2019 ARML Local Problems and Solutions

Team Round (45 minutes)

T-1 Compute the number of ordered pairs of positive integers (x, y) that satisfy the equality $x^y = 2^{9!}$.

Answer: 160

Solution: The given equation implies x is a power of 2, so let $x = 2^z$ for some nonnegative integer z. Then $2^{zy} = 2^{9!}$, so $yz = 9!$. The number of pairs satisfying the equality is equal to the number of positive divisors of $9! = 2^7 \times 3^4 \times 5 \times 7$, and $\sigma(9!) = (7+1)(4+1)(1+1)(1+1) = \boxed{160}$.

T-2 Let $PQRS$ be a rectangle and let A, B, C, and D be points such that A is on $\overline{PQ}$, D is on $\overline{RS}$, and B and C are inside the rectangle such that $\overline{AB} \parallel \overline{PS}$, $\overline{AB} \perp \overline{BC}$, and $\overline{BC} \perp \overline{CD}$. Point E lies on $\overline{RS}$ such that $\overline{AE}$ intersects $\overline{BC}$ and $[PABCDS] = [PAES]$. Given that $AB = 30$, $BC = 24$, and $CD = 10$, compute DE.

Answer: 12

Solution: Let F be the point of intersection of $\overline{AE}$ and $\overline{BC}$. Let $y = DE$ and G be on FC such that $EG \perp BC$. Since the two polygon areas are the same, the area of $\triangle AFB$ and trapezoid $FEDC$ are the same. Note $EG = CD = 10$ is the height of the trapezoid. Furthermore, $\triangle AFB \sim EFG$ so $\frac{AB}{EG} = \frac{BF}{GF}$, implying $BF = 3GF$. Letting $x = GF$, it follows that $BF = 3x$ so $BC = 4x + GC$. Since $GC = DE$, it follows $4x + y = 24$. Then, the area of $\triangle AFB$ is $\frac{30 \cdot 3x}{2} = 45x$ and the area of $FEDC$ is $\frac{10(y+x+y)}{2} = 5(x + 2y)$. Equating these two areas yields $9x = x + 2y$ or $y = 4x$. Therefore, $24 = y + 4x = 2y$ so $y = \boxed{12}$.

T-3 Given that the numbers $\log_2(\log_2 x)$, $\log_4(\log_4 x)$, and $\log_{16}(\log_{16} x)$ form an arithmetic progression in that order, compute x.

Answer: $\sqrt[4]{2}$ or $2^{1/4}$

Solution: The given condition translates to the equation $\log_2(\log_2 x) + \log_{16}(\log_{16} x) = 2\log_4(\log_4 x)$. To solve this equation, let $y = \log_2 x$. Then, by the change-of-base formula, $\log_{16} x = \frac{\log_2 x}{\log_2 16} = \frac{y}{4}$ and $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{y}{2}$, so it follows that $\log_2 y + \log_{16} \frac{y}{4} = 2\log_4 \frac{y}{2}$, or $\log_2 y + \log_{16} y - \log_{16} 4 = 2\log_4 y - 2\log_4 2$, which simplifies to $\log_2 y + \log_{16} y + \frac{1}{2} = 2\log_4 y$. Let $z = \log_2 y$. Then, by similar logic, $\log_{16} y = \frac{z}{4}$ and $\log_4 = \frac{z}{2}$, so it follows that $z + \frac{z}{4} + \frac{1}{2} = 2 \cdot \frac{z}{2}$. Solving for z gives $z = -2$. Then $y = 2^z = 2^{-2} = \frac{1}{4}$, so $x = 2^y = \boxed{\sqrt[4]{2}}$.

T-4 Compute the maximum value of the function $f(x) = \sin(x) + \sin(x + \cos^{-1}\frac{4}{7})$ as x varies over all real numbers.

Answer: $\frac{\sqrt{154}}{7}$

Solution: Note that

$$\begin{aligned} f(x) &= \sin(x)[1+\cos(\cos^{-1}\tfrac{4}{7})] + \cos(x)[\sin\cos^{-1}\tfrac{4}{7}] \\ &\le \sqrt{(1+\cos(\cos^{-1}\tfrac{4}{7}))^2 + (\sin(\cos^{-1}\tfrac{4}{7}))^2} \\ &= \sqrt{2+2\cos(\cos^{-1}\tfrac{4}{7})} = \sqrt{2+\tfrac{8}{7}} = \sqrt{\tfrac{22}{7}} = \boxed{\tfrac{\sqrt{154}}{7}}. \end{aligned}$$

T-5 Lulu the Magical Pony has developed a wagering strategy for seven days at the casino. He plays the same game five times each day, but wagers different amounts each day. For $1 \le N \le 7$, on day N, for each play of the game he either wins $\$2 \times 5^{N-1}$ or loses $\$5^{N-1}$. At the end of the week, Lulu has a profit of \$20,119. Compute the total number of games that Lulu won over the week.

Answer: 14 or 18

Solution: Let a_i be the number of times Lulu wins on day i. Then, his profit, in dollars, on day i is $2a_i5^{i-1} - (5-a_i)\,5^{i-1} = 3a_i \cdot 5^{i-1} - 5^i$. Summing this expression from $i=1$ to $i=7$ yields $3\sum_{i=1}^{7} a_i5^{i-1} - \frac{5^8-5}{5-1} = 3\sum_{i=1}^{7} a_i5^{i-1} - 97655 \to 3\sum_{i=1}^{7} a_i5^{i-1} = 97655+20119 = 117774$. Therefore, $\sum_{i=1}^{7} a_i5^{i-1} = 39258$ and the goal is to compute $\sum_{i=1}^{7} a_i$. Each a_i can be determined by representing 39258 in base 5. Since $39258 = 2224013_5$, Lulu won $\sum_{i=1}^{7} a_i = 2+2+2+4+0+1+3 = \boxed{14}$ games.[1]

T-6 Let $\mathcal{T}$ be an isosceles trapezoid such that there exists a point P in the plane of $\mathcal{T}$ whose distances to the four vertices of $\mathcal{T}$ are 5, 6, 8, and 11. Compute the greatest possible ratio of the length of the longer base of $\mathcal{T}$ to the length of its shorter base.

