AMERICAN REGIONS MATH LEAGUE & Power & Local Contests 2009–2014

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ISBN 978-194309167-6

Published by ARML. Website: http://www.arml.com/
Errata and additional material available at http://www.arml.com/books.

Printed in the United States of America.

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Preface: American Regions Math League

When I was a high school math student, in Chicago in the mid-1980's, ARML was *the* contest. From our local, somewhat esoteric tryout rituals to the experience of being at Penn State with teams from dozens of other states and regions, there was nothing quite like ARML. What makes ARML unique? The quirkiness of the format (the combination of traditional number-answer and extended, proof-based problems, low-time-pressure team questions and high-pressure relays), and, I'd like to think, the quirkiness and quality of the problems (I hope that others agree that certain problems just "feel like" ARML problems) are contributing factors. But ultimately I think what makes ARML special, as a competitor, coach, or author, is the joy of joining together with so many other people, from around a "region" and around the country, for a day of exciting mathematics. There's nothing like it.

As a new teacher at Phillips Andover in 1996, I vividly remember when the previous head author, Don Barry, suggested I help out by reviewing a draft of that year's contest. Figuring that the contest itself is about two and a half hours of math, I settled down one winter afternoon in front of the fireplace and opened Don's packet for what I expected would be a "three-hour tour" of intriguing math. You can imagine my surprise when I discovered that, before the *final* version that I and everyone else had seen at the end of May, Don would prepare three or four candidates for every problem! That afternoon turned into an evening, and then several more; but I realized that as challenging as it is to participate in the contest, the work of crafting it is even more difficult—and even more rewarding.

ARML was founded in 1976 as the Atlantic Region Mathematics League. It was an outgrowth of NYSML, the New York State Mathematics League, founded in 1973, and designed to serve as a competition for the best of the teams in the math leagues in New York. Both were the joint vision of Alfred Kalfus and Steve Adrian. Alfred served as the first president of ARML and Steve, along with Joe Quartararo (from 1976 to 1982), did the organizational work, site preparations, and publicity. Marty Badoian as vice president and Eric Walstein also played crucial roles in ARML's early days. The impetus for ARML came when Marty Badoian brought a Massachusetts team to NYSML in 1974 and 1975 and they did so well, losing by a point in 1974 and winning by a point in 1975, that an expanded competition seemed desirable. ARML was conceived as an interstate competition covering the eastern seaboard, formed through the joint action of the New York State Mathematics League, the New England Association of Mathematics Leagues, and leagues from New Jersey, Pennsylvania, Maryland, and Virginia. But ARML is a great contest and it soon attracted math students and math teachers from all over the country. By 1984 ARML was renamed the American Regions Mathematics League. Currently, it takes place simultaneously at four sites, Penn State University in State College, Pennsylvania, the University of Iowa in Iowa City, Iowa, the University of Georgia in Athens, Georgia, and the University of Nevada at Las Vegas. It brings together some 2200 of the finest young mathematicians in the United States and Canada. In the past teams from Russia have competed and in the last few years teams from Taiwan, the Philippines, Columbia, and Hong Kong have been taking part. Taiwanese educators even created a similar competition for schools in Taiwan called TRML.

ARML is a competition between regions. A region may be as large as a state–there are teams from Texas, Minnesota, and Georgia, it may be half a state–eastern and western Massachusetts field teams, it may be a county such as Suffolk county in New York, it may be a city such as Chicago or New York, or a region may even be a high school. Each team consists of 15 students and a region may send more than one team.

The contest consists of 6 parts. First is the Team Round. A team's 15 students are in one classroom, they receive 10 problems, they have 20 minutes to solve them as a group, and up through 2008 they could use calculators. Typically, problems are divided up so that every problem is being worked on by at least one student, answers are posted, and hurried consultations take place if there is disagreement. Next comes the Power Question. Teams are given 60 minutes to solve a series of in-depth questions on one topic and/or prove a number of theorems on that topic. The teams' papers are graded by a hardworking group of teachers tucked away in a corner of the auditorium. Following the Power Question, the teams come together in a large auditorium for the Individual Round. Here, starting in 2009, students individually solved 10 questions, given in pairs, with 10 minutes for each pair, right answer only. Neither calculators nor collaboration are allowed. Initial problems are easier, but an easier problem is generally paired with a more difficult one. We try to write problems so that 80–90% of the students can solve the first one and less than 5% can solve the last one. The next round is the Relay Round. The teams of 15 are divided up into 5 groups of 3. Each 1^{st} person in a group gets the same problem, each 2^{nd} person gets the same problem, and each 3^{rd} person gets the same problem. The first person's problem has all the information necessary to solve it, but the second person's problem requires the first person's answer, and the third person needs the second person's answer. None of the three knows the other's questions. A well-written relay problem enables the second and third team members to do considerable work while waiting. They may even be able to discover a candidate pool of likely answers to their problems.

Those four rounds are the only rounds that count for the Team and Individual competition. Scoring is as follows: each correct answer on the Team Round earns 5 points. Sometimes a team gets all 50 points, but 40 or 45 has typically been the top score and the average for all the teams has been between 20 and 25. The Power Question is worth 50 points; often one or more teams earn a perfect score. Each individual problem is worth 1 point per contestant, meaning that the team can earn as many as $10 \cdot 15 = 150$ points on this round. Generally, 100 is a great score. On the Relay Round, only the third person's answer is scored. If the group of 3 gets the problem correct within 3 minutes they earn 5 points, if within 6 minutes they earn 3 points. Thus, each relay is worth a maximum of $5 \cdot 5 = 25$ points; a score of 14 is quite good. The winning team's score for the entire contest usually lies between 225 and 250 points.

The Super Relay has followed the relay races the past few years. It is just for fun and is a relay race for the whole team involving 15 questions. The team that gets the correct answer to the last question is the winner, earning a round of applause and bragging rights 'til next year. To keep the Super Relay from getting bogged down, we usually slip one or two problems into the relay that can be solved without requiring the previous person's answer. Students pass answers from both ends into position 8. That student's question requires two answers in order to be solved.

The Tiebreaker Round follows the relays. Until 2006 ARML gave only 3 top individual prizes, forcing a playoff took place between the top scorers. With increased sponsorship, ARML was able to increase the number of awards to 20. The Tiebreaker is a dramatic moment. At each site the contenders come to the front of the auditorium, they receive the problem at the same time that it is flashed on an overhead screen. Each student is timed and the winners are determined by who gets the correct answer most quickly. Three tiebreakers are now given. A site will give tiebreakers until it either has 20 correct answers or has exhausted the problems and students. Afterwards times are compared with the other sites.

ARML is quite an undertaking. It brings together great students, it is one of the few mathematics contests that involves exciting travel, meeting lots of other students, and renewing friendships established during summer programs. It relies on the work of a large number of dedicated adults who do all the organizing and spend hours finding or writing problems to use in practice sessions. It is a contest that promotes and demands creativity and imagination. The students who take part in ARML are very experienced problem solvers, quick and insightful. Those of us who write problems for ARML respect the abilities of our participants and, consequently, we spend hours developing problems that spring from high school mathematics, yet are out of the ordinary, problems that can't be attacked in rote fashion. We don't just take a theorem and write a problem that employs it. We ask questions, we imagine situations we'd never thought of before, we try out this idea and that possibility, we run into dead-ends, and we occasionally stumble across a really neat idea and that's the one that makes it into an ARML competition. There are many different topics on each year's ARML, but more important, there is a wide variety of ways of thinking.

This publication makes available the problem sets and solutions for the 2009 to 2014 ARML contests. Five earlier publications contain ARML problems from its beginning in 1976 to 2008 and NYSML problems from its beginning in 1973 to 1992. The first was *NYSML-ARML Contests 1973–1982*, published in 1983 by Mu Alpha Theta. The second was *NYSML-ARML Contests 1983–1988* by Gilbert Kessler and Lawrence Zimmerman, published by the National Council of Teachers of Mathematics in 1989. The third was *ARML-NYSML Contests 1989–1994* by Lawrence Zimmerman and Gilbert Kessler, published by MathPro Press in 1995. The fourth was *American Regions Math League & ARML Power Contests* by Don Barry and Thomas Kilkelly, published by ARML in 2003. The fifth was *American Regions Math League & Power & Local Contests* by Don Barry, Paul Dreyer, and Thomas Kilkelly, published by ARML in 2009.

ARML has an executive board that organizes each contest. Alfred Kalfus was president from 1976 until 1989 when Mark Saul of Bronxville High School, NY took over. Mark retired in 2001 and Tim Sanders, director of the Great Plains Mathematics League, took over. In June of 2004, J. Bryan Sullivan became president. In June of 2013, Paul Dreyer became president. Current board members include Steve Adrian, Marty Badoian, Linda Berman, Barbara Chao, Steve Condie, Mike Curry, Paul Dreyer, Matthew Schneider, and Don Slater.

It has been my privilege to serve as a co-author of the contest since 1998, and as head author since Don Barry's retirement from the contest in 2008. In that time, we have made a few minor adjustments to the contest. After much discussion, the committee decided to add two questions to the individual round in order to reduce the number of students receiving zeroes without lowering the difficulty of the problems themselves. And, after even more discussion, the committee decided to eliminate calculators from the contest entirely—even though as mathematicians we heartily endorse using whatever technology is available. That last decision is less of a contradiction than it may seem. We saw the disparities between teams' technological arsenals grow rather than shrink, and we found more and more problems for which the elegant solution we intended shrank into irrelevance next to sheer number-crunching. One last adjustment that (unlike the others) received little feedback from coaches was an update of the list of "Some Mathematical Ideas Used in the Competition," to include updated notation and a few cautions for the inexperienced. (The number one is now explicitly excluded from the list of primes, because it is a unit.)

