ and $x = 176-21 = 155$ so $x+y = 170$. This will continue as one moves along the line in the direction of the vector $\binom{11}{-21}$, with each succeeding lattice point having coordinates such that $x+y$ is $21-11 = 10$ less than the preceding lattice point.

Note: in general, to see how one might choose whether to maximize x or y with the constraint $ax+by = k$, rewrite the constraint equation as $y = \frac{k}{b} - \frac{a}{b}x$. To maximize $x+y = \left(1-\frac{a}{b}\right)x + \frac{k}{b}$, consider $\frac{a}{b}$. If $\frac{a}{b} < 1$, maximize x. If $\frac{a}{b} > 1$, minimize x.

R3-B Let T be the number you will receive. Three cubes of side length 1, 7, and n cm are to be glued together at their faces. Given that the minimum possible surface area of the resulting solid is $2T$ square cm, compute n.

Answer: 4

Solution: Consider three cubes of side length $a \le b \le c$ cm that are to be glued together at their faces. The sum of the surface areas of the three cubes is $A = 6(a^2+b^2+c^2)$ square cm. Let S be the surface area of the resulting solid after gluing. Obviously $S < A$, but by how much could S be less than A? The cube with side length b cm could be glued at most by one full face to the cube with side length c cm, resulting in subtracting at most twice the smaller cube's face area from A (because one gluing removes that face's area twice). The cube with side length a cm could be glued at most by one full face to each of the other two cubes, resulting in subtracting at most four times its face area from A (because two gluings remove that face's area twice each). Thus, $S \ge A - 2b^2 - 4a^2 = 2a^2 + 4b^2 + 6c^2$. Note that in the case where $a+b \le c$ it is possible to glue the cube with side length b cm by exactly one full face to the cube with side length c cm (by making them sharing a vertex), and then glue the cube with side length a cm by exactly one full face to each of the other two cubes. In this case exactly $2b^2+4a^2$ would be subtracted from A, resulting in $S = 2a^2+4b^2+6c^2$, so the minimum possible value of S in the case where $a+b \le c$ would be $2a^2+4b^2+6c^2$ square cm.

In this problem, the sum of the surface areas of the three cubes is $A = 6(1^2+7^2+n^2) = 300+6n^2$.

Suppose $n \le 1$. Then $n+1 \le 7$, so the minimum surface area of the configuration is $300+6n^2 - 4\cdot n^2 - 2\cdot 1^2 = 298+2n^2 = 2T$ square cm, which implies $n = \sqrt{T-149}$. With $T = 180$, $n = \sqrt{31}$, which is not less than or equal to 1. This is a contradiction.

Suppose $n \ge 6$. Then the total surface area of the unglued cubes is at least $6\cdot(1^2+6^2+7^2) = 516$ square cm. If the cubes are glued such that the maximum area is removed by gluing, that removed area is bounded above by $4\cdot 1^2 + 2\cdot 7^2 = 102$ square cm, and that leaves an exposed surface area of at least $516-102 = 414$ square cm. With $T = 180$, this is a contradiction because $414 > 360 = 2\cdot 180$.

Suppose instead that $1 < n < 6$. Then $1+n \le 7$, so the minimum surface area of the configuration is $300 + 6n^2 - 4 \cdot 1^2 - 2 \cdot n^2 = 296 + 4n^2 = 2T$, which implies $n = \sqrt{\frac{T-148}{2}}$. With $T = 180$, the answer is $n = \sqrt{\frac{32}{2}} = \mathbf{4}$, which lies in the interval $1 < n < 6$, as needed.

RA-1 Compute the least positive integer x for which $6x^2 - 7x > 20$.

Answer: 3

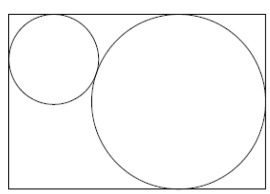
Solution: Note that $6x^2 - 7x - 20 > 0$ implies $(3x+4)(2x-5) > 0$ so $x < -4/3$ or $x > 5/2$. Thus the least positive integer in the solution set is $x = \mathbf{3}$.