Answer: $\frac{57}{11}$

Solution: Suppose $ABCD$ is the trapezoid with $BC \parallel AD$ and $BC < AD$. Let $PA = a$, $PB = b$, $PC = c$, and $PD = d$; the problem stipulates that $\{a,b,c,d\} = \{5,6,8,11\}$.

Let Q be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and set $QB = QC = s$, $BA = CD = t$, and $PQ = r$. Stewart's Theorem on $\triangle APQ$ and $\triangle BPQ$ yields the system of equations

$$r^2t + a^2s = b^2(s+t) + st(s+t) \qquad \text{and} \qquad r^2t + d^2s = c^2(s+t) + st(s+t).$$

[1]Since $0 \le a_i \le 5$ for all a_i, there is another base-5 representation of 39258 where fives are allowed as a digit: 2223513_5. As a result, 18 was also accepted.

Subtracting these two equations and utilizing $\triangle QBC \sim \triangle QAD$ yields

$$\frac{AD}{BC} = \frac{s+t}{s} = \frac{d^2 - a^2}{c^2 - b^2}.$$

This ratio has a maximum of $\boxed{\frac{57}{11}}$, achieved at $(a, b, c, d) = (8, 5, 6, 11)$.

T-7 Let $f(x) = x^2 - 2\sqrt{2}x + 3$. Define the functions $g_n\big(f(x)\big)$ such that $g_1\big(f(x)\big) = f(x)$ and $g_{n+1}\big(f(x)\big) = f(x)^{g_n(f(x))}$ for all $n \geq 1$. Compute the sum of all distinct real values x such that $g_{2019}(f(x)) = 2019$.

Answer: $2\sqrt{2}$

Solution: If $g_{2019}(f(x)) = 2019$, it follows that $f(x) > 1$. Let $a = f(x)$ so $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = 2019$ (where the exponent is taken 2019 times), and it also follows that $a > 1$. By definition, x is a solution to $g_{2019}(f(x)) = 2019$ if and only if x is a solution to $x^2 - 2\sqrt{2}x + (3 - a) = 0$. The discriminant of the left hand side shows that there are two distinct real solutions, and therefore, by Vieta, the sum of all such x is $\boxed{2\sqrt{2}}$.

T-8 Integers a, b, c, d, e, and f are chosen uniformly at random and with replacement from the set $\{1, 2, \ldots, 12\}$. Compute the probability that $a^b c^d e^f - 1$ is divisible by 3.

Answer: $\frac{1}{6}$

Solution: Consider the expression $a^b c^d e^f - 1$ modulo 3. The integers a, c, and e are equally likely to be any of the three residues modulo 3. If any of them is $0 \pmod 3$, then the entire expression becomes $0 - 1 \equiv 2 \pmod 3$, so none of a, c, and e can be $0 \pmod 3$; they must each be either $1 \pmod 3$ or $2 \pmod 3$.

With probability $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$, all three of a, c, and e are $1 \pmod 3$, which makes

$$a^b c^d e^f - 1 \equiv 1^b 1^d 1^f - 1 = 0 \pmod 3,$$

as desired. The probability that each variable is either $1 \pmod 3$ or $2 \pmod 3$ is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$; the probability that that occurs *but* not all the variables are $1 \pmod 3$ is $\frac{8}{27} - \frac{1}{27} = \frac{7}{27}$. In this case, the expression reduces to

$$a^b c^d e^f - 1 \equiv 2^x - 1 \pmod 3,$$

where x is the sum of the exponents on the variables which are $2 \pmod 3$. (For example, if $a \equiv 1 \pmod 3$ and $c \equiv e \equiv 2 \pmod 3$, then $a^b c^d e^f - 1 \equiv 2^{d+f} - 1 \pmod 3$, so $x = d + f$.) Because b, d, and f are equally likely to be odd or even, the variable x is also equally likely to be odd or even. (In general, if $a_1, \ldots, a_k$ are equally likely to be odd or even, then their sum $a_1 + \cdots + a_k$ is *also* equally likely to be odd or even, because regardless of the values of $a_1, \ldots, a_{k-1}$, the parity of a_k is equally likely to be the same as the parity of $a_1 + \cdots + a_{k-1}$, making the sum even, or different, making the sum odd.)

Since $2^2 \equiv 1 \pmod 3$, it follows that $2^x - 1$ is equally likely to be $2^1 - 1 \equiv 1 \pmod 3$ or $2^2 - 1 \equiv 0 \pmod 3$. So, the probability that $a^b c^d e^f - 1$ is divisible by 3 in this case is $\frac{7}{27} \cdot \frac{1}{2} = \frac{7}{54}$.

In total, the desired probability is $\frac{1}{27} + \frac{7}{54} = \boxed{\frac{1}{6}}$.

T-9 Positive numbers x, y, and z satisfy $x^3y^2z = 36$. Compute the least possible value of $2x+y+3z$.

Answer: $6\sqrt{2}$

Solution: Note that $2x+y+3z = \left(\frac{2x}{3} + \frac{2x}{3} + \frac{2x}{3}\right) + \left(\frac{y}{2} + \frac{y}{2}\right) + 3z$. Therefore, by the AM-GM inequality, $\frac{2x+y+3z}{6} \geq \sqrt[6]{\frac{8}{27} \times \frac{1}{4} \times 3 \times x^3y^2z} = \sqrt[6]{8} = \sqrt{2}$. Equality occurs when $\frac{2x}{3} = \frac{y}{2} = 3z$ so $(x, y, z) = \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}, \frac{\sqrt{2}}{3}\right)$ and $2x + y + 3z = \boxed{6\sqrt{2}}$.