While writing and solving math problems is of course a great source of excitement, the problem committee meetings themselves are greatly enlivened by the good spirits and collaborative energy of a tremendous team of problem writers, solvers, and editors. These tremendously hard-working, creative, and dedicated people are a testament to ARML's power to inspire terrific mathematics: almost all of them are former ARML competitors. The problem committee consists of Paul Dreyer, Edward Early, Zuming Feng, Benji Fisher, Zac Franco, Chris Jeuell, Winston Luo, Andy Niedermaier, George Reuter, Andy Soffer, Eric Wepsic, and of course myself. Given the collaborative nature of our processes of question selection, editing, and rewriting, it's safe to say that every problem on the contest is the work of many minds. One of the joys of working on the contest these last years has been getting to spend so much time with these people, and to learn so much from them; for their patience, boundless energy, and passion for getting young people to encounter terrific mathematics, they have my thanks.

There are literally hundreds of other people who should be acknowledged here, including the dedicated coaches who spend days of unpaid time (and often hundreds of their own dollars) making sure that students arrive safely and ready to do terrific mathematics. My own mathematical journey began in fifth grade, when my father (himself not a mathematician) took me to the local bookstore to show me that "real math" looked totally different from the subject I hated in school; it continued under the auspices of four great teachers, Steve Viktora, Arnold Ross, David Kelly, and Paul J. Sally, Jr., whose names have a way of materializing in Power Questions. Don Barry brought me into the contest, and did me the great honor of asking me to succeed him as head author; even afterwards, his advice and counsel have been invaluable. Co-author Chris Jeuell has stepped into the (largely informal, and now formal) role of head editor and proofreader; the contest owes much to his thoughtful, exacting eye, which has caught many errors (despite my predilection for inserting them in late drafts). Longtime author Leo Schneider (1937–2010) and indispensable coach Marlys Henke (1943–2014) became friends of mine by their friendship to ARML, and are terribly missed. But most of all, I'd like to thank and dedicate this book to my wife, Allison, and my three children, Emma, Jonah, and Helen, who have put up with hundreds of hours (and now something like 17 after-Memorial-Day weekends) of mentally- and physically-absent parenting so that the contest can get done, and get done well. My gratitude for their love and support is, if not measureable, at least unbounded.

Paul J. Karafiol

Preface: ARML Power Contest

In 1994, under the leadership of its president, Mark Saul, ARML introduced the ARML Power Contest. Modeled on the Power Question which involves, as Mark noted, "cooperative effort in exploring a problem situation through the solution of chains of related problems", ARML began to offer a competition by mail. In November and February, each participating team receives a set of problems based on a major theme. Some of the problems require a numerical answer, some require justification, while others require that a proof be written. The mathematics of the problems has been geared so that students in an honors class, a math club, or on a math team can have a unique, challenging, yet mostly successful problem solving and mathematics writing experience. There is no limit to the size of the team, but the time for solving the problem set is limited to 45 minutes. Currently, the team's solutions are mailed to the author where they are graded by the author and some of his colleagues. There are 40 points per contest; the team score is the sum of the scores of the two contests. A team may represent a region, not just a school, but the contest must be taken by all members of the team at the same time and place. The contest has grown steadily. The contest boasts teams from single schools as well as from regional ARML teams, and competitors represent almost all 50 states, Canada, and Colombia. An agreement with the Mu Alpha Theta mathematics honor society to grant a few of its chapters free entry to the contest has spurred even more interest as teams return after their free year to keep their students engaged in the challenging and fun mathematics.

The Power Question and the Power Contest are both designed to simulate actual mathematical research activity. This is not easy. As Mark noted,

> "Power Contest problems are difficult to write. They must provide meaningful problem situations both for the novice and veteran mathletes. They must attract schools with strong traditions in mathematics competitions, yet offer experiences for students new to such events. Thus, they must build mathematically significant results out of mathematically trivial materials."

The idea for a Power Contest question usually starts off quite small and simple, such as a special case of a theorem the author has just read in a journal or an extension of another problem. The idea then grows and the parts take shape, leading students through simple calculations, explanation, and examples to deeper, more powerful results. The goal is to make the mathematics inviting to all participating teams yet also challenge and discriminate amongst the top mathematical talent in the competition, while keeping it assessable by the graders. Creating the problems is as much a challenge for the author as solving them is to the students!

For the majority of its existence, through the 2012–2013 competition, Thomas Kilkelly was the main author, coordinator, and grader for the ARML Power Contest. In developing the problems he has been assisted at times by Andy Niedermaier, a former ARML contestant from Minnesota, and by Ivaylo Kertezov from Sofia, Bulgaria. Many others have contributed ideas, and helped proofread and grade the contest. Starting with the 2013–2014 competition, stewardship of the contest passed to Micah Fogel of the Illinois Mathematics and Science Academy. His goals are to maintain the quality of the contest while modernizing the administration by creating electronic registration, distribution, and scoring systems. Please visit the contest on the Web at `www.armlpower.com`.

The current Power Contest author wishes to thank the ARML Board for entrusting him with the work, Thomas Kilkelly for making the contest what it is today, the Illinois Mathematics and Science Academy for supporting the work, and the students and coaches who provide wonderful feedback and encouragement. We also wish to thank Rileigh Luczak for typesetting the Power Contests in LaTeX.

Thomas Kilkelly
Micah Fogel

Preface: ARML Local

ARML Local was created to offer schools an opportunity to either practice for ARML or to take part in a contest similar to ARML at a local site. If a school is not already taking part in ARML, we hope that if it finds ARML Local to be a valuable experience, a school will be encouraged to support its students to participate in ARML. A team consists of up to 6 students. The local site administrator downloads the problems on the date of the contest, administers the test, corrects the problems, and enters the team's results online. The answers to all problems are numerical, there are no proofs to correct.

ARML Local consisted of four parts until 2014, when it dropped to three.

> The Team Round consisted of 10 questions until 2014, at which point it expanded to 15 questions. The team of students works together for 45 minutes to find the answers to the problems.
>
> The Theme Round takes 60 minutes and consists of around ten questions based on a common topic. Unlike the Power Question at ARML, the Theme Round does not involve proof. The reason for this is that it would be impossible to ensure consistency of grading across so many different sites. This round was discontinued in 2014.
>
> The Individual Round consists of 10 questions given in pairs with 10 minutes allowed for each pair.
>
> The Relay Rounds depart from the approach used at ARML to some degree by taking advantage of the fact that teams consist of 6 members. First of all, there are three rounds not two. Secondly, the first relay is for three teams of two, the second is for two teams of three, and the third is for one team of all six students. The times allotted for the relays are 6, 8, and 10 minutes respectively.

More information about the contest is available at `http://arml.com/arml_local/`.

Over the period of 2009 to 2014, ARML Local has been written by Paul Dreyer who is also the head coordinator of ARML Local as well as Andy Niedermaier, Richard Olson, and George Reuter, and has been edited by Oleg Kryzhanovsky and Ryan Martin.

Paul Dreyer

Sponsor and Supporters of ARML

Sponsor: D.E. Shaw group (http://deshaw.com/)

The D.E. Shaw group is a global investment and technology development firm. It employs a substantial number of top-notch computer scientists and mathematicians, many of whom joined the firm directly from leading university math and science programs. A number of them are former ARML competitors. Given the nature and challenges of its work, D.E. Shaw has a particular interest in promoting excellence in mathematics. It is especially proud of its years-long association with ARML and of its current sponsorship of this national competition.

Supporters:

American Mathematical Society	http://www.ams.org/
Canada/USA Mathcamp	http://www.mathcamp.org/
Design Science	http://www.dessci.com/
Expii	http://www.expii.com/
John Wiley & Sons	http://www.wiley.com/
Mathematical Association of America	http://www.maa.org/
Mu Alpha Theta	http://www.mualphatheta.org/
Penguin Group	http://www.penguin.com/services-academic/
Pi Mu Epsilon	http://www.pme-math.org/
Princeton University Press	http://www.press.princeton.edu/
PROMYS	http://www.promys.org/
Star League	https://www.starleague.us/
Texas Instruments	http://www.education.ti.com/
Wolfram Research	http://www.wolfram.com/
Yu's Elite Education	http://www.yuselite.org/

Acknowledgements: Problem Writers and Reviewers

Problem Writers and Reviewers for ARML 2009–2014

2009: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Andy Niedermaier, Leo Schneider, Andy Soffer, and Eric Wepsic

Reviewers: Chris Jeuell. Walter Carlip and Chris Jeuell helped with formatting the contest in LaTeX.