RA-2 *Let T be the number you will receive.* Consider the geometric sequence $4, -\dfrac{3}{2}, \dfrac{9}{16}, \ldots$. The sum of the infinite series of these numbers is S and the sum of the first T numbers of the sequence is S_T. Compute $S \cdot S_T$.

Answer: $\frac{98}{11}$ or $8\frac{10}{11}$

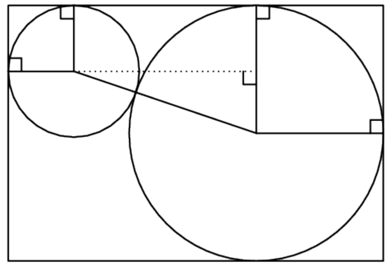
Solution: For this sequence, $a = 4$ and $r = -\frac{3}{8}$. The value of S is $S = \dfrac{4}{1-(-3/8)} = \dfrac{32}{11}$. The sum of the first three terms is $S_3 = 4 - \frac{3}{2} + \frac{9}{16}$, or $S_3 = \frac{49}{16}$. The desired product is $\dfrac{32}{11} \cdot \dfrac{49}{16} = \dfrac{\mathbf{98}}{\mathbf{11}}$.

RA-3 *Let T be the number you will receive.* Two circles inside a rectangle are externally tangent to each other, one with radius 3 and the other with radius $r > 3$. The smaller circle is tangent to two sides of the rectangle while the larger circle is tangent to three sides, as shown. Given that the rectangle has area $11 \cdot T$, compute r.



Answer: $\frac{17-\sqrt{93}}{2}$

Solution: Draw in segments which connect the center of each circle to the points of tangency, as well as the segment between the centers of the two circles. By the Pythagorean Theorem, the width of the rectangle is divided by these lines into segments of length 3, $\sqrt{(r+3)^2 - (r-3)^2} = \sqrt{12r} = 2\sqrt{3r}$, and r.



It follows that the area of the rectangle is $2r(3+2\sqrt{3r}+r) = 2r(\sqrt{3}+\sqrt{r})^2 = 2(\sqrt{3r}+r)^2$. Substituting and simplifying,

$$r+\sqrt{3r} = 7$$

so then $\sqrt{r} = \frac{-\sqrt{3}+\sqrt{31}}{2}$. It follows that $r = \mathbf{\frac{17-\sqrt{93}}{2}}$, as desired.

RB-1 Compute the value of $\dfrac{2020^2 - 17\cdot 2022 - 4}{2022}$.

Answer: 2001

Solution: Let $x = 2020$. It follows that the given expression is equivalent to $\dfrac{x^2-17x-38}{x+2} = \dfrac{(x-19)(x+2)}{x+2} = x - 19$. The answer is $2020 - 19 = \mathbf{2001}$. Similarly, the expression can be rearranged as $\frac{2020^2-4}{2022} - \frac{17\cdot 2022}{2022} = \frac{(2020+2)(2020-2)}{2022} - 17 = 2018 - 17 = \mathbf{2001}$.

RB-2 Let T be the number you will receive. Let $f(n)$ denote the $2n$-digit number that results from adjoining n 20s together. For example, $f(4) = 20202020$. Compute the greatest single-digit divisor of $f(T)$.

Answer: 7

Solution: Note that

$$f(n) = 2\cdot \underbrace{1010\ldots10}_{2n \text{ digits}} = 2\cdot(10^{2n-1}+10^{2n-3}+\cdots+10^1).$$

The sum in parentheses is a geometric series with common ratio 100; its sum is $\frac{10(100^n-1)}{99}$. Note that 8 cannot be a divisor of $f(n)$ for any n because the last three digits of $f(n)$ are always 020, and 8 is not a divisor of 20. Note also that 5 is clearly a divisor of $f(n)$ and that 9 will be a divisor of $f(n)$ if and only if n is a multiple of 9 because the sum of all digits of $f(n)$ equals $2n$ and 2 and 9 are relatively prime. Now, consider whether 7 is a factor of $\frac{10(100^n-1)}{99}$ by considering its residue modulo 7. Noting that 99, 100, and 10 are congruent to 1, 2, and 3 modulo 7, respectively,

$$(99)^{-1}(10)(100^n-1) \equiv 3(2^n-1) \pmod{7}.$$

For $n = 1, 2, 3$, it follows that $\frac{10(100^n-1)}{9}$ is equivalent to 3, 2, and 0 (mod 7), respectively. Therefore if n is a multiple of 3, it follows that $f(n)$ is a multiple of 7. With $T = 2001$, it follows that $f(T)$ is a multiple of 7 and not a multiple of 9, so the greatest single-digit factor of $f(T)$ is **7**.