T-10 Compute the least positive value of θ in radians for which $\sin\theta + \sin(2\theta) + \cdots + \sin(2019\theta) = 0$.

Answer: $\frac{\pi}{1010}$

Solution: In general:

$$\begin{aligned}
\sum_{k=1}^{n} \sin k\theta &= \frac{\sum_{k=1}^{n} 2\sin\frac{\theta}{2}\sin k\theta}{2\sin\frac{\theta}{2}} \\
&= \frac{\sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right)\theta - \cos(k+\frac{1}{2})\theta\right)}{2\sin\frac{\theta}{2}} \\
\textit{(telescoping sum)} \quad &= \frac{\cos\frac{\theta}{2} - \cos\left(n\theta + \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}
\end{aligned}$$

Therefore, for the above expression to be equal to zero, for some integer k either $n\theta = 2\pi k$ or $n\theta + \frac{\theta}{2} = 2\pi k - \frac{\theta}{2} \to (n+1)\theta = 2\pi k$. When $n = 2019$, the least positive value of θ that satisfies one of these equations occurs when $k = 1$ and $\theta = \boxed{\frac{\pi}{1010}}$.

T-11 In $\triangle ABC$, D is on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. Let P and Q be the incenters of $\triangle ABD$ and $\triangle ADC$, respectively. Given that $BC = 14$, $AD = 12$, $CD = 5$, and O is the circumcenter of $\triangle ABC$, compute $[OPQ]$.

Answer: $\frac{53}{16}$

Solution: Using coordinates: let $B = (0, 0)$, $C = (14, 0)$, $A = (9, 12)$, and $D = (9, 0)$. Suppose $O = (7, y)$, then $7^2 + y^2 = OC^2 = OA^2 = 2^2 + (y - 12)^2 \to y = \frac{33}{8}$ so $O = (7, \frac{33}{8})$. Then, the inradius of $\triangle ABD$ is $\frac{9+12-15}{2} = 3$ and the inradius of $\triangle ADC$ is $\frac{5+12-13}{2} = 2$ since both are right triangles. Therefore, $P = (6, 3)$ and $Q = (11, 2)$. By the shoelace formula, the area of OPQ is $\frac{1}{2}\left|6(2) + 11(\frac{33}{8}) + 7(3) - 3(11) - 2(7) - (\frac{33}{8})(6)\right| = \boxed{\dfrac{53}{16}}$.

T-12 For integers $n \geq 2$, let $h(n)$ be the number of positive integers $x \leq n$ such that $x^2 \equiv 1 \pmod{n}$. Compute the greatest integer $n \geq 2$ such that $h(n) = \phi(n)$, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

Answer: 24

Solution: The function h is multiplicative so it suffices to determine when $h(p^\alpha) = \phi(p^\alpha)$ for prime p. If p is odd, then p^α divides $x^2 - 1 = (x-1)(x+1)$ implies either p^α divides $x - 1$ or p^α divides $x + 1$ (otherwise, p divides $(x+1) - (x-1) = 2$, contradiction). Therefore $x^2 \equiv 1 \bmod p^\alpha$ if and only if $x \equiv \pm \bmod p^\alpha$, so $h(p^\alpha) = 2$. Since $\phi(p^\alpha) = p^{\alpha-1}(p-1)$, it follows that $\phi(p^\alpha) = 2$ implies $p = 3$ and $\alpha = 1$. Consider the case where $h(2^\alpha) = \phi(2^\alpha)$. This is true for $0 \leq \alpha \leq 3$. When $\alpha > 3$, 2^α divides $(x-1)(x+1)$. Since only one of these factors are divisible by 4, it follows that $x \equiv \pm 1 \bmod 2^{\alpha-1}$. Therefore, $h(2^\alpha) = 4 < 2^{\alpha-1} = \phi(2^\alpha)$ so there are no such powers of two. Therefore, $h(n) = \phi(n)$ if and only if $n = 2^a \cdot 3^b$ where $a \in \{0, 1, 2, 3\}$ and $b \in \{0, 1\}$. Therefore, the greatest such integer is $n = \boxed{24}$.

T-13 Compute the number of functions $f : \{1, 2, \ldots, 20\} \to \{1, 2, 3, 4\}$ such that $f(m)$ divides $f(n)$ whenever m divides n, and m and $f(m)$ have the same parity for all $m \in \{1, 2, \ldots, 20\}$.

Answer: 3296

Solution: Note that $f(n) = 1$ for all odd $n \leq 10$ (if $f(n) = 3$, then $f(2n)$ is even and divisible by 3, which is a contradiction because no value in the range is a multiple of 6). The function values for odd numbers greater than 10 can either be 1 or 3, so there are 2^5 possible assignments of values to the odd integers in the domain.

Consider the greatest positive integer α such that $f(2^\alpha) = 2$.

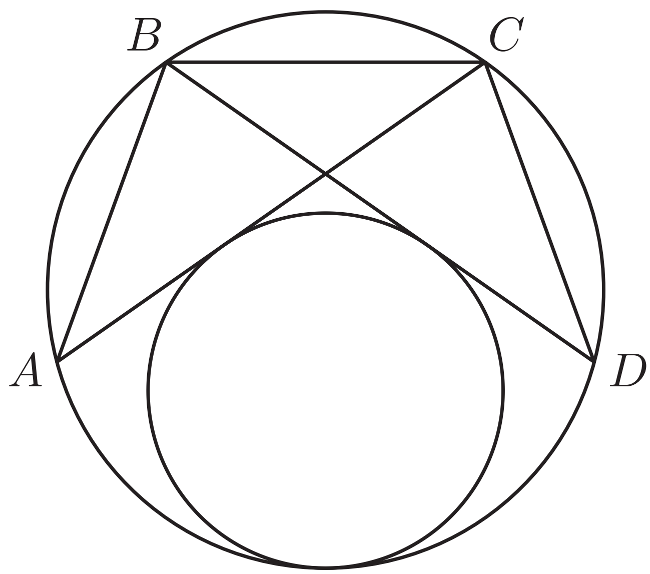
$f(2) = 4$ In this case $f(n) = 4$ for all even values of n.