2010: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Andy Niedermaier, Leo Schneider, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher and Chris Jeuell

2011: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, Leo Schneider, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher and Chris Jeuell

2012: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher and Chris Jeuell

2013: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher and Chris Jeuell

2014: Writers: Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, George Reuter, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher and Chris Jeuell

Problem Writers for the ARML Power Contest

Feb. 2008: Thomas Kilkelly	Nov. 2009: Thomas Kilkelly
Feb. 2009: Thomas Kilkelly	Nov. 2010: Thomas Kilkelly
Feb. 2010: Thomas Kilkelly	Nov. 2011: Thomas Kilkelly
Feb. 2011: Thomas Kilkelly	Nov. 2012: Thomas Kilkelly
Feb. 2012: Thomas Kilkelly	Nov. 2013: Micah Fogel
Feb. 2013: Micah Fogel	Nov. 2014: Micah Fogel

Prize Winners

ARML Prize Winners 2009–2014

2009 Team Competition

Division A

1.	Lehigh Valley A1: Fire	215
2.	Phillips Exeter A1: Red	201
3.	SFBA A	197

Division B

1.	Thomas Jefferson B1	156
2.	Phillips Academy Andover B1	150
3.	Lehigh Valley B1: Ice	144

2009 Individual Competition D.E. Shaw Winners

1.	Alex Song	Ontario West
2.	Warut Suksompong	Phillips Exeter
3.	Qinxuan Pan	Montgomery
4.	Chong Gu	NYC
5.	Junu Bae	Western Penn
6.	David Yang	Southern California
7.	Bayley Wang	San Francisco Bay Area
8.	Zhifan (Ivan) Zhang	Southern California
9.	Aki Hogge	Montgomery
10.	Robert Yang	Utah

2010 Team Competition

Division A

1.	Lehigh Valley: A1	204
2.	Southern California: A1	192
3.	San Francisco: A1	190

Division B

1.	Nassau County: B1	124
2.	Montgomery: B1	112
3.	Minnesota: B1	112

2010 Individual Competition D.E. Shaw Winners

1.	Ben Gunby	Georgetown B1
2.	Bobby Shen	Texas A1
3.	Thomas Lu	North Carolina A1
4.	George Xing	Southern California A1
5.	Lyndon Ji	Indiana A1
6.	Allen Liu	Upstate NY
7.	Peter Cha	New York City A1
8.	Bryce Taylor	North Carolina A1
8.	Neil Gurram	Michigan A1
8.	Kerrick Staley	Iowa A1
8.	Kirin Sinha	Georgetown B1
8.	Peter Rassolov	South Carolina A1

2011 Team Competition

Division A

1. Lehigh Valley: A1 232
2. Exeter: A1 225
3. North Carolina: A1 222

Division B

1. Michigan: B1 169
2. Florida: B1 145
3. Connecticut: B1 137

2011 Individual Competition D.E. Shaw Winners

1.	Zhou Qun (Alex) Song	Michigan A1
2.	Shyam Narayanan	Kansas B1
3.	Spencer Kwon	Exeter A1
4.	Calvin Deng	North Carolina A1
5.	Lewis Chen	San Francisco A1
6.	Ben Alpert	Colorado A1
7.	Che-Cheng Lien	Washington A1
8.	Jiaqi Xie	Florida A1
9.	Bryan Cai	Lehigh Valley A1
10.	Allen Liu	Upstate NY A1
11.	Jong Wook Kim	Phillips Academy A1
12.	Seung In Sohn	Thomas Jefferson A1
13.	Edward Park	Georgia A1
13.	Eric Schneider	Lehigh Valley A1
15.	Victor Wang	Missouri A1
16.	Tim Wu	Georgia A1
17.	Douglas Chen	San Francisco A1
18.	Adam Hood	Thomas Jefferson A1
19.	Brandon Tran	San Francisco B1
20.	Tiangi (David) Lu	Michigan A1

2012 Team Competition

Division A

1. North Carolina A1 223
2. Lehigh Valley A1 213
3. SFBA/Norcal A1 211

Division B

1. WWP CML Legends B1 173
2. ALTERNATE 2 B1 136
3. Western Pennsylvania B1 126

2012 Individual Competition D.E. Shaw Winners

1.	Allen Liu	Upstate New York A1
2.	Shyam Narayanan	Kansas B1
3.	Lewis Chen	SFBA/Norcal A1
4.	Thomas Lu	North Carolina A1
5.	Andre Arslan	New York City A1
6.	Anderson Wang	Lehigh Valley A1
7.	Max Schindler	Missouri A1
8.	Alex Song	PEARL A1
9.	Robi Bhattacharjee	San Diego A1
10.	Thomas Swayze	San Diego A1
11.	Richard Yip	New York City A1
12.	Calvin Deng	North Carolina A1
13.	Alexander Smith	Montgomery A1
13.	Spencer Kwon	PEARL A1
15.	Brian Wagner	Phillips Academy Andover A1
16.	Mark Sellke	Indiana A1
17.	Brice Huang	WWP CML Legends B1
18.	Dai Yang	PEARL A1
19.	SeungIn Sohn	Thomas Jefferson A1
20.	George Du	San Diego A1

2013 Team Competition

Division A

1.	SFBA/NorCal A1	234
2.	North Carolina A1	219
3.	WWP A1	216

Division B

1.	Toronto Math Circles B1	122
2.	Lehigh Valley B1	117
3.	Minnesota B1	114

2013 Individual Competition D.E. Shaw Winners

1.	Allen Liu	Upstate New York A1
2.	Eric Schneider	Lehigh Valley A1
3.	Yang Liu	Missouri A1
4.	Alan Zhou	Eastern Massachusetts A1
5.	Darryl Wu	Washington A1
6.	Andrei Arsian	New York City A1
7.	Ben Taylor	West Virginia A1
8.	Brice Huang	WWP A1
9.	Calvin Deng	North Carolina A1
10.	James Tao	Chicago A1
11.	Mark Sellke	Indiana A1
12.	Robin Park	Thomas Jefferson A1
13.	Zachary Polansky	Eastern Massachusetts A1
14.	Thomas Swayze	San Diego A1
15.	Michael Ma	Texas A1
16.	David Xing	Georgia A1
17.	Akshaj Kadaveru	Fairfax Math Circle A1
18.	Brian Gu	Washington A1
19.	David Stoner	South Carolina A1
20.	Jerry Wu	SFBA/Northern California A1

2014 Team Competition

Division A

1.	PEARL A1	260
2.	SFBA/NorCal A1	242
3.	New York City Math Team A1	238

Division B

1.	WWP B1	152
2.	Howard County B1	132
3.	OHIO B1	125

2014 Individual Competition D.E. Shaw Winners

1.	Darryl Wu	Washington A1
2.	Yannick Yao	PEARL
3.	Andy Wei	PEARL
4.	Mark Sellke	Indiana A1
5.	Shyam Narayanan	Kansas A1
6.	James Lin	PEARL
7.	Jack Gurev	SFBA / NorthCal A1
8.	Zack Polansky	Eastern MA A1
9.	Brian Burks	SFBA / NorthCal A1
10.	Alex Song	PEARL
11.	Ravi Jagadeesan	PEARL
12.	Dennis Zhao	Montgomery A1
13.	Akshaj Kadavevu	Fairfax
14.	Richard Chen	South Carolina A1
15.	Haoquing Wang	Florida A1
16.	David Stoner	South Carolina A1
17.	Zachary Obsniuk	Michigan A1 Reals
18.	Owen Goff	Washington A1
19.	James Shi	SFBA / NorthCal A1
20.	Ben Wei	Toronto A1

ARML Power Contest Prize Winners

2008-09

1. Long Beach Math Circle	Long Beach, CA	Coach: Kent Merryfield
1. Wayzata High School	Plymouth, MN	Coach: Bill Skerbitz
3. Academy for the Advancement of Science and Technology	Hackensack, NJ	Coach: Joe Holbrook

2009-10

1. Wayzata High School	Plymouth, MN	Coach: Bill Skerbitz
2. Western Washington ARML	Seattle, WA	Coach: Ming Hu
3. Lynbrook High School	San Jose, CA	Coach: Rita Korsunsky

2010-11

1. Western Washington ARML	Seattle, WA	Coach: Ming Hu
2. Iowa City West High School	Iowa City, IA	Coach: Joye Walker
2. Wayzata High School	Plymouth, MN	Coach: Bill Skerbitz

2011-12

1. Lynbrook High School	San Jose, CA	Coach: Rita Korsunsky
2. Long Beach Math Circle	Long Beach, CA	Coach: Kent Merryfield
2. San Diego Math Circle	San Diego, CA	Coach: David W. Brown

2012-13

1. Western Washington ARML	Seattle, WA	Coach: Ming Hu & Monica Sung
2. Indus Center for Academic Excellence	Lathrup Village, MI	Coach: Dr. Raghunath Khetan
2. Long Beach Math Circle	Long Beach, CA	Coach: Kent Merryfield

2013-14

1. Lynbrook High School	San Jose, CA
2. Bergen County Academies	Hackensack, NJ
2. West Windsor-Plainsboro HS North	Plainsboro, NJ
2. Western Washington ARML	Seattle, WA

ARML Local Prize Winners

2009 Competition

Archimedes Division

1. AAST Mu AAA	136	Hackensack, NJ	Joe Holbrook/Ken Mayers
2. Math Zoom Green	123	Irvine, CA	Kevin Wang
3. Thomas Jefferson A	117	Alexandria, VA	Jen Allard

Bernoulli Division

1. Odle Middle School	90	Bellevue, WA	Lon-Chan Chu
2. Nashoba High School	68	Bolton, MA	Mary Redford

Individual Winners

Brian Hamrick	10	Thomas Jefferson A	Alexandria, VA
Mitchell Lee	10	Thomas Jefferson C	Alexandria, VA
Daniel Li	10	Thomas Jefferson A	Alexandria, VA
Matt Mayers	10	AAST Mu AAA	Hackensack, NJ
In-Sung Na	10	SK Telecom	Bergen County, NJ
George Xing	10	Math Zoom Green	Irvine, CA
David Yang	10	Math Zoom Green	Irvine, CA