RB-3 Let T be the digit you will receive. Compute the number of T-digit positive integers such that each digit from 1 to T inclusive appears exactly once and the sum of any three consecutive digits is a multiple of 3.

Answer: 48

Solution: To satisfy the conditions of the problem, it must be that in every group of three consecutive digits of a number, either they are all equivalent to each other modulo 3, or they all have different residues modulo 3. The latter condition is the only relevant one for this problem. To see this, consider that if each group of three consecutive digits is equivalent modulo 3, then all digits of the integer would be equivalent modulo 3, but this is impossible if $T \geq 2$ because 1 and 2 are not equivalent modulo 3.

Note that if a, b, c, and d are four consecutive digits of N (in that order), then $a+b+c$ and $b+c+d$ are both multiples of 3, and therefore $a-d$ is a multiple of 3 which implies $a \equiv d \pmod 3$. This argument implies that the first, fourth, seventh, etc., digits of N have the same residue modulo 3. Similarly, the second, fifth, eighth, etc., digits of N have the same residue modulo 3, and the third, sixth, ninth, etc., digits of N have the same residue modulo 3. This pattern will be important in this solution.

Therefore, let N be a T-digit number that satisfies the conditions of the problem, and let S_0, S_1, and S_2 be the sets of digits between 1 and T inclusive that have 0, 1, and 2 as their residue modulo 3, respectively. If $T \in S_1$, then S_1 has one more element than both S_0 and S_2, and therefore N must begin and end with an element in S_1. The second and third digits of N would be in S_0 and S_2 in some order, and then the pattern would repeat. Once a pattern is determined, there are $\lfloor \frac{T}{3} \rfloor!$ ways to arrange the digits of S_0 and S_2 in N, and $\lceil \frac{T}{3} \rceil!$ ways to arrange the digits of S_1 in N, for a total of $2\lfloor \frac{T}{3} \rfloor!^2 \lceil \frac{T}{3} \rceil!$ ways to arrange the digits of N. Similarly, if $T \in S_2$, then N must begin and end with a digit in S_1 and S_2 or vice versa, and there is a total of $2\lfloor \frac{T}{3} \rfloor! \lceil \frac{T}{3} \rceil!^2$ ways to arrange the digits of N. If $T \in S_0$, then the three leftmost digits of N are from three different sets S_0, S_1, and S_2 in any of $3! = 6$ possible orderings, and the pattern repeats, for a total of $6\frac{T}{3}!^3$ ways to arrange the digits of N. With $T = 7 \equiv 1 \pmod 3$, it follows that $T \in S_1$, and so the answer is $2 \times 3! \times 2! \times 2! = \mathbf{48}$.

R3-1 Line ℓ has a slope of 3, and is tangent to the circle $x^2 + y^2 = 40$ at a lattice point (i.e., a point with integer coordinates) in the second quadrant. Given that this lattice point has coordinates (a, b), compute $a + b$.

Answer: -4

Solution: The tangent line has a slope of 3, so the radius to which it is drawn has a slope of $-1/3$. This radius "runs" 3 units for every unit it "drops". To land on a lattice point on the circle requires 6 units over and 2 units up from the origin. Thus $a + b = -6 + 2 = \mathbf{-4}$.

R3-2 *Let T be the number you will receive.* The point (T, T) is the midpoint of the line segment whose endpoints are $(-3, 5)$ and (A, B). Compute $A + B$.

Answer: -18

Solution: Use the definition of midpoint to arrive at two equations: $T = \frac{A-3}{2}$ and $T = \frac{B+5}{2}$. Therefore $A = 2T + 3$ and $B = 2T - 5$. The desired sum is $A + B = 4T - 2$. Substituting, it follows that $A + B = 4(-4) - 2 = \mathbf{-18}$.