$2^\alpha = 2$ In this case, $f(4) = f(8) = f(12) = f(16) = f(20) = 4$ while $f(10)$ and $f(14)$ can be either 2 or 4, but $f(6) \mid f(18)$. This gives 12 possibilities.

$2^\alpha \geq 4$ In these cases, $f(14)$ can be 2 or 4, $f(6) \mid f(12)$, $f(6) \mid f(18)$, and $f(10) \mid f(20)$. This results in 30 possible cases for each.

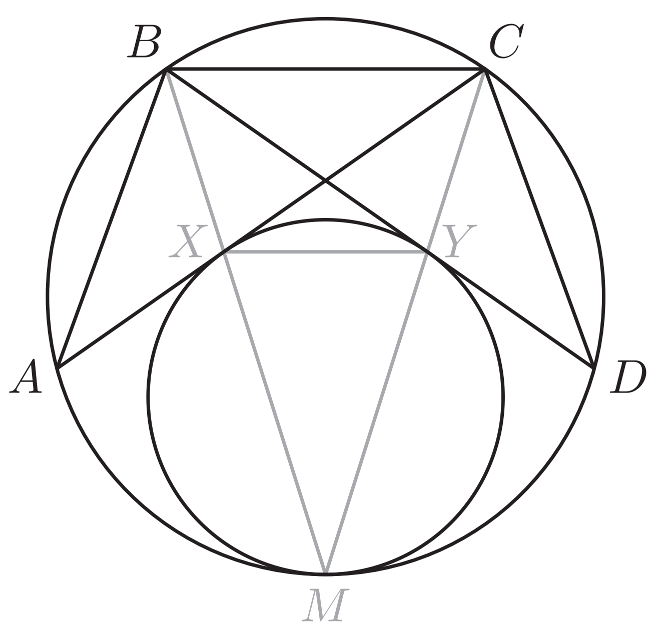
Therefore, there is a total of $2^5(1 + 12 + 3 \cdot 30) = \boxed{3296}$ such functions.

T-14 Let Ω be a circle with radius 5, and suppose that A, B, C, and D are points on Ω in that order such that $AB = BC = CD = 4$. Compute the radius of the circle shown in the figure below that is tangent to $\overline{AC}$, $\overline{BD}$, and minor arc $\widehat{AD}$.



Answer: $2\sqrt{21}-5$

Solution: Let X, Y, and M denote the tangency points of the circle to $\overline{AC}$, $\overline{BD}$, and $\widehat{AD}$ respectively. Note that by Archimedes' Lemma, B, X, and M are collinear; C, Y, and M are collinear; and M is the midpoint of $\widehat{AD}$.



The crucial claim is that $AX = XY = YD$. Indeed, note that from $\angle ABM = \angle DBM$ and $\angle BAC = \angle DAC$, X is the incenter of $\triangle ABD$; and therefore M is the circumcenter of $\triangle AXD$[2]

[2]See http://web.evanchen.cc/handouts/Fact5/Fact5.pdf

which gives $AM = MX = MD$. By the same argument MY is also equal to all these lengths. But now $\angle AMB = \angle BMC = \angle CMD$ yields $\triangle AMX \cong \triangle XMY \cong \triangle YMD$, which gives $AX = XY = YD$.

To finish, note that $\triangle AMX$ being isosceles implies that $\triangle BCX$ is isosceles, so $BX = CX = 4$. The Law of Sines yields $\sin\angle BCA = \frac{2}{5}$, and it follows that $AC = \frac{8}{5}\sqrt{21}$. Thus $XY = AX = 4(\frac{2}{5}\sqrt{21} - 1)$, so letting r denote the radius of the desired circle, using homothety gives

$$\tfrac{r}{5} = \tfrac{XY}{4} = \tfrac{2}{5}\sqrt{21} - 1 \quad \Rightarrow \quad r = \boxed{2\sqrt{21} - 5}.$$

T-15 Given that $\displaystyle\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{5^k} = \sqrt{5}$, compute the value of the sum

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k}.$$

Answer: $\frac{5}{2}(\sqrt{5} - 1)$ or $\dfrac{5\sqrt{5} - 5}{2}$

Solution: Note that from the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, it follows that

$$\binom{2k+1}{k} = \frac{k+1}{2k+2}\binom{2k+2}{k+1} = \frac{1}{2}\binom{2k+2}{k+1}.$$

Thus

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k} = \frac{1}{2}\sum_{k=0}^{\infty} \binom{2k+2}{k+1} \frac{1}{5^k} = \frac{5}{2}\sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \boxed{\frac{5}{2}(\sqrt{5} - 1)}.$$

Individual Round (10 minutes per pair)

I-1 Compute the least positive integer n such that $2n$ and $3n$ both contain the digit 7 when written in base ten.

Answer: 39

Solution: Since $2n$ is always even, if $2n$ contains the digit 7, then $2n \geq 70$, or $n \geq 35$. Trying values of n starting with 35, $n = \boxed{39}$ is the first one that works, with $2n = 78$ and $3n = 117$.