2010 Competition

Archimedes Division

1. San Diego A	141	San Diego, CA	David Brown
2. Hwa Chong Institution Team A	137	Singapore	Lai Chiang Ang
3. Thomas Jefferson A	132	Alexandria, VA	Pat Gabriel

Bernoulli Division

1. Albany High	74	Albany, CA	Margaret Buck-Bauer
1. Waterford	74	Sandy, UT	Thomas Brennan/Tihomir Asparah

Individual Winners

Pooja Chandrashekar[1]	10	Fairfax Math Circle	Fairfax, VA
Tian Nie	10	Hwa Chong Institution Team A	Singapore
Linfeng Xu	10	Hwa Chong Institution Team A	Singapore
Fan Yin	10	Massey SL2	Windsor, ON
David Yang	10	Math Zoom Green	Irvine, CA
Casey Fu	10	San Diego A	San Diego, CA
Adam Hood	10	Thomas Jefferson A	Alexandria, VA
Sin Kim	10	Thomas Jefferson A	Alexandria, VA
Allen Park	10	Walton High School	Marietta, GA
Edward Park	10	Walton High School	Marietta, GA

2011 Competition

Archimedes Division

1. AAST Mu A	151	Hackensack, NJ	Michael Abramson
2. Thomas Jefferson 3	146	Alexandria, VA	Pat Gabriel
3. San Diego A	145	San Diego, CA	David W. Brown

Bernoulli Division

1. Albany High	102	Albany, CA	Marguerite Buck-Bauer
1. A* SSA – Tucson	102	Tucson, AZ	Nizamettin Yildiz

Individual Winners

Joshua Clark[2]	10	Mentor Cardinals	Mentor, OH

2012 Competition

Archimedes Division

1. A* – Chen Chen Chen	148	Sunnyvale, CA	Alper Halbutogullari
2. Bergen 1	129	Hackensack, NJ	Michael Abramson
3. FMC Black	124	Fairfax, VA	Jane Andraka

[1] Poola Chandrashekar was the only student to answer the tiebreaker question correctly.

[2] Joshua Clark was among 41 students who scored a perfect 10 on the individual round. Six of these students answered the tiebreaker correctly and Joshua correctly answered the tiebreaker the fastest.

Bernoulli Division

1.	A∗ – Sonoran Science Academy – Tucson	82	Tucson, AZ	Ismail Koksal
2.	AMSA	65	Marlborough, MA	Matt Vea

Individual Winners

Pooja Chandrashekar[3]	10	Thomas Jefferson High School 3	Alexandria, VA
Jongwhan Park	10	Bergen 1	Hackensack, NJ
Eugene Chen	10	A∗ – Chen Chen Chen	Sunnyvale, CA

2013 Competition

Archimedes Division

1.	A∗ Chen Chen Chen	136	Santa Clara, CA	Ali Gurel
2.	Bergen Sun	115	Hackensack, NJ	Michael Abramson
3.	Math Zoom Blue	114	Irvine, CA	Kevin Wang

Individual Winners

Zach Markos	10	Math Zoom Green	Irvine, CA
Jongwhan Park	10	Bergen Sun	Hackensack, NJ

2014 Competition

Archimedes Division

1.	Star League A-Star Chen	124	Santa Clara, CA	Ali Gurel
2.	BCA AJ ARML Local	122	Hackensack, NJ	Michael Abramson
3.	OCMC Team Alpha	116	Irvine, CA	Kent Merryfield

Individual Winners

Tomas Choi	10	Star League A-Star Chen	Santa Clara, CA
Paolo Gentili	10	San Diego Math Circle A	San Diego, CA

[3] Poola Chandrashekar correctly answered the tiebreaker question the fastest. Jongwhan Park also correctly answered the tiebreaker question.

ARML Awards

The *Alfred Kalfus Founder's Award* is given each year for long-term service to ARML. Al Kalfus was the founder and first president of both NYSML and ARML. He was devoted to his students and dedicated to excellence. Some measure of the man can be found in a preface that he wrote to the first book of ARML problems covering the 1976 to 1982 contests: "Meanwhile, enjoy the problems given in these pages. Remember, they were posed not to stump you, but to provoke your mathematical ingenuity, to guide you to clever tricks and new methods of attack, to open new facets of topics to the uninitiated, and to offer you the unique joy that only the solving of a truly challenging problem can bring." This award was first given in 1985.

The *Samuel L. Greitzer Coach Award* is given each year for outstanding service to a regional team. For many years Dr. Samuel Greitzer, a professor at Rutgers University, coached the United States IMO team. He also served as the editor for Arbelos, a small journal designed for high school students that was full of challenging mathematics problems and investigations. The award was first given in 1987.

The *Harry and Ruth Ruderman Award* is awarded at ARML to the winning team of the ARML Power Contest. Harry Ruderman influenced and inspired generations of students and teachers. He was a prolific writer, problem poser, and lecturer. For over a decade, he served as a contributor and reviewer of ARML questions as well as being the head judge. It was first awarded in 1996.

Part I

ARML Contests

2009 Contest

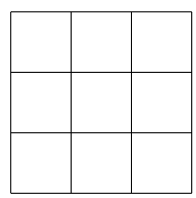
Contents

2009 Team Problems

T-1. Let N be a six-digit number formed by an arrangement of the digits $1, 2, 3, 3, 4, 5$. Compute the smallest value of N that is divisible by 264.

T-2. In triangle ABC, $AB = 4$, $BC = 6$, and $AC = 8$. Squares $ABQR$ and $BCST$ are drawn external to and lie in the same plane as $\triangle ABC$. Compute QT.

T-3. The numbers $1, 2, \ldots, 8$ are placed in the 3×3 grid below, leaving exactly one blank square. Such a placement is called *okay* if in every pair of adjacent squares, either one square is blank or the difference between the two numbers is at most 2 (two squares are considered adjacent if they share a common side). If reflections, rotations, etc. of placements are considered distinct, compute the number of distinct okay placements.



T-4. An ellipse in the first quadrant is tangent to both the x-axis and y-axis. One focus is at $(3, 7)$, and the other focus is at $(d, 7)$. Compute d.

T-5. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let $\mathbf{u}$ be the vector from A_1 to A_2 and let $\mathbf{v}$ be the vector from A_1 to A_8. The vector from A_1 to A_4 can be written as $a\mathbf{u} + b\mathbf{v}$ for a unique ordered pair of real numbers (a, b). Compute (a, b).

T-6. Compute the integer n such that $2009 < n < 3009$ and the sum of the odd positive divisors of n is 1024.

T-7. Points A, R, M, and L are consecutively the midpoints of the sides of a square whose area is 650. The coordinates of point A are $(11, 5)$. If points R, M, and L are all lattice points, and R is in Quadrant I, compute the number of possible ordered pairs (x, y) of coordinates for point R.

T-8. The *taxicab distance* between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

The region $\mathcal{R}$ is obtained by taking the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$ and removing every point whose taxicab distance to any vertex of the cube is less than $\frac{3}{5}$. Compute the volume of $\mathcal{R}$.

T-9. Let a and b be real numbers such that

$$a^3 - 15a^2 + 20a - 50 = 0 \quad \text{and} \quad 8b^3 - 60b^2 - 290b + 2575 = 0.$$

Compute $a + b$.

T-10. For a positive integer n, define $s(n)$ to be the sum of n and its digits. For example, $s(2009) = 2009 + 2 + 0 + 0 + 9 = 2020$. Compute the number of elements in the set $\{s(0), s(1), s(2), \ldots, s(9999)\}$.

2009 Team Answers

T-1. 135432

T-2. $2\sqrt{10}$

T-3. 32

T-4. $\dfrac{49}{3}$

T-5. $\left(2+\sqrt{2}, 1+\sqrt{2}\right)$

T-6. 2604

T-7. 10

T-8. $\dfrac{179}{250}$

T-9. $\dfrac{15}{2}$

T-10. 9046

2009 Team Solutions

T-1. Note that $264 = 3 \cdot 8 \cdot 11$, so we will need to address all these factors. Because the sum of the digits is 18, it follows that 3 divides N, regardless of how we order the digits of N. In order for 8 to divide N, we need N to end in $\underline{O}12, \underline{O}52, \underline{E}32$, or $\underline{E}24$, where O and E denote odd and even digits. Now write $N = \underline{U}\,\underline{V}\,\underline{W}\,\underline{X}\,\underline{Y}\,\underline{Z}$. Note that N is divisible by 11 if and only if $(U + W + Y) - (V + X + Z)$ is divisible by 11. Because the sum of the three largest digits is only 12, we must have $U + W + Y = V + X + Z = 9$.

Because Z must be even, this implies that V, X, Z are $2, 3, 4$ (in some order). This means $Y \neq 2$, and so we must have $Z \neq 4 \Rightarrow Z = 2$. Of the three remaining possibilities, $\underline{E}32$ gives the smallest solution, **135432**.

T-2. Set $\mathrm{m}\angle ABC = x$ and $\mathrm{m}\angle TBQ = y$. Then $x + y = 180^\circ$ and so $\cos x + \cos y = 0$. Applying the Law of Cosines to triangles ABC and TBQ gives $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos x$ and $QT^2 = BT^2 + BQ^2 - 2BT \cdot BQ \cos y$, which, after substituting values, become $8^2 = 4^2 + 6^2 - 48 \cos x$ and $QT^2 = 4^2 + 6^2 - 48 \cos y$.