R3-3 *Let T be the number you will receive.* Consider the triangle bounded by the coordinate axes and the graph of $3x + 2y = T$. Compute the area of this triangle.

Answer: 27

Solution: The triangle has vertices at $(0, 0)$, $(\frac{T}{3}, 0)$, and $(0, \frac{T}{2})$. The area of the triangle is $\frac{1}{2} \cdot \frac{|T|}{3} \cdot \frac{|T|}{2}$, which is $\frac{T^2}{12}$. Substituting, it follows that $\frac{(-18)^2}{12} = \mathbf{27}$.

R3-4 *Let T be the number you will receive.* Consider the triangle bounded by the coordinate axes and the graph of $5x + 2y = T + 3$. Compute the area of this triangle.

Answer: 45

Solution: The triangle has vertices at $(0, 0)$, $(\frac{T+3}{5}, 0)$, and $(0, \frac{T+3}{2})$. The area of the triangle is $\frac{1}{2} \cdot \frac{T+3}{5} \cdot \frac{T+3}{2}$, which is $\frac{(T+3)^2}{20}$. Substituting, it follows that $\frac{(30)^2}{20} = \mathbf{45}$.

R3-5 *Let T be the number you will receive.* Compute the distance between the point $(8, 4)$ and the line $3x + 9y = 2T$.

Answer: $\sqrt{10}$

Solution: The line $3x + 9y = 2T$ has slope $-\frac{1}{3}$, so the perpendicular line through $(8, 4)$ that will be used to measure minimum distance has slope 3. That line has equation $y = 3x - 20$. The two lines intersect at $(\frac{2T+180}{30}, \frac{2T-20}{10})$. Substituting $T = 45$, it follows that the intersection point is $(9, 7)$. The distance between the points in question is $\sqrt{3^2 + 1^2} = \mathbf{\sqrt{10}}$.

R3-6 *Let T be the number you will receive.* Given that A and B are acute angles such that $\tan A = \dfrac{1}{3}$ and $\tan B = T$, compute $\sin(2A) + \cos(2B)$.

Answer: $-\frac{12}{55}$

Solution: Construct a right triangle with acute angle A and legs of length 1 and 3. Then the hypotenuse of this triangle has length $\sqrt{10}$ and so $\sin A = \dfrac{1}{\sqrt{10}}$ and $\cos A = \dfrac{3}{\sqrt{10}}$. Similarly, construct a right triangle with acute angle B and legs of length T and 1; the hypotenuse is of length $\sqrt{T^2+1}$ so $\sin B = \dfrac{T}{\sqrt{T^2+1}}$ and $\cos B = \dfrac{1}{\sqrt{T^2+1}}$. Thus $\sin(2A)+\cos(2B) = 2\sin A\cos A + \cos^2 B - \sin^2 B = 2\cdot\dfrac{1}{\sqrt{10}}\cdot\dfrac{3}{\sqrt{10}} + \dfrac{1}{T^2+1} - \dfrac{T^2}{T^2+1}$. Substituting, the answer is $\dfrac{3}{5} + \dfrac{-9}{11} = -\mathbf{\dfrac{12}{55}}$.

Individual Round (10 minutes per pair)

I-1 There are four distinct prime factors of 999039. Compute the sum of these four prime factors.

Answer: 1070

Solution: Rewrite $999039 = 1000000 - 961 = 1000^2 - 31^2 = (1031)(969)$. Factor 969 as $969 = 3 \cdot 323 = 3 \cdot 17 \cdot 19$. The number 1031 is prime, so the sum of the four prime factors is $3 + 17 + 19 + 1031 = \mathbf{1070}$.

I-2 Suppose that m and n are real numbers such that $\frac{7}{2020} < \frac{m}{m+n} < \frac{9}{2020}$. Compute the number of different possible integer values of $\frac{n}{m}$.

Answer: 64

Solution: Taking reciprocals yields $\frac{2020}{7} > \frac{m+n}{m} > \frac{2020}{9}$, which implies $287\frac{4}{7} > \frac{n}{m} > 223\frac{4}{9}$. Thus $\frac{n}{m}$ can be any integer between 224 and 287 inclusive, so the answer is $287-224+1 = \mathbf{64}$.