I-2 Circle $\mathcal{C}$ lies entirely inside rectangle $\mathcal{R}$, and $\mathcal{C}$ is tangent to three of the sides of $\mathcal{R}$. Given that the ratio of the area of $\mathcal{C}$ to the area of $\mathcal{R}$ is $\frac{\pi}{5}$, compute the ratio of the circumference of $\mathcal{C}$ to the perimeter of $\mathcal{R}$.

Answer: $\frac{2\pi}{9}$

Solution: Let r be the radius of $\mathcal{C}$. Then one of the side lengths of $\mathcal{R}$ is $2r$. Let the other side length be s. Then the area of $\mathcal{C}$ is πr^2 and the area of $\mathcal{R}$ is $2rs$, so $\frac{\pi r^2}{2rs} = \frac{\pi}{5}$. This gives $s = \frac{5r}{2}$. Now, the circumference of $\mathcal{C}$ is $2\pi r$ and the perimeter of $\mathcal{R}$ is $2(2r + s) = 2(2r + \frac{5r}{2}) = 9r$, so the desired ratio is $\frac{2\pi r}{9r} = \boxed{\frac{2\pi}{9}}$.

I-3 Compute the sum of all integers k with $1 \leq k \leq 2019$ such that $(2k-1)^2 = 1 + k \times 10^{k/13}$.

Answer: 26

Solution: Expanding the left hand side gives $4k^2 - 4k = k \times 10^{k/13}$ so $4k - 4 = 10^{k/13}$. Note that $10^{k/13}$ must be an integer so k is divisible by 13. Therefore, when $k \geq 39$, $10^{k/13} > 4k$ so it suffices to check $k = 13$ and $k = 26$. Only $k = \boxed{26}$ is a solution.

I-4 Exactly five distinct vertices of a regular 12-gon are randomly colored red, with all vertices equally likely to be colored. Compute the probability that no edge of the 12-gon has both its vertices colored red.

Answer: $\frac{1}{22}$

Solution: Label the vertices in order $v_1, v_2, \ldots, v_{12}$. Let R be the set of red vertices. Assume without loss of generality that $v_1 \in R$, then there are $\binom{11}{4} = 330$ ways to select the remaining four vertices in R. Call a set R *edge-free* if it does not contain both vertices of an edge of the 12$-$gon. Consider the five possible cases of the vertex (or vertices) in an edge-free R that is/are furthest from v_1: either v_6, v_7, v_8, v_6 and v_8, or v_5 and v_9. For any other cases, it is impossible to have the remaining elements in R not share an edge with another vertex in R. Note that neither v_2 nor v_{12} can be in an edge-free R that contains v_1.

– v_6: R cannot contain v_5, v_7, or v_8, which forces one of v_3 or v_4, plus v_9 and v_{11} to be in R for it to be edge-free. By symmetry, there are only two edge-free sets for the case of v_8 being furthest as well.
– v_7: R cannot contain v_6 or v_8. There will be two remaining vertices of R clockwise from v_7 to v_1 and one counter-clockwise, or vice versa. The side that has two is forced to be v_9 and v_{11} or v_3 and v_5, the vertex on the other side can be any of the three not adjacent to v_1 or v_7. This makes six edge-free sets in total.
– v_6 and v_8: R cannot contain v_5, v_7, or v_9. The two remaining vertices must be one of v_{10} or v_{11} and v_3 or v_4. This results in four edge-free sets.
– v_5 and v_9: R can only contain v_3 and v_{11} and be edge-free.

In total, there are 15 edge-free sets of size five, so the probability is $\frac{15}{330} = \boxed{\frac{1}{22}}$.

I-5 Triangle SAM lies in the coordinate plane with $S = (-15, 0)$, $A = (0, 36)$, and $M = (15, 0)$. There is a unique point $B = (x, y)$ in the interior of $\triangle SAM$ such that the perimeters of $\triangle BSA$, $\triangle BAM$, and $\triangle BMS$ are equal. Compute $x + y$.

Answer: 20

Solution: Note that $\triangle SAM$ is isosceles with $SM = 15 - (-15) = 30$ and

$$AS = AM = \sqrt{(0-(-15))^2 + 36^2} = \sqrt{(0-15)^2 + 36^2} = \sqrt{15^2 + 36^2} = \sqrt{3^2(5^2 + 12^2)} = 3 \cdot 13 = 39.$$

Note that point B must lie on the altitude $\overline{AH}$ from A to $\overline{SM}$, for if this were not the case, then by symmetry, the reflection of B across $\overline{AH}$—call it B'—would also satisfy the given constraints, implying that B was not unique. Hence $BS = BM$, and it therefore suffices to equate the perimeters of $\triangle BAM$ and $\triangle BSM$. It also follows that H is the midpoint of $\overline{SM}$, hence $x = \frac{-15+15}{2} = 0$. Thus

$$\begin{aligned}
& BA + BM + AM = BS + BM + SM \\
\Longrightarrow\ & BA + AM = BS + SM \\
\Longrightarrow\ & (36 - y) + 39 = \sqrt{(0-(-15))^2 + (y-0)^2} + 30 \\
\Longrightarrow\ & 45 - y = \sqrt{15^2 + y^2} \\
\Longrightarrow\ & y^2 - 90y + 45^2 = y^2 + 15^2 \\
\Longrightarrow\ & 90y = 45^2 - 15^2.
\end{aligned}$$

Factoring the right-hand side of the last equation gives $90y = (45-15)(45+15) = 30 \cdot 60$ and dividing each side by 90 yields $y = 20$, hence $x + y = \boxed{20}$.

I-6 The function $f(x) = x^4 + ax^3 + bx^2 + 2000x + d$ has four distinct roots. Two of the roots sum to 5; the other two roots also sum to 5. Compute b.

