2019 ARML Local Problems

Team Round (45 minutes)

T-1 Compute the number of ordered pairs of positive integers (x, y) that satisfy the equality $x^y = 2^{9!}$.

T-2 Let $PQRS$ be a rectangle and let A, B, C, and D be points such that A is on $\overline{PQ}$, D is on $\overline{RS}$, and B and C are inside the rectangle such that $\overline{AB} \parallel \overline{PS}$, $\overline{AB} \perp \overline{BC}$, and $\overline{BC} \perp \overline{CD}$. Point E lies on $\overline{RS}$ such that $\overline{AE}$ intersects $\overline{BC}$ and $[PABCDS] = [PAES]$. Given that $AB = 30$, $BC = 24$, and $CD = 10$, compute DE.

T-3 Given that the numbers $\log_2(\log_2 x)$, $\log_4(\log_4 x)$, and $\log_{16}(\log_{16} x)$ form an arithmetic progression in that order, compute x.

T-4 Compute the maximum value of the function $f(x) = \sin(x) + \sin(x + \cos^{-1} \frac{4}{7})$ as x varies over all real numbers.

T-5 Lulu the Magical Pony has developed a wagering strategy for seven days at the casino. He plays the same game five times each day, but wagers different amounts each day. For $1 \leq N \leq 7$, on day N, for each play of the game he either wins $\$2 \times 5^{N-1}$ or loses $\$5^{N-1}$. At the end of the week, Lulu has a profit of \$20,119. Compute the total number of games that Lulu won over the week.

T-6 Let $\mathcal{T}$ be an isosceles trapezoid such that there exists a point P in the plane of $\mathcal{T}$ whose distances to the four vertices of $\mathcal{T}$ are 5, 6, 8, and 11. Compute the greatest possible ratio of the length of the longer base of $\mathcal{T}$ to the length of its shorter base.

T-7 Let $f(x) = x^2 - 2\sqrt{2}x + 3$. Define the functions $g_n\big(f(x)\big)$ such that $g_1\big(f(x)\big) = f(x)$ and $g_{n+1}\big(f(x)\big) = f(x)^{g_n(f(x))}$ for all $n \geq 1$. Compute the sum of all distinct real values x such that $g_{2019}(f(x)) = 2019$.

T-8 Integers a, b, c, d, e, and f are chosen uniformly at random and with replacement from the set $\{1, 2, \ldots, 12\}$. Compute the probability that $a^b c^d e^f - 1$ is divisible by 3.

T-9 Positive numbers x, y, and z satisfy $x^3y^2z = 36$. Compute the least possible value of $2x + y + 3z$.

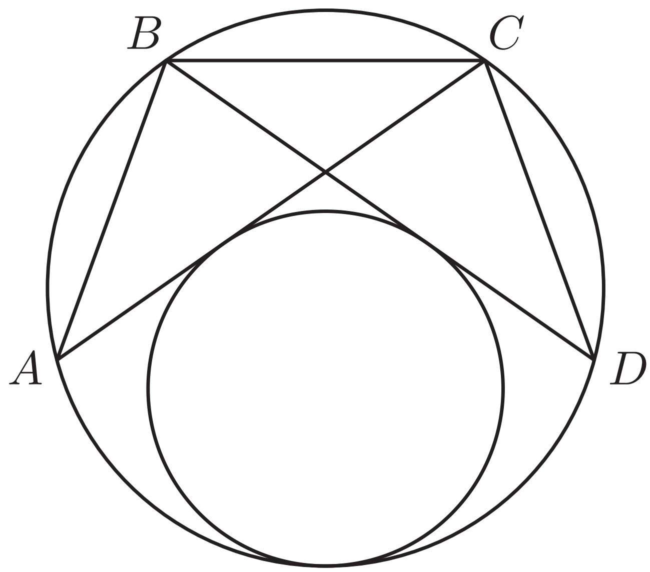
T-10 Compute the least positive value of θ in radians for which $\sin\theta + \sin(2\theta) + \cdots + \sin(2019\theta) = 0$.

T-11 In $\triangle ABC$, D is on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. Let P and Q be the incenters of $\triangle ABD$ and $\triangle ADC$, respectively. Given that $BC = 14$, $AD = 12$, $CD = 5$, and O is the circumcenter of $\triangle ABC$, compute $[OPQ]$.

T-12 For integers $n \geq 2$, let $h(n)$ be the number of positive integers $x \leq n$ such that $x^2 \equiv 1 \pmod{n}$. Compute the greatest integer $n \geq 2$ such that $h(n) = \phi(n)$, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

T-13 Compute the number of functions $f : \{1, 2, \ldots, 20\} \to \{1, 2, 3, 4\}$ such that $f(m)$ divides $f(n)$ whenever m divides n, and m and $f(m)$ have the same parity for all $m \in \{1, 2, \ldots, 20\}$.