I-3 In rectangle $ARML$, diagonal $\overline{RL}$ is drawn along with altitudes $\overline{AX}$ of $\triangle LAR$ and $\overline{MY}$ of $\triangle RML$. Given that $AR = 3$ and $RM = 4$, compute XY.

Answer: $\frac{7}{5}$ or $1\frac{2}{5}$ or 1.4

Solution: The Pythagorean Theorem implies that $RL = 5$. Because $\triangle LAR \sim \triangle MYL$ and $\triangle MYL \cong \triangle AXR$, it follows that $\frac{LY}{LM} = \frac{RA}{RL}$ and $RX = LY = \frac{9}{5}$. Thus $XY = 5 - 2 \cdot \frac{9}{5} = \mathbf{\frac{7}{5}}$.

I-4 A real number x is selected uniformly at random between 0 and π. Compute the probability that $\sin(x)\cos(x) < \frac{1}{4}$.

Answer: $\frac{2}{3}$

Solution: The quantity $\sin(x)\cos(x)$ is $\frac{1}{2}\sin(2x)$, so the requested probability is the same as the probability that $\sin(x) < \frac{1}{2}$ between 0 and 2π. The only part of that interval where $\sin(x) \geq \frac{1}{2}$ is between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus the desired probability is $1 - \frac{2\pi/3}{2\pi} = \mathbf{\frac{2}{3}}$.

I-5 In a Taekwondo class of 21 students, the students compete in "sparring teams" of three students. Suppose that there are 56 different sparring teams that have ever been formed by the students in the class (here, two teams are the same if they consist of the same three students and different otherwise). Given that each student has been, on average, a part of N different sparring teams, compute N.

Answer: 8

Solution: Suppose that a student S has been on $T(S)$ teams. Consider the quantity $\sum T(S)$, where the sum is taken over all 21 students S. Each team that has been formed will be counted 3 times by this sum, once in the count $T(S)$ for each student S in the team. Because there are 56 teams, it follows that $\sum T(S) = 3 \cdot 56 = 168$. So the average value of $T(S)$ is $\frac{168}{21} = \mathbf{8}$.

I-6 Define points P_k in the plane so that $P_n = (n^2, n^2)$ if n is even and $P_n = (n^2, n^2 - 1)$ if n is odd. Given that points X_k lie on the x-axis, compute the least possible length of the path $P_1X_1P_2X_2 \cdots P_{98}X_{98}P_{99}$.

Answer: 646898

Solution: By the reflection principle, the shortest distance $P_1X_1P_2$ is the distance between $P_1 = (1, 0)$ and the reflection of $P_2 = (4, 4)$ across the x-axis which is $P_2' = (4, -4)$. Notice that

$$P_1P_2' = \sqrt{(4-1)^2 + (-4-0)^2} = 5 = 1^2 + 2^2.$$

Similarly, the shortest distance $P_2X_2P_3$ is the distance between $P_2 = (4, 4)$ and $P_3' = (9, -8)$ which is the reflection of $P_3 = (9, 8)$ across the x-axis. Thus

$$P_2P_3' = \sqrt{(9-4)^2 + (-8-4)^2} = 13 = 2^2 + 3^2.$$

It can be verified that

$$P_kP_{k+1}' = \sqrt{((k+1)^2 - k^2)^2 + (1 - (k+1)^2 - k^2)^2} = \sqrt{4k^4 + 8k^3 + 8k^2 + 4k + 1} = k^2 + (k+1)^2,$$

and so

$$\min(P_1X_1P_2X_2P_3 \cdots P_{99}) = 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + \cdots + 98^2 + 98^2 + 99^2 = 2(1^2 + 2^2 + \cdots + 99^2) - 1^2 - 99^2$$

or $2 \cdot \dfrac{99 \cdot 100 \cdot 199}{6} - 9802 = \mathbf{646898}$.