Answer: -375

Solution: Let r_1, r_2, s_1, and s_2 be the four roots, with $r_1 + r_2 = s_1 + s_2 = 5$. By Vieta's formulas:

$$-2000 = r_1r_2s_1 + r_1r_2s_2 + r_1s_1s_2 + r_2s_1s_2.$$

This equation factors as follows:

$$-2000 = r_1r_2(s_1 + s_2) + (r_1 + r_2)s_1s_2.$$

The terms in parentheses both equal 5, so $-400 = r_1r_2 + s_1s_2$. Now, by Vieta again,

$$\begin{aligned} b &= r_1r_2 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2 + s_1s_2 \\ &= (r_1r_2 + s_1s_2) + (r_1 + r_2)(s_1 + s_2) \\ &= -400 + (5)(5) = \boxed{-375}. \end{aligned}$$

I-7 Compute the sum of all possible values for d that satisfy the following property: there exists a positive integer n with exactly 15 divisors such that $2n$ has exactly d divisors.

Answer: 84

Solution: Since n has $15 = 3^1 5^1$ divisors, either $n = p^{14}$ for some prime p, or $n = p^4q^2$ for some distinct primes p and q. If $n = p^{14}$, then either $p = 2$, in which case $2n = 2^{15}$ has 16 divisors, or $p \neq 2$, in which case $2n = 2^1p^{14}$ has $2 \cdot 15 = 30$ divisors. If $n = p^4q^2$ and $p = 2$, then $2n = 2^5q^2$ has $6 \cdot 3 = 18$ divisors. If $n = p^4q^2$ and $q = 2$, then $2n = p^42^3$ has $5 \cdot 4 = 20$ divisors. If $p \neq 2$ and $q \neq 2$, then $2n = 2^1p^4q^2$ has $2 \cdot 5 \cdot 3 = 30$ divisors, a value already found. Therefore the answer is $16 + 30 + 18 + 20 = \boxed{84}$.

I-8 Let $\theta = \frac{1}{2}\sin^{-1}\frac{2}{3}$. Compute $\sin^4\theta + \cos^4\theta$.

Answer: $\frac{7}{9}$

Solution: Notice that

$$\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta.$$

Using the double-angle formula, this can be rewritten as

$$1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{1}{2}(2\sin\theta\cos\theta)^2 = 1 - \frac{1}{2}\sin^2(2\theta).$$

Since $2\theta = \sin^{-1}\frac{2}{3}$, it follows that $\sin(2\theta) = \frac{2}{3}$, so $1 - \frac{1}{2}\sin^2(2\theta) = 1 - \frac{1}{2}\left(\frac{2}{3}\right)^2 = \boxed{\frac{7}{9}}$.

I-9 Let P be a set of five points in the plane, no three of which are colinear. Let S be the set of all line segments between two points in P. Compute the number of subsets of S such that exactly four triangles have all their edges in S.

Answer: 40

Solution: The segments in S form ten triangles. The removal of any single segment from S eliminates three triangles. Accordingly, removing any pair of segments that share no points in common will eliminate six triangles, leaving four. Removing two edges that share a point (say p) in common removes five triangles in total, leaving five. If another edge containing p as a vertex is removed, another triangle is removed, leaving four. Finally, if the last edge containing p is removed, no additional triangles are removed. There are 15 pairs of disjoint segments in S, and there are 5 choices of p and 5 choices of a single (or no) edge to retain containing p, so there are $15 + 25 = \boxed{40}$ subsets of S that contain exactly four triangles.

I-10 Let m and n be positive integers. When the point $(15, 17)$ is reflected across the line $y = mx$ and then the reflected point is reflected across the line $y = nx$, the resulting point is $(17, 15)$. Compute $m + n$.

Answer: 256

Solution: Let θ be the angle from the ray $y = mx$ to the ray $y = nx$ in the first quadrant. Then a reflection of $(15, 17)$ over these two lines in succession constitutes a rotation through an angle of 2θ. Therefore, θ must be in the clockwise direction, so $m > n$, and 2θ must equal the angle between ℓ_1 and ℓ_2, where ℓ_1 is the line $y = \frac{17}{15}x$ and ℓ_2 is the line $y = \frac{15}{17}x$. Letting ℓ denote the line $y = x$, it follows that ℓ_1 and ℓ_2 are reflections of each other over ℓ, so θ must equal the angle between ℓ and ℓ_2.

The angle between the positive x-axis and the line $y = kx$ is $\arctan k$, so it follows that

$$\arctan m - \arctan n = \arctan 1 - \arctan \tfrac{15}{17}.$$

Taking the tangent of both sides and using the difference identity for tangents gives

$$\frac{m-n}{1+mn} = \frac{1 - \frac{15}{17}}{1 + \frac{15}{17}} = \frac{1}{16}.$$

Cross-multiplying gives $16m - 16n = 1 + mn$, or $mn - 16m + 16n + 1 = 0$. Subtracting 257 from both sides, this factors as

$$(m + 16)(n - 16) = -257.$$

Since 257 is a prime number and $m, n > 0$, the only possibility is that $m + 16 = 257$ and $n - 16 = -1$, which gives $(m, n) = (241, 15)$, so $m + n = \boxed{256}$.

Relay Round (6, 8, 10 minutes)

R1-1 Three 1×1 squares in a 3×3 grid are chosen randomly and colored orange. Compute the probability that no row or column of the grid contains more than one orange square.

Answer: $\frac{1}{14}$

Solution: There are $\binom{9}{3} = 84$ possible colorings of 3 out of 9 squares in the grid. Exactly six of them have exactly one orange square in each row and column. Such a coloring would have to have one orange square in each column, let r_i be the row of the orange square in column i, the r_i must be distinct, and there are $3! = 6$ possible assignments of the r_i. Therefore the desired probability is $\frac{6}{84} = \boxed{\frac{1}{14}}$.

R1-2 Let $T =$ TNYWR. Compute the sum of the prime factors of $\dfrac{1-T^2}{T^3}$.