T-14 Let Ω be a circle with radius 5, and suppose that A, B, C, and D are points on Ω in that order such that $AB = BC = CD = 4$. Compute the radius of the circle shown in the figure below that is tangent to $\overline{AC}$, $\overline{BD}$, and minor arc $\widehat{AD}$.



T-15 Given that $\displaystyle\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{5^k} = \sqrt{5}$, compute the value of the sum

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k}.$$

Individual Round (10 minutes per pair)

I-1 Compute the least positive integer n such that $2n$ and $3n$ both contain the digit 7 when written in base ten.

I-2 Circle $\mathcal{C}$ lies entirely inside rectangle $\mathcal{R}$, and $\mathcal{C}$ is tangent to three of the sides of $\mathcal{R}$. Given that the ratio of the area of $\mathcal{C}$ to the area of $\mathcal{R}$ is $\frac{\pi}{5}$, compute the ratio of the circumference of $\mathcal{C}$ to the perimeter of $\mathcal{R}$.

I-3 Compute the sum of all integers k with $1 \leq k \leq 2019$ such that $(2k-1)^2 = 1 + k \times 10^{k/13}$.

I-4 Exactly five distinct vertices of a regular 12-gon are randomly colored red, with all vertices equally likely to be colored. Compute the probability that no edge of the 12-gon has both its vertices colored red.

I-5 Triangle SAM lies in the coordinate plane with $S = (-15, 0)$, $A = (0, 36)$, and $M = (15, 0)$. There is a unique point $B = (x, y)$ in the interior of $\triangle SAM$ such that the perimeters of $\triangle BSA$, $\triangle BAM$, and $\triangle BMS$ are equal. Compute $x + y$.

I-6 The function $f(x) = x^4 + ax^3 + bx^2 + 2000x + d$ has four distinct roots. Two of the roots sum to 5; the other two roots also sum to 5. Compute b.

I-7 Compute the sum of all possible values for d that satisfy the following property: there exists a positive integer n with exactly 15 divisors such that $2n$ has exactly d divisors.

I-8 Let $\theta = \frac{1}{2}\sin^{-1}\frac{2}{3}$. Compute $\sin^4\theta + \cos^4\theta$.

I-9 Let P be a set of five points in the plane, no three of which are colinear. Let S be the set of all line segments between two points in P. Compute the number of subsets of S such that exactly four triangles have all their edges in S.

I-10 Let m and n be positive integers. When the point $(15, 17)$ is reflected across the line $y = mx$ and then the reflected point is reflected across the line $y = nx$, the resulting point is $(17, 15)$. Compute $m + n$.

Relay Round (6, 8, 10 minutes)

R1-1 Three 1×1 squares in a 3×3 grid are chosen randomly and colored orange. Compute the probability that no row or column of the grid contains more than one orange square.

R1-2 Let $T =$ TNYWR. Compute the sum of the prime factors of $\dfrac{1-T^2}{T^3}$.

R2-1 Let $a_1, a_2, a_3, \ldots$ be a nonconstant arithmetic sequence. Suppose that a_1, a_{11}, a_{111} is a geometric sequence. Compute a_{11}/a_1.

R2-2 Let $T =$ TNYWR. Frank has a $3 \times T$ grid of squares. Compute the number of ways that Frank can shade some (possibly none) of the squares in the grid, such that each row and column contains at most one shaded square.

R2-3 Let $T =$ TNYWR. Let N be the positive integer created by joining T copies of T. (For example, if $T = 15$, then N is the 30-digit integer $151515\ldots15$.) Compute the remainder when N is divided by 99.

R3-1 Given that $(20^{19} + 100)^2 - (20^{19} - 100)^2 = 20^N$, compute N.

R3-2 Let $T =$ TNYWR. Given that a rectangle has area T and perimeter $T - 1$, compute the length of the shorter side of the rectangle.

R3-3 Let $T =$ TNYWR. Let N be the sum of all positive integers which divide $2^7(T^8 - 1)$. Compute the remainder when N is divided by 100.

R3-4 Let $T =$ TNYWR. Fran flips four fair coins and rolls a standard, fair six-sided die. Let p be the probability that the number rolled on the die is equal to the number of heads flipped. Compute pT.

R3-5 Let $T =$ TNYWR. In $\triangle ABC$, point M is the midpoint of $\overline{BC}$ and D is the foot of the altitude from A to $\overline{BC}$, with D between B and M. Given that $AC = T$, $AM = 4$, and $BD = \dfrac{3}{4}$, compute AB.

R3-6 Let $T =$ TNYWR. For a positive integer n, let S_n be the sum of the first n positive integers. Compute the remainder of S_{T^4} when divided by 9.

Tiebreaker (10 minutes)

TB Compute the number of orderings of the numbers $1, 2, \ldots, 9$ with the following property: if m comes before n in the ordering, then $m < n^2$.