I-7 Let $ABCD$ be a unit square. Points E and F are chosen randomly, uniformly, and independently on $\overline{AB}$ and $\overline{CD}$, respectively, with all points on each segment equally likely to be chosen. Compute the probability that

$$\max\{[AEFD], [BEFC]\} \geq \frac{3}{4}.$$

Answer: $\frac{1}{4}$ or 0.25

Solution: Let $AE = x$ and $DF = y$, in which case $[AEFD] = \frac{x+y}{2}$ and $[BEFC] = 1 - \frac{x+y}{2}$. Then the larger of those two areas being at least $\frac{3}{4}$ corresponds to either $x + y \le \frac{1}{2}$ or $x + y \ge \frac{3}{2}$. Considering the unit square of possible values of (x, y), these two regions have combined area $\mathbf{\frac{1}{4}}$.

I-8 Amy adds some positive numbers together and gets 17. Bella multiplies those same positive numbers together and gets N. Compute the least positive integer that cannot be N.

Answer: 518

Solution: Suppose that n positive numbers add to 17. Define $a_n = \left(\frac{17}{n}\right)^n$ for natural numbers n, and suppose that $M = \max a_n$ over all natural n. By the AM-GM Inequality, it follows that $N \le M$. Consider the following calculations to establish the value of n for which $\left(\frac{17}{n}\right)^n$ is the greatest.

$\left(\frac{17}{1}\right)^1 = 17,$

$\left(\frac{17}{2}\right)^2 < 9^2 = 81,$

$\left(\frac{17}{3}\right)^3 < 6^3 = 216,$

$\left(\frac{17}{4}\right)^4 = ((4.25)^2)^2 < 20^2 = 400,$

$\left(\frac{17}{5}\right)^5 = ((3.4)^2)^2 \cdot 3.4 < 12^2 \cdot 3.4 = 144 \cdot 3.4 < 490,$

$\left(\frac{17}{6}\right)^6 = \left(\frac{289}{36}\right)^3 > \left(\frac{288}{36}\right)^3 = 8^3 = 512,$

$\left(\frac{17}{7}\right)^7 = \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right) = \frac{289}{49} \cdot \frac{289}{49} \cdot \frac{289}{49} \cdot \frac{17}{7} < 5.9 \cdot 5.9 \cdot 6 \cdot \frac{17}{7} < 35 \cdot 6 \cdot \frac{17}{7} = 5 \cdot 6 \cdot 17 = 510,$

$\left(\frac{17}{8}\right)^8 = ((2.125)^2)^4 < 4.6^4 < 22^2 = 484,$

$\left(\frac{17}{9}\right)^9 < 2^9 = 512,$

$\left(\frac{17}{10}\right)^{10} = ((1.7)^2)^5 = (2.89)^5 < 3^5 = 243,$

$\left(\frac{17}{11}\right)^{11} = \left(\frac{17}{11}\right)^{10} \cdot \frac{17}{11} < 243 \cdot 2 = 486,$

$\left(\frac{17}{12}\right)^{12} = \left(\frac{289}{144}\right)^6 < ((2.1)^2)^3 < 5^3 = 125,$

$\left(\frac{17}{13}\right)^{13} = \left(\frac{17}{13}\right)^{12} \cdot \frac{17}{13} < 125 \cdot 2 = 250,$

$\left(\frac{17}{14}\right)^{14} = \left(\frac{17}{14}\right)^{13} \cdot \frac{17}{14} < 250 \cdot 2 = 500,$

$\left(\frac{17}{15}\right)^{15} < \left(\frac{18}{15}\right)^{15} = (1.2)^{15} = ((1.2)^3)^5 < 2^5 = 32,$

$\left(\frac{17}{16}\right)^{16} = \left(\frac{17}{16}\right)^{15} \cdot \frac{17}{16} < 32 \cdot 2 = 64.$

It is clear that for $n \ge 17$, it follows that $\left(\frac{17}{n}\right)^n \le \left(\frac{17}{17}\right)^n = 1$, so $M = a_6$. By the Binomial Theorem, $\left(\frac{17}{6}\right)^6 = \left(3 - \frac{1}{6}\right)^6 = \sum_{k=0}^{6} \binom{6}{k} 3^{6-k} \left(-\frac{1}{6}\right)^k = \sum_{k=0}^{6} (-1)^k \binom{6}{k} 9^{3-k} \left(\frac{1}{2}\right)^k = 729 - 6 \cdot \frac{81}{2} + 15 \cdot \frac{9}{4} - 20 \cdot \frac{1}{8} + 15 \cdot \frac{1}{9 \cdot 16} - \cdots$, which is approximately $486 + 33.75 - 2.5 + 0.1 - \cdots \approx 517.35$, so M is between 517 and 518. Thus 518 cannot be N for any set of positive numbers.