Answer: 30

Solution: Let $N = \dfrac{1-T^2}{T^3}$, then $N = \dfrac{1}{T^3} - \dfrac{1}{T} = \dfrac{1}{T}\left(\dfrac{1}{T^2} - 1\right) = \dfrac{1}{T}\left(\dfrac{1}{T} - 1\right)\left(\dfrac{1}{T} + 1\right)$. As $T = \frac{1}{14}$, $N = 14 \times 13 \times 15$, which has prime factors 2, 3, 5, 7, and 13. The sum of these prime factors is $\boxed{30}$.

R2-1 Let $a_1, a_2, a_3, \ldots$ be a nonconstant arithmetic sequence. Suppose that a_1, a_{11}, a_{111} is a geometric sequence. Compute a_{11}/a_1.

Answer: 10

Solution: If r is the common ratio, then $a_{11} = ra_1$ and $a_{111} = r^2 a_1$, so $a_{11} - a_1 = (r-1)a_1$ and $a_{111} - a_{11} = (r^2 - r)a_1$. Therefore,

$$\frac{a_{111} - a_{11}}{a_{11} - a_1} = \frac{(r^2 - r)a_1}{(r-1)a_1} = r.$$

If d is the common difference of the arithmetic sequence, then $a_{111} - a_{11} = 100d$ and $a_{11} - a_1 = 10d$, so

$$\frac{a_{111} - a_{11}}{a_{11} - a_1} = \frac{100d}{10d} = 10.$$

Thus, $r = \boxed{10}$.

R2-2 Let $T = $ TNYWR. Frank has a $3 \times T$ grid of squares. Compute the number of ways that Frank can shade some (possibly none) of the squares in the grid, such that each row and column contains at most one shaded square.

Answer: 1021

Solution: Frank can shade 0, 1, 2, or 3 squares. There is 1 way for Frank to shade 0 squares, and there are $3T$ ways for Frank to shade 1 square. To shade 2 squares, Frank can pick the first square in $3T$ ways and the second square in $2(T-1)$ ways; but the order of the squares does not matter, so the total number of ways to shade 2 squares is $3T \cdot 2(T-1) \cdot \frac{1}{2} = 3T(T-1)$. Similarly, to shade 3 squares, Frank can pick the squares in $3T \cdot 2(T-1) \cdot 1(T-2)$ ways, but since the order of the squares does not matter, the total number of ways is $3T \cdot 2(T-1) \cdot 1(T-2) \cdot \frac{1}{3!} = T(T-1)(T-2)$. So the answer is

$$1 + 3T + 3T(T-1) + T(T-1)(T-2) = T^3 + 2T + 1.$$

With $T = 10$, the answer is $10^3 + 2 \cdot 10 + 1 = \boxed{1021}$.

R2-3 Let $T = $ TNYWR. Let N be the positive integer created by joining T copies of T. (For example, if $T = 15$, then N is the 30-digit integer $151515\ldots15$.) Compute the remainder when N is divided by 99.

Answer: 70

Solution: It suffices to determine N modulo 9 and modulo 11. To find N modulo 9, note that if the sum of the digits of T is s, then the sum of the digits of N is $T \cdot s$. But by the divisibility rule for 9, $T \equiv s \pmod{9}$, so $N \equiv T \cdot s \equiv s^2 \pmod{9}$.

To find N modulo 11, use the divisibility rule for 11. First, if T has an odd number of digits and T is even, then in the alternating sum of digits of N, all the terms will cancel out, making $N \equiv 0 \pmod{11}$. If T has an odd number of digits and T is odd, then all the terms will cancel except for the rightmost appearance of T, which means that $N \equiv T \pmod{11}$. On the other hand, if T has an even number of digits, then the alternating sum of digits of N equals T times the alternating sum of digits of T. If s' is the alternating sum of digits of T, then $N \equiv T \cdot s' \equiv (s')^2 \pmod{11}$.

With $T = 1021$, the digit sum is $s = 4$, so $N \equiv s^2 \equiv 7 \pmod{9}$. Also, T has an even number of digits and $s' = 1 - 2 + 0 - 1 = -2$, so $N \equiv (s')^2 \equiv 4 \pmod{11}$. These two congruences imply that $N \equiv \boxed{70} \pmod{99}$.

Alternate Solution: Because $100 \equiv 1 \pmod{99}$, if $N = \overline{a_{2k}a_{2k-1}\ldots a_2a_1}$, then $N \equiv \overline{a_{2k}a_{2k-1}} + \cdots + \overline{a_2a_1} \pmod{99}$. If T has an even number of digits, and S is the pairwise-digit sum of T, then $N \equiv S^2 \pmod{99}$. If T has an odd number of digits (say k), then let R be the $2k$ digit number consisting of two copies of T, and $N \equiv T + (T-1)/2 \times R \pmod{99}$. Since $T = 1021$, $S = 31$ and $N \equiv 961 \pmod{99} \equiv 9 + 61 \equiv \boxed{70} \pmod{99}$.

R3-1 Given that $(20^{19}+100)^2-(20^{19}-100)^2=20^N$, compute N.

Answer: 21

Solution: Let $x=20^{19}$, then the left hand side is $(x+100)^2-(x-100)^2=(x^2+200x+10000)-(x^2-200x+10000)=400x=20^2x$, so $N=\boxed{21}$.

R3-2 Let $T=$ TNYWR. Given that a rectangle has area T and perimeter $T-1$, compute the length of the shorter side of the rectangle.