Adding the last two equations yields $QT^2 + 8^2 = 2(4^2 + 6^2)$ or $QT = \mathbf{2\sqrt{10}}$.

Remark: This problem is closely related to the fact that in a parallelogram, the sum of the squares of the lengths of its diagonals is the equal to the sum of the squares of the lengths of its sides.

T-3. We say that two numbers are *neighbors* if they occupy adjacent squares, and that a is a *friend* of b if $0 < |a - b| \leq 2$. Using this vocabulary, the problem's condition is that every pair of neighbors must be friends of each other. Each of the numbers 1 and 8 has two friends, and each number has at most four friends.

If there is no number written in the center square, then we must have one of the cycles in the figures below. For each cycle, there are 8 rotations. Thus there are 16 possible configurations with no number written in the center square.

2	1	3
4	—	5
6	8	7

3	1	2
5	—	4
7	8	6

Now assume that the center square contains the number n. Because n has at least three neighbors, $n \neq 1$ and $n \neq 8$. First we show that 1 must be in a corner. If 1 is a neighbor of n, then one of the corners neighboring 1 must be empty, because 1 has only two friends (2 and 3). If c is in the other corner neighboring 1, then $\{n, c\} = \{2, 3\}$. But then n must have three more friends (n_1, n_2, n_3) other than 1 and c, for a total of five friends, which is impossible, as illustrated below. Therefore 1 must be in a corner.

—	1	c
n_1	n	n_2
	n_3	

Now we show that 1 can only have one neighbor, i.e., one of the squares adjacent to 1 is empty. If 1 has two neighbors, then we have, up to a reflection and a rotation, the configuration shown below. Because 2 has only one more friend, the corner next to 2 is empty and $n = 4$. Consequently, $m_1 = 5$ (refer to the figure below). Then 4 has one friend (namely 6) left with two neighbors m_2 and m_3, which is impossible. Thus 1 must have exactly one neighbor. An analogous argument shows that 8 must also be at a corner with exactly one neighbor.

1	2	—
3	n	m_3
m_1	m_2	

Therefore, 8 and 1 must be in non-opposite corners, with the blank square between them. Thus, up to reflections and rotations, the only possible configuration is the one shown at left below.

1	—	8
m		

1	—	8
2/3	4/5	6/7
3/2	5/4	7/6

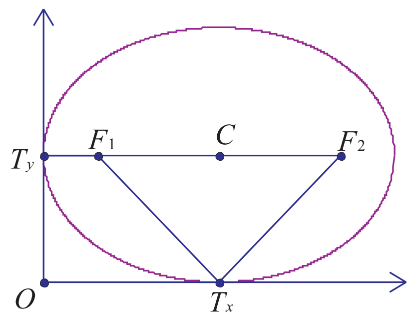
There are two possible values for m, namely 2 and 3. For each of the cases $m = 2$ and $m = 3$, the rest of the configuration is uniquely determined, as illustrated in the figure above right. We summarize our process: there are four corner positions for 1; two (non-opposite) corner positions for 8 (after 1 is placed); and two choices for the number in the square neighboring 1 but not neighboring 8. This leads to $4 \cdot 2 \cdot 2 = 16$ distinct configurations with a number written in the center square.

Therefore, there are 16 configurations in which the center square is blank and 16 configurations with a number in the center square, for a total of **32** distinct configurations.

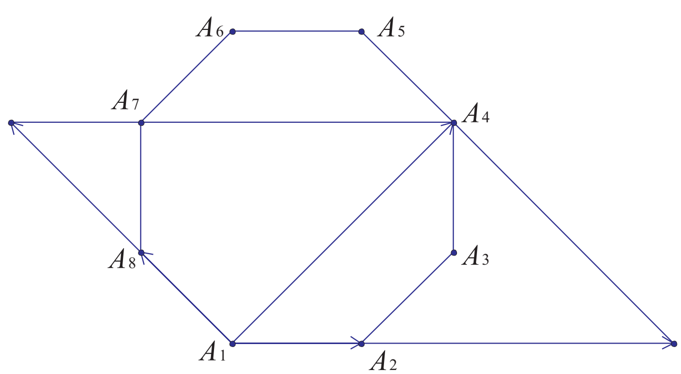
T-4. See the diagram below. The center of the ellipse is $C = \left(\frac{d+3}{2}, 7\right)$. The major axis of the ellipse is the line $y = 7$, and the minor axis is the line $x = \frac{d+3}{2}$. The ellipse is tangent to the coordinate axes at $T_x = \left(\frac{d+3}{2}, 0\right)$ and $T_y = (0, 7)$. Let $F_1 = (3, 7)$ and $F_2 = (d, 7)$. Using the locus definition of an ellipse, we have $F_1T_x + F_2T_x = F_1T_y + F_2T_y$; that is,

$$2\sqrt{\left(\frac{d-3}{2}\right)^2 + 7^2} = d + 3 \quad \text{or} \quad \sqrt{(d-3)^2 + 14^2} = d + 3.$$

Squaring both sides of the last equation gives $d^2 - 6d + 205 = d^2 + 6d + 9$ or $196 = 12d$, so $d = \mathbf{\frac{49}{3}}$.



T-5. We can scale the octagon so that $A_1A_2 = \sqrt{2}$. Because the exterior angle of the octagon is $45°$, we can place the octagon in the coordinate plane with A_1 being the origin, $A_2 = (\sqrt{2}, 0)$, and $A_8 = (1, 1)$.



Then $A_3 = (1 + \sqrt{2}, 1)$ and $A_4 = (1 + \sqrt{2}, 1 + \sqrt{2})$. It follows that $\mathbf{u} = \left\langle \sqrt{2}, 0 \right\rangle$, $\mathbf{v} = \langle -1, 1 \rangle$, and

$$\overrightarrow{A_1A_4} = \left\langle 1 + \sqrt{2}, 1 + \sqrt{2} \right\rangle = a\left\langle \sqrt{2}, 0 \right\rangle + b\left\langle -1, 1 \right\rangle = \left\langle a\sqrt{2} - b, b \right\rangle.$$

Thus $b = \sqrt{2} + 1$ and $a\sqrt{2} - b = \sqrt{2} + 1$, or $a = 2 + \sqrt{2}$, so $(a, b) = \mathbf{(2 + \sqrt{2}, \sqrt{2} + 1)}$.

Alternate Solution: Extend $\overline{A_1A_2}$ and $\overline{A_5A_4}$ to meet at point Q; let P be the intersection of $\overline{A_1Q}$ and $\overleftrightarrow{A_6A_3}$. Then $A_1A_2 = \|\mathbf{u}\|$, $A_2P = \|\mathbf{u}\|\sqrt{2}$, and $PQ = \|\mathbf{u}\|$, so $A_1Q = \left(2 + \sqrt{2}\right)\|\mathbf{u}\|$.

Because A_1QA_4 is a $45°{-}45°{-}90°$ right triangle, $A_4Q = \dfrac{A_1Q}{\sqrt{2}} = \left(\sqrt{2}+1\right)\|\mathbf{u}\|$. Thus $\overrightarrow{A_1A_4} = \overrightarrow{A_1Q} + \overrightarrow{QA_4}$, and because $\|\mathbf{u}\| = \|\mathbf{v}\|$, we have $(a,b) = \mathbf{(2+\sqrt{2}, \sqrt{2}+1)}$.

T-6. Suppose that $n = 2^k p_1^{a_1} \cdots p_r^{a_r}$, where the p_i are distinct odd primes, k is a nonnegative integer, and $a_1, \ldots, a_r$ are positive integers. Then the sum of the odd positive divisors of n is equal to

$$\prod_{i=1}^{r}(1+p_i+\cdots+p_i^{a_i}) = \prod_{i=1}^{r}\frac{p_i^{a_i+1}-1}{p_i-1} = 1024 = 2^{10}.$$

Note that $1+p_i+\cdots+p_i^{a_i}$ is the sum of a_i+1 odd numbers. Because the product of those sums is a power of two, each sum must be even (in fact, a power of 2). Thus, each a_i must be odd.

Because $1+11+11^2+11^3 > 1024$, if $p_i \geq 11$, then $a_i = 1$ and $1+p_i$ must be a power of 2 that is no greater than 1024. The possible values of p_i, with $p_i \geq 11$, are 31 and 127 (as 5 divides 255, 7 divides 511, and 3 divides 1023).

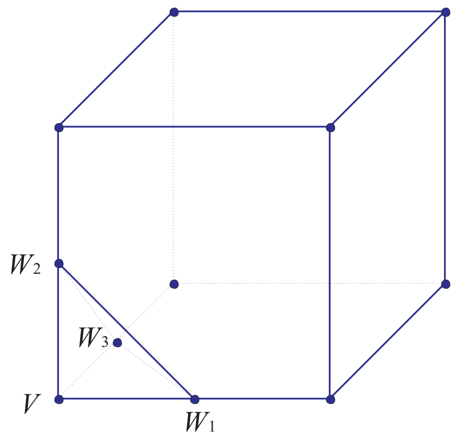
If $p_1 < 11$, then p_i can be 3, 5, 7. It is routine to check that $a_i = 1$ and $p_i = 3$ or 7.