For $0 \le x \le 17$, consider $f(x) = \frac{17^4 \cdot x(34 - x)}{6^6}$. Note that f is continuous with $f(0) = 0$ and

$f(17) = a_6$. By the Intermediate Value Theorem, for every natural number K strictly between 0 and a_6, there exists x such that $f(x) = K$. Therefore every K strictly between 0 and a_6 could be N for some set of positive numbers. Thus the least natural number that cannot be N is **518**.

Alternate Solution: Define M as above. Consider $F(x) = \frac{\ln x}{x}$ for positive x. Because $F'(x) = \frac{1-\ln x}{x^2}$, it follows that F is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$. Note that the function $f(x) = e^x$ is strictly increasing for all real numbers, and thus the function $e^{F(x)} = x^{1/x}$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$. Note also that the function $g(x) = x^{17}$ is strictly increasing for all real numbers, so the function $G(x) = x^{17/x}$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$.

Now consider that $a_n = \left(\frac{17}{n}\right)^n = \left(\frac{17}{n}\right)^{17\cdot(n/17)} = G\left(\frac{17}{n}\right)$. Because $\ldots < \frac{17}{8} < \frac{17}{7} < 2.5 < e < 2.8 < \frac{17}{6} < \frac{17}{5} < \ldots < \frac{17}{1}$, and because $G(x)$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$, it follows that $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$ and $a_7 > a_8 > a_9 > \ldots$. As above, it can be shown that $a_6 > a_7$. Thus $M = a_6$, and the solution follows as above.

I-9 Let $N = 2020^{20}$. Given that $10^{0.3010} < 2 < 10^{0.3011}$, compute the number of digits in the base-10 expansion of N.

Answer: 67

Solution: The number of digits in N is $\lfloor \log_{10} N \rfloor + 1$. Suppose that $\epsilon = \log_{10} 1.01$. Note that

$$\begin{aligned} \log_{10} N &= 20 \log_{10}(2020) \\ &= 20 \left(\log_{10} 101 + \log_{10} 2 + \log_{10} 10\right) \\ &< 20 \left(2 + \epsilon + 0.3011 + 1\right), \text{ where } 0 \le \epsilon \le 0.01 \\ &= 20 \cdot (3.3011 + \epsilon) \\ &= 66.022 + 20\epsilon < 67. \end{aligned}$$

Accordingly, N has **67** digits.
Note: To justify $\epsilon \le 0.01$, note that $\log_{10}(1 + x) < \ln(1 + x) < x$ for positive x.

I-10 Compute the number of five-term sequences of positive integers a_1, a_2, a_3, a_4, a_5 that satisfy $\dfrac{a_k}{a_{k-1}} \ge k$ for $2 \le k \le 5$ and $a_5 < 250$.

Answer: 5745

Solution: Given the increasing sequence of positive integers $\mathcal{S} = (a_1, a_2, a_3, a_4, a_5)$, define the subsequence (a_1, a_2, a_3) to be *left-acceptable* if $a_2 \ge 2a_1$ and $a_3 \ge 3a_2$, and define the subsequence (a_3, a_4, a_5) to be *right-acceptable* if $a_4 \ge 4a_3$ and $a_5 \ge 5a_4$. Then N is the number of integer sequences $(a_1, a_2, a_3, a_4, a_5)$ such that (a_1, a_2, a_3) is left-acceptable, (a_3, a_4, a_5) is right-acceptable, $a_1 \ge 1$, and $a_5 < 250$. Note that $a_3 < 13$, as otherwise, $a_4 \ge 52$ and $a_5 \ge 260$, which violates the requirement that $a_5 < 250$. Similarly, it follows that (a_1, a_2) must be one of $(1,2)$, $(1,3)$, $(1,4)$, $(2,4)$, and $a_3 \geq 6$. For convenience, define R_k to be the number of right-acceptable triples (a_3, a_4, a_5) with $a_3 = k$. To count R_k, first observe that if (x, y, z) is right-acceptable, then for $0 < x' < x$, the triple (x', y, z) is also right-acceptable. For $6 \leq k < 12$, consider counting $R_k - R_{k+1}$, that is, the number of right-acceptable triples with $a_3 = k$, but where (a_4, a_5) is not one of the R_{k+1} possible ordered pairs for a right-acceptable triple in which $a_3 = k+1$. For these triples, $4k \leq a_4 \leq 4k+3$, and these values of a_4 yield a total of $(250 - 5 \cdot 4k) + (250 - 5 \cdot (4k+1)) + (250 - 5 \cdot (4k+2)) + (250 - 5 \cdot (4k+3)) = 970 - 80k$ possible ordered pairs (a_4, a_5). Thus