Answer: 3

Solution: Let x and y be the lengths of the sides of the rectangle, then $2x+2y=T-1$ and $xy=T$, combining the two equations gives $x=\dfrac{2y+1}{y-2}$. Provided $y\neq 2$, plugging this expression into the rectangle area formula gives $2y^2+y=T(y-2)$. As $T=21$, this simplifies to $2y^2+y=21y-42\to 2y^2-20y+42=0$, which factors to $2(y-3)(y-7)=0$. Therefore the shorter edge of the rectangle has length $\boxed{3}$.

R3-3 Let $T=$ TNYWR. Let N be the sum of all positive integers which divide $2^7(T^8-1)$. Compute the remainder when N is divided by 100.

Answer: 32

Solution: Let $S=2^7(T^8-1)$. By difference of squares, $S=2^7(T^2-1)(T^2+1)(T^4+1)=2^{12}\cdot 5\cdot 41$ for $T=3$. Note that $N=\sigma(S)=\sigma(2^{12})\sigma(5)\sigma(41)=\left(2^{13}-1\right)(6)(42)$. Therefore, $N\equiv 91\cdot 52\equiv \boxed{32} \bmod 100$.

R3-4 Let $T=$ TNYWR. Fran flips four fair coins and rolls a standard, fair six-sided die. Let p be the probability that the number rolled on the die is equal to the number of heads flipped. Compute pT.

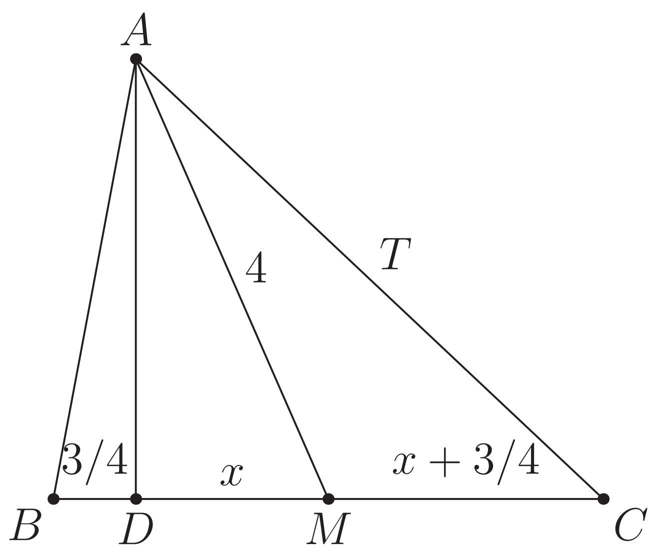
Answer: 5

Solution: If Fran flips 0 heads, then it is impossible for the result of the die roll to equal the number of heads flipped. On the other hand, if Fran flips a nonzero number of heads $(1,2,3,$ or 4 heads), then there is always a $\frac{1}{6}$ probability that the die roll matches the number of heads flipped. The probability that Fran flips 0 heads is $\left(\frac{1}{2}\right)^4=\frac{1}{16}$, so the probability that Fran flips a nonzero number of heads is $1-\frac{1}{16}=\frac{15}{16}$. Therefore, $p=\frac{15}{16}\cdot\frac{1}{6}=\frac{5}{32}$, and $pT=\boxed{5}$.

R3-5 Let T = TNYWR. In $\triangle ABC$, point M is the midpoint of $\overline{BC}$ and D is the foot of the altitude from A to $\overline{BC}$, with D between B and M. Given that $AC = T$, $AM = 4$, and $BD = \dfrac{3}{4}$, compute AB.

Answer: $\sqrt{15}$

Solution: Let $DM = x$. Since M is the midpoint of $\overline{BC}$, it follows that $CM = BM = x + \frac{3}{4}$.



By the Pythagorean theorem on $\triangle ADM$ and $\triangle ADC$, it follows that $AD^2 + x^2 = 4^2$ and $AD^2 + (2x + \frac{3}{4})^2 = T^2$. Subtracting these two equations, $(2x + \frac{3}{4})^2 - x^2 = T^2 - 4^2$, which for $T = 5$ simplifies to $3x^2 + 3x - \frac{135}{16} = 0$, or $x^2 + x - \frac{45}{16} = 0$. This factors as $(x + \frac{9}{4})(x - \frac{5}{4}) = 0$, so $x = \frac{5}{4}$.

Then $AD = \sqrt{4^2 - (\frac{5}{4})^2} = \sqrt{\frac{231}{16}}$, so $AB = \sqrt{AD^2 + (\frac{3}{4})^2} = \sqrt{\frac{240}{16}} = \boxed{\sqrt{15}}$.

R3-6 Let T = TNYWR. For a positive integer n, let S_n be the sum of the first n positive integers. Compute the remainder of S_{T^4} when divided by 9.

Answer: 0

Solution: The numbers described are the triangular numbers, whose residues modulo 9 form a 9-cycle: $1, 3, 6, 1, 6, 3, 1, 0, 0, 1, 3, 6, \ldots$. $T = \sqrt{15}$, so $T^4 = 15^2 = 225$, a multiple of 9, so the remainder is $\boxed{0}$.

Tiebreaker (10 minutes)

TB Compute the number of orderings of the numbers $1, 2, \ldots, 9$ with the following property: if m comes before n in the ordering, then $m < n^2$.

Answer: 3240

Solution: The given condition amounts to the following: 1 must come first in the permutation; 2 must come before $4, 5, 6, \ldots, 9$; and 3 must come before 9. After placing 1, it follows that 2 must either be in the second or third spot. If 2 is in the second spot, then of the 7! possible permutations of $3, 4, \ldots, 9$, exactly $\frac{1}{2}$ of them have 3 before 9, so there are $\frac{7!}{2}$ possible permutations. If 2 is in the third spot, then 3 must be in the second spot, and all 6! possible permutations of the remaining numbers $4, 5, \ldots, 9$ work. So the total number of permutations is $\frac{7!}{2} + 6! = \frac{5040}{2} + 720 = 2520 + 720 = \boxed{3240}$.