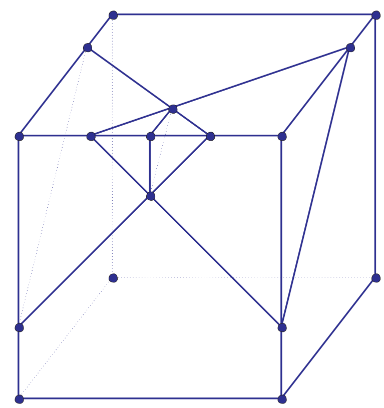
Thus $a_i = 1$ for all i, and the possible values of p_i are $3, 7, 31, 127$. The only combinations of these primes that yield 1024 are $(1+3)\cdot(1+7)\cdot(1+31)$ (with $n = 2^k \cdot 3 \cdot 7 \cdot 31 = 651 \cdot 2^k$) and $(1+7)\cdot(1+127)$ (with $n = 7 \cdot 127 = 889 \cdot 2^k$). Thus $n = 651 \cdot 2^2 = \mathbf{2604}$ is the unique value of n satisfying the conditions of the problem.

T-7. Write $x = 11 + c$ and $y = 5 + d$. Then $AR^2 = c^2 + d^2 = \frac{1}{2} \cdot 650 = 325$. Note that $325 = 18^2 + 1^2 = 17^2 + 6^2 = 15^2 + 10^2$. Temporarily restricting ourselves to the case where c and d are both positive, there are three classes of solutions: $\{c,d\} = \{18,1\}$, $\{c,d\} = \{17,6\}$, or $\{c,d\} = \{15,10\}$. In fact, c and d can be negative, so long as those values do not cause x or y to be negative. So there are **10** solutions:

(c,d)	(x,y)
$(18,1)$	$(29,6)$
$(18,-1)$	$(29,4)$
$(1,18)$	$(12,23)$
$(-1,18)$	$(10,23)$
$(17,6)$	$(28,11)$
$(6,17)$	$(17,22)$
$(-6,17)$	$(5,22)$
$(15,10)$	$(26,15)$
$(10,15)$	$(21,20)$
$(-10,15)$	$(1,20)$

T-8. For a fixed vertex V on the cube, the locus of points on or inside the cube that are at most $\frac{3}{5}$ away from V form a *corner* at V (that is, the right pyramid $VW_1W_2W_3$ in the figure shown at left below, with equilateral triangular base $W_1W_2W_3$ and three isosceles right triangular lateral faces $VW_1W_2, VW_2W_3, VW_3W_1$). Thus $\mathcal{R}$ is formed by removing eight such congruent corners from the cube. However, each two neighboring corners share a common region along their shared edge. This common region is the union of two smaller right pyramids, each similar to the original corners. (See the figure shown at right below.)





We compute the volume of $\mathcal{R}$ as

$$1 - 8 \cdot \frac{1}{6}\left(\frac{3}{5}\right)^3 + 12 \cdot 2 \cdot \frac{1}{6}\left(\frac{1}{10}\right)^3 = \mathbf{\frac{179}{250}}.$$

T-9. Each cubic expression can be depressed—that is, the quadratic term can be eliminated—by substituting as follows. Because $(a-p)^3 = a^3 - 3a^2p + 3ap^2 - p^3$, setting $p = -\frac{(-15)}{3} = 5$ and substituting $c + p = a$ transforms the expression $a^3 - 15a^2 + 20a - 50$ into the equivalent expression $(c+5)^3 - 15(c+5)^2 + 20(c+5) - 50$, which simplifies to $c^3 - 55c - 200$. Similarly, the substitution $d = b - \frac{5}{2}$ yields the equation $d^3 - 55d = -200$. [This procedure, which is analogous to completing the square, is an essential step in the algebraic solution to the general cubic equation.]

Consider the function $f(x) = x^3 - 55x$. It has three zeros, namely, 0 and $\pm\sqrt{55}$. Therefore, it has a relative maximum and a relative minimum in the interval $[-\sqrt{55}, \sqrt{55}]$. Note that for $0 \le x \le 5.5$, $|f(x)| < |x^3| < 5.5^3 = 166.375$, and for $5.5 < x \le \sqrt{55} < 8$, we have

$$|f(x)| = |x^3 - 55x| < x|x^2 - 55| < 8(55 - 5.5^2) = 198.$$

Because $f(x)$ is an odd function of x (its graph is symmetric about the origin), we conclude that for $-\sqrt{55} \le x \le \sqrt{55}$, $|f(x)| < 198$. Therefore, for constant m with $|m| > 198$, there is a unique real number x_0 such that $f(x_0) = m$.

In particular, since $200 > 198$, the values of c and d are uniquely determined. Because $f(x)$ is odd, we conclude that $c = -d$, or $a + b = \mathbf{\frac{15}{2}}$.

Alternate Solution: Set $a = x - b$ and substitute into the first equation. We get

$$\begin{aligned}(x-b)^3 - 15(x-b)^2 + 20(x-b) - 50 &= 0\\ -b^3 + b^2(3x-15) + b(-3x^2+30x-20) + (x^3-15x^2+20x-50) &= 0\\ 8b^3 + b^2(-24x+120) + b(24x^2-240x+160) - 8(x^3-15x^2+20x-50) &= 0.\end{aligned}$$

If we equate coefficients, we see that

$$\begin{aligned}-24x+120 &= -60\\ 24x^2-240x+160 &= -290\\ -8(x^3-15x^2+20x-50) &= 2575\end{aligned}$$

are all satisfied by $x = \frac{15}{2}$. This means that any real solution b to the second equation yields a real solution of $\frac{15}{2} - b$ to the first equation. We can follow the reasoning of the previous solution to establish the existence of exactly one real solution to the second cubic equation. Thus a and b are unique, and their sum is $(\frac{15}{2} - b) + b = \boldsymbol{\frac{15}{2}}$.

T-10. If $s(10x) = a$, then the values of s over $\{10x+0, 10x+1, \ldots, 10x+9\}$ are $a, a+2, a+4, \ldots, a+18$. Furthermore, if x is not a multiple of 10, then $s(10(x+1)) = a + 11$. This indicates that the values of s "interweave" somewhat from one group of 10 to the next: the sets alternate between even and odd. Because the s-values for starting blocks of ten differ by 11, consecutive blocks of the same parity differ by 22, so the values of s do not overlap. That is, s takes on 100 distinct values over any range of the form $\{100y+0, 100y+1, \ldots, 100y+99\}$.

First determine how many values are repeated between consecutive hundreds. Let y be an integer that is not a multiple of 10. Then the largest value for $s(100y+k)$ $(0 \le k \le 99)$ is $100y + (s(y) - y) + 99 + s(99) = 100y + s(y) - y + 117$, whereas the smallest value in the next group of 100 is for

$$\begin{aligned}s(100(y+1)) &= 100(y+1) + (s(y+1) - (y+1)) = 100y + (s(y)+2) - (y+1) + 100\\ &= 100y + s(y) - y + 101.\end{aligned}$$

This result implies that the values for $s(100y+91)$ through $s(100y+99)$ match the values of $s(100y+100)$ through $s(100y+108)$. So there are 9 repeated values.
Now determine how many values are repeated between consecutive thousands. Let z be a digit, and consider $s(1000z+999)$ versus $s(1000(z+1))$. The first value equals

$$1000z + (s(z) - z) + 999 + s(999) = 1000z + z + 1026 = 1001z + 1026.$$

The latter value equals $1000(z+1) + (s(z+1) - (z+1)) = 1001(z+1) = 1001z + 1001$. These values differ by an odd number. We have overlap between the $982, 983, \ldots, 989$ terms and the $000, 001, \ldots, 007$ terms. We also have overlap between the $992, 993, \ldots, 999$ terms and the $010, 011, \ldots, 017$ terms, for a total of 16 repeated values in all.

There are 90 instances in which we have 9 repeated terms, and 9 instances in which we have 16 repeated terms, so there are a total of $10000 - 90 \cdot 9 - 9 \cdot 16 = \mathbf{9046}$ unique values.

2009 Individual Problems

I-1. Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

I-2. Define a *reverse prime* to be a positive integer N such that when the digits of N are read in reverse order, the resulting number is a prime. For example, the numbers 5, 16, and 110 are all reverse primes. Compute the largest two-digit integer N such that the numbers $N, 4 \cdot N$, and $5 \cdot N$ are all reverse primes.

I-3. Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

I-4. Point P is on the hypotenuse $\overline{EN}$ of right triangle BEN such that $\overline{BP}$ bisects $\angle EBN$. Perpendiculars $\overline{PR}$ and $\overline{PS}$ are drawn to sides $\overline{BE}$ and $\overline{BN}$, respectively. If $EN = 221$ and $PR = 60$, compute $\frac{1}{BE} + \frac{1}{BN}$.

I-5. Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

I-6. Let k be the least common multiple of the numbers in the set $\mathcal{S} = \{1, 2, \ldots, 30\}$. Determine the number of positive integer divisors of k that are divisible by exactly 28 of the numbers in the set $\mathcal{S}$.

I-7. Let A and B be digits from the set $\{0, 1, 2, \ldots, 9\}$. Let r be the two-digit integer $\underline{A}\,\underline{B}$ and let s be the two-digit integer $\underline{B}\,\underline{A}$, so that r and s are members of the set $\{00, 01, \ldots, 99\}$. Compute the number of ordered pairs (A, B) such that $|r - s| = k^2$ for some integer k.

I-8. For $k \geq 3$, we define an ordered k-tuple of real numbers $(x_1, x_2, \ldots, x_k)$ to be *special* if, for every i such that $1 \leq i \leq k$, the product $x_1 \cdot x_2 \cdot \ldots \cdot x_k = x_i^2$. Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

I-9. A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$.