$$R_k = 970 - 80k + R_{k+1}. \qquad (*)$$

Furthermore, note that if $a_3 = 12$, then it follows that $a_4 \geq 48$ and $5 \cdot 48 = 240 \leq a_5 < 250$, so $(a_4, a_5) = (48, a_5)$ or $(49, a_5)$. If $a_4 = 48$, then there are 10 possible values for a_5: $240, 241, \ldots, 249$. If $a_4 = 49$, then there are 5 possible values for a_5: $245, 246, 247, 248, 249$. Thus $R_{12} = 15$. Now consider the possible left-acceptable triples. If (a_1, a_2) is $(1,2)$, then a_3 must be at least 6, and if (a_1, a_2) is $(1,3)$, then a_3 must be at least 9. If (a_1, a_2) is either $(1,4)$ or $(2,4)$, then a_3 must equal 12. Using $(*)$ to compute the values of R_k, the following calculations give values for R_{11}, R_{10}, R_9, R_8, R_7, and R_6: $R_{11} = 970 - 80 \cdot 11 + 15 = 105$, $R_{10} = 970 - 80 \cdot 10 + 105 = 275$, $R_9 = 970 - 80 \cdot 9 + 275 = 525$, $R_8 = 970 - 80 \cdot 8 + 525 = 855$, $R_7 = 970 - 80 \cdot 7 + 855 = 1265$, and $R_6 = 970 - 80 \cdot 6 + 1265 = 1755$.
It therefore follows that the desired number of sequences is

$$\begin{aligned} & R_6 + R_7 + R_8 + 2(R_9 + R_{10} + R_{11}) + 4R_{12} \\ = {} & 1755 + 1265 + 855 + 2(525 + 275 + 105) + 4 \cdot 15 \\ = {} & \mathbf{5745}. \end{aligned}$$

Tiebreaker (10 minutes)

TB Triangle ABC has side lengths $AB = 8$, $BC = 5$, and $AC = 7$. Point P lies inside $\triangle ABC$ so that $m\angle APB = m\angle BPC = m\angle CPA = 120^\circ$. Compute $AP + BP + CP$.

Answer: $\sqrt{129}$

Solution: Let $AP = x$, $BP = y$, and $CP = z$. Apply the Law of Cosines to $\triangle APB$, $\triangle APC$, and $\triangle BPC$, to obtain $25 = y^2+z^2+yz$, $49 = x^2+z^2+xz$, and $64 = x^2+y^2+xy$. Add these three equations to obtain $138 = 2(x^2+y^2+z^2)+(xy+yz+xz) = 2(x+y+z)^2-3(xy+yz+xz)$. Now note that the areas of $\triangle APB$, $\triangle APC$, and $\triangle BPC$ are $\frac{\sqrt{3}}{4}xy$, $\frac{\sqrt{3}}{4}xz$, and $\frac{\sqrt{3}}{4}yz$, respectively, so if K is the area of $\triangle ABC$, then $K = \frac{\sqrt{3}}{4}(xy + xz + yz)$. By Heron's Formula, $K = 10\sqrt{3}$, so $xy + xz + yz = 40$. Substituting into the equation above yields $138 = 2(x + y + z)^2 - 120$, from which $x + y + z = \boldsymbol{\sqrt{129}}$.