I-10. If $6 \tan^{-1} x + 4 \tan^{-1}(3x) = \pi$, compute x^2.

2009 Individual Answers

I-1. 28

I-2. 79

I-3. 7

I-4. $\dfrac{1}{60}$

I-5. $\sqrt{2}$

I-6. 23

I-7. 42

I-8. 12

I-9. 6

I-10. $\dfrac{15-8\sqrt{3}}{33}$

2009 Individual Solutions

I-1. Let x, $2x$, and $4x$ be the ages of the children p years ago. Then $x+2x+4x = p$, so $7x = p$. Since p is prime, $x = 1$. Thus the sum of the children's current ages is $(1+7)+(2+7)+(4+7) = \mathbf{28}$.

I-2. Because $N < 100$, $5 \cdot N < 500$. Since no primes end in 4, it follows that $5 \cdot N < 400$, hence $N \le 79$. The reverses of $5 \cdot 79 = 395$, $4 \cdot 79 = 316$, and 79 are 593, 613, and 97, respectively. All three of these numbers are prime, thus **79** is the largest two-digit integer N for which N, $4 \cdot N$, and $5 \cdot N$ are all reverse primes.

I-3. Let r and b be the number of students wearing red and blue jerseys, respectively. Then either we choose two blues and one red or one blue and two reds. Thus

$$\begin{aligned} \binom{b}{2}\binom{r}{1} + \binom{b}{1}\binom{r}{2} &= 25 \\ \Rightarrow \quad \frac{rb(b-1)}{2} + \frac{br(r-1)}{2} &= 25 \\ \Rightarrow \quad rb((r-1)+(b-1)) &= 50 \\ \Rightarrow \quad rb(r+b-2) &= 50. \end{aligned}$$

Now because r, b, and $r+b-2$ are positive integer divisors of 50, and $r, b \ge 2$, we have only a few possibilities to check. If $r = 2$, then $b^2 = 25$, so $b = 5$; the case $r = 5$ is symmetric. If $r = 10$, then $b(b+8) = 5$, which is impossible. If $r = 25$, then $b(b+23) = 2$, which is also impossible. So $\{r, b\} = \{2, 5\}$, and $r + b = \mathbf{7}$.

I-4. We observe that $\frac{1}{BE} + \frac{1}{BN} = \frac{BE+BN}{BE \cdot BN}$. The product in the denominator suggests that we compare areas. Let $[BEN]$ denote the area of $\triangle BEN$. Then $[BEN] = \frac{1}{2} BE \cdot BN$, but because $PR = PS = 60$, we can also write $[BEN] = [BEP] + [BNP] = \frac{1}{2} \cdot 60 \cdot BE + \frac{1}{2} \cdot 60 \cdot BN$. Therefore $BE \cdot BN = 60(BE + BN)$, so $\frac{1}{BE} + \frac{1}{BN} = \frac{BE+BN}{BE \cdot BN} = \mathbf{\frac{1}{60}}$. Note that this value does not depend on the length of the hypotenuse $\overline{EN}$; for a given location of point P, $\frac{1}{BE} + \frac{1}{BN}$ is invariant.

Alternate Solution: Using similar triangles, we have $\frac{ER}{PR} = \frac{PS}{SN} = \frac{BE}{BN}$, so $\frac{BE-60}{60} = \frac{60}{BN-60} = \frac{BE}{BN}$ and $BE^2 + BN^2 = 221^2$. Using algebra, we find that $BE = 204$, $BN = 85$, and $\frac{1}{204} + \frac{1}{85} = \mathbf{\frac{1}{60}}$.

I-5. If $y = \log_a(\log_a x)$, then $a^{a^y} = x$. Let $y = \log_2(\log_2 x) = \log_4(\log_4 x)$. Then $2^{2^y} = 4^{4^y} = (2^2)^{(2^2)^y} = 2^{2^{2y+1}}$, so $2y + 1 = y$, $y = -1$, and $x = \mathbf{\sqrt{2}}$. (This problem is based on one submitted by ARML alum James Albrecht, 1986–2007.)

Alternate Solution: Raise 4 (or 2^2) to the power of both sides to get $(\log_2 x)^2 = \log_4 x$. By the change of base formula, $\dfrac{(\log x)^2}{(\log 2)^2} = \dfrac{\log x}{2\log 2}$, so $\log x = \dfrac{\log 2}{2}$, thus $x = 2^{1/2} = \sqrt{\mathbf{2}}$.

Alternate Solution: Let $x = 4^a$. The equation then becomes $\log_2(2a) = \log_4 a$. Raising 4 to the power of each side, we get $4a^2 = a$. Since $a \neq 0$, we get $4a = 1$, thus $a = \frac{1}{4}$ and $x = \sqrt{\mathbf{2}}$.

I-6. We know that $k = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$. It is not difficult to see that the set $\mathcal{T}_1 = \left\{\frac{k}{2}, \frac{k}{3}, \frac{k}{5}, \frac{k}{17}, \frac{k}{19}, \frac{k}{23}, \frac{k}{29}\right\}$ comprises all divisors of k that are divisible by exactly 29 of the numbers in the set $\mathcal{S}$. Let $\mathcal{P} = \{2, 3, 5, 17, 19, 23, 29\}$. Then

$$\mathcal{T}_2 = \left\{\frac{k}{p_1 p_2}, \text{ where } p_1 \text{ and } p_2 \text{ are distinct elements of } \mathcal{P}\right\}$$

consists of divisors of k that are divisible by exactly 28 of the numbers in the set $\mathcal{S}$. There are $\binom{7}{2} = 21$ elements in $\mathcal{T}_2$.

Furthermore, note that $\frac{k}{7}$ is only divisible by 26 of the numbers in $\mathcal{S}$ (since it is not divisible by 7, 14, 21, or 28) while $\frac{k}{11}$ and $\frac{k}{13}$ are each divisible by 28 of the numbers in $\mathcal{S}$. We can also rule out $\frac{k}{4}$ (27 divisors: all but 8, 16, and 24), $\frac{k}{9}$ (27 divisors), $\frac{k}{25}$ (24 divisors), and all other numbers, thus the answer is $21 + 2 = \mathbf{23}$.

I-7. Because $|(10A+B) - (10B+A)| = 9|A - B| = k^2$, it follows that $|A - B|$ is a perfect square.
$|A - B| = 0$ yields 10 pairs of integers: $(A, B) = (0, 0), (1, 1), \ldots, (9, 9)$.
$|A - B| = 1$ yields 18 pairs: the nine $(A, B) = (0, 1), (1, 2), \ldots, (8, 9)$, and their reverses.
$|A - B| = 4$ yields 12 pairs: the six $(A, B) = (0, 4), (1, 5), \ldots, (5, 9)$, and their reverses.
$|A - B| = 9$ yields 2 pairs: $(A, B) = (0, 9)$ and its reverse.
Thus the total number of possible ordered pairs (A, B) is $10 + 18 + 12 + 2 = \mathbf{42}$.

I-8. The given conditions imply k equations. By taking the product of these k equations, we have $(x_1 x_2 \ldots x_k)^{k-1} = x_1 x_2 \ldots x_k$. Thus it follows that either $x_1 x_2 \ldots x_k = 0$ or $x_1 x_2 \ldots x_k = \pm 1$. If $x_1 x_2 \ldots x_k = 0$, then some $x_j = 0$, and by plugging this into each of the equations, it follows that all of the x_i's are equal to 0. Note that we cannot have $x_1 x_2 \ldots x_k = -1$, because the left hand side equals $x_1(x_2 \ldots x_k) = x_1^2$, which can't be negative, because the x_i's are all given as real. Thus $x_1 x_2 \ldots x_k = 1$, and it follows that each x_i is equal to either 1 or -1. Because the product of the x_i's is 1, there must be an even number of -1's. Furthermore, by picking any even number of the x_i's to be -1, it can be readily verified that the ordered k-tuple $(x_1, x_2, \ldots, x_k)$ is special. Thus there are

$$\binom{k}{0} + \binom{k}{2} + \binom{k}{4} + \ldots + \binom{k}{2\lfloor k/2 \rfloor}$$

special non-zero k-tuples. By considering the binomial expansion of $(1 + 1)^k + (1 - 1)^k$, it is clear that the above sum of binomial coefficients equals 2^{k-1}. Thus there are a total of

$2^{k-1}+1$ special k-tuples. Because $2^{10}=1024$ and $2^{11}=2048$, the inequality $2^{k-1}+1 \geq 2009$ is first satisfied when $k = \mathbf{12}$.

Alternate Solution: Use a recursive approach. Let S_k denote the number of special non-zero k-tuples. From the analysis in the above solution, each x_i must be either 1 or -1. It can easily be verified that $S_3 = 4$. For $k > 3$, suppose that $x_k = 1$ for a given special k-tuple. Then the k equations that follow are precisely the equation $x_1x_2\ldots x_{k-1} = 1$ and the $k-1$ equations that follow for the special $(k-1)$-tuple $(x_1, x_2, \ldots, x_{k-1})$. Because $x_1x_2\ldots x_{k-1} = 1$ is consistent for a special $(k-1)$-tuple, and because this equation imposes no further restrictions, we conclude that there are S_{k-1} special k-tuples in which $x_k = 1$.

If, on the other hand, $x_k = -1$ for a given special k-tuple, then consider the k equations that result, and make the substitution $x_1 = -y_1$. Then the k resulting equations are precisely the same as the k equations obtained in the case where $x_k = 1$, except that x_1 is replaced by y_1. Thus $(x_1, x_2, \ldots, x_{k-1}, -1)$ is special if and only if $(y_1, x_2, \ldots, x_{k-1})$ is special, and thus there are S_{k-1} special k-tuples in which $x_k = -1$.

Thus the recursion becomes $S_k = 2S_{k-1}$, and because $S_3 = 4$, it follows that $S_k = 2^{k-1}$.

I-9. Since $\pi r^2 h = 1$, we have $h = \frac{1}{\pi r^2}$ and $\pi r^2 = \frac{1}{h}$. Consequently,

$$2\pi rh + 2\pi r^2 = 12 \Rightarrow (2\pi r)\left(\frac{1}{\pi r^2}\right) + 2\left(\frac{1}{h}\right) = 12 \Rightarrow \frac{2}{r} + \frac{2}{h} = 12 \Rightarrow \frac{1}{r} + \frac{1}{h} = \mathbf{6}.$$

Alternate Solution: The total surface area is $2\pi rh + 2\pi r^2 = 12$ and the volume is $\pi r^2 h = 1$. Dividing, we obtain $\dfrac{12}{1} = \dfrac{2\pi rh + 2\pi r^2}{\pi r^2 h} = \dfrac{2}{r} + \dfrac{2}{h}$, thus $\dfrac{1}{r} + \dfrac{1}{h} = \dfrac{12}{2} = \mathbf{6}$.

I-10. Let $z = 1 + xi$ and $w = 1 + 3xi$, where $i = \sqrt{-1}$. Then $\tan^{-1} x = \arg z$ and $\tan^{-1}(3x) = \arg w$, where $\arg z$ gives the measure of the angle in standard position whose terminal side passes through z. By DeMoivre's theorem, $6\tan^{-1} x = \arg(z^6)$ and $4\tan^{-1}(3x) = \arg(w^6)$. Therefore the equation $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$ is equivalent to $z^6 \cdot w^4 = a$, where a is a real number (and, in fact, $a < 0$). To simplify somewhat, we can take the square root of both sides, and get $z^3 \cdot w^2 = 0 + bi$, where b is a real number. Then $(1 + xi)^3(1 + 3xi)^2 = 0 + bi$. Expanding each binomial and collecting real and imaginary terms in each factor yields $((1 - 3x^2) + (3x - x^3)i)((1 - 9x^2) + 6xi) = 0 + bi$. In order that the real part of the product be 0, we have $(1 - 3x^2)(1 - 9x^2) - (3x - x^3)(6x) = 0$. This equation simplifies to $1 - 30x^2 + 33x^4 = 0$, yielding $x^2 = \frac{15 \pm 8\sqrt{3}}{33}$. Notice that $\frac{15 \pm 8\sqrt{3}}{33} \approx 1$, which would mean that $x \approx 1$, and so $\tan^{-1}(x) \approx \frac{\pi}{4}$, which is too large, since $6 \cdot \frac{\pi}{4} > \pi$. (It can be verified that this value for x yields a value of 3π for the left side of the equation.) Therefore we are left with $x^2 = \dfrac{\mathbf{15 - 8\sqrt{3}}}{\mathbf{33}}$. To verify that this answer is reasonable, consider that $\sqrt{3} \approx 1.73$, so that $15 - 8\sqrt{3} \approx 1.16$, and so $x^2 \approx \frac{7}{200} = 0.035$. Then x itself is a little less than 0.2, and so

$\tan^{-1} x \approx \frac{\pi}{15}$. Similarly, $3x$ is about 0.6, so $\tan^{-1}(3x)$ is about $\frac{\pi}{6}$. $6 \cdot \frac{\pi}{15} + 4 \cdot \frac{\pi}{6}$ is reasonably close to π.

Alternate Solution: Recall that $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$, thus $\tan(2a) = \dfrac{2\tan a}{1 - \tan^2 a}$ and

$$\tan(3a) = \tan(2a + a) = \frac{\frac{2\tan a}{1-\tan^2 a} + \tan a}{1 - \frac{2\tan a}{1-\tan^2 a} \cdot \tan a} = \frac{2\tan a + \tan a - \tan^3 a}{1 - \tan^2 a - 2\tan^2 a} = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}.$$

Back to the problem at hand, divide both sides by 2 to obtain $3\tan^{-1} x + 2\tan^{-1}(3x) = \frac{\pi}{2}$. Taking the tangent of the left side yields $\dfrac{\tan(3\tan^{-1} x) + \tan(2\tan^{-1}(3x))}{1 - \tan(3\tan^{-1} x)\tan(2\tan^{-1}(3x))}$. We know that the denominator must be 0 since $\tan\frac{\pi}{2}$ is undefined, thus $1 = \tan(3\tan^{-1} x)\tan(2\tan^{-1}(3x)) = \dfrac{3x - x^3}{1 - 3x^2} \cdot \dfrac{2 \cdot 3x}{1 - (3x)^2}$ and hence $(1 - 3x^2)(1 - 9x^2) = (3x - x^3)(6x)$. Simplifying yields $33x^4 - 30x^2 + 1 = 0$, and applying the quadratic formula gives $x^2 = \dfrac{15 \pm 8\sqrt{3}}{33}$. The "+" solution is extraneous: as noted in the previous solution, $x = \dfrac{15 + 8\sqrt{3}}{33}$ yields a value of 3π for the left side of the equation), so we are left with $x^2 = \boldsymbol{\dfrac{15 - 8\sqrt{3}}{33}}$.

Power Question 2009: Sign on the Label

An ***n*-label** is a permutation of the numbers 1 through n. For example, $J = 35214$ is a 5-label and $K = 132$ is a 3-label. For a fixed positive integer p, where $p \leq n$, consider consecutive blocks of p numbers in an n-label. For example, when $p = 3$ and $L = 263415$, the blocks are $263, 634, 341$, and 415. We can associate to each of these blocks a p-label that corresponds to the relative order of the numbers in that block. For $L = 263415$, we get the following:

$$\underline{263}415 \to 132; \quad 2\underline{634}15 \to 312; \quad 26\underline{341}5 \to 231; \quad 263\underline{415} \to 213.$$

Moving from left to right in the n-label, there are $n - p + 1$ such blocks, which means we obtain an $(n - p + 1)$-tuple of p-labels. For $L = 263415$, we get the 4-tuple $(132, 312, 231, 213)$. We will call this $(n - p + 1)$-tuple the ***p*-signature** of L (or **signature**, if p is clear from the context) and denote it by $S_p[L]$; the p-labels in the signature are called **windows**. For $L = 263415$, the windows are $132, 312, 231$, and 213, and we write

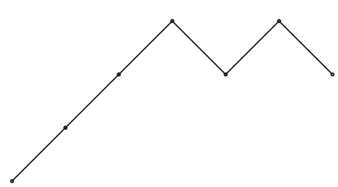
$$S_3[263415] = (132, 312, 231, 213).$$

More generally, we will call *any* $(n - p + 1)$-tuple of p-labels a p-signature, even if we do not know of an n-label to which it corresponds (and even if no such label exists). A signature that occurs for exactly one n-label is called **unique**, and a signature that doesn't occur for any n-labels is called **impossible**. A **possible** signature is one that occurs for at least one n-label.
In this power question, you will be asked to analyze some of the properties of labels and signatures.

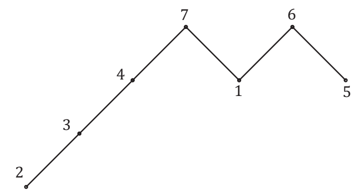
The Problems

1. (a) Compute the 3-signature for 52341. [1]
   (b) Find another 5-label with the same 3-signature as in part (a). [2]
   (c) Compute two other 6-labels with the same 4-signature as 462135. [2]
2. (a) Explain why the label 1234 has a unique 3-signature. [1]
   (b) List three other 4-labels with unique 3-signatures. [1]
   (c) Explain why the 3-signature $(123, 321)$ is impossible. [1]
   (d) List three other impossible 3-signatures that have exactly two windows. [1]

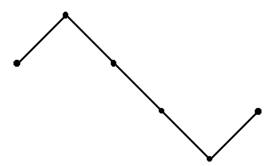
We can associate a **shape** to a given 2-signature: a diagram of up and down steps that indicates the relative order of adjacent numbers. For example, the following shape corresponds to the 2-signature $(12, 12, 12, 21, 12, 21)$:



A 7-label with this 2-signature corresponds to placing the numbers 1 through 7 at the nodes above so that numbers increase with each up step and decrease with each down step. The 7-label 2347165 is shown below:



3. Consider the shape below:



   (a) Find the 2-signature that corresponds to this shape. [2]

   (b) Compute two different 6-labels with the 2-signature you found in part (a). [3]

4. (a) List all 5-labels with 2-signature $(12, 12, 21, 21)$. [2]

   (b) Find a formula for the number of $(2n+1)$-labels with the 2-signature [3]

$$(\underbrace{12, 12, \ldots, 12}_{n}, \underbrace{21, 21, \ldots, 21}_{n}).$$

5. (a) Compute the number of 5-labels with 2-signature $(12, 21, 12, 21)$. [2]

   (b) Determine the number of 9-labels with 2-signature

$$(12, 21, 12, 21, 12, 21, 12, 21).$$

   Justify your answer. [3]

6. (a) Determine whether the following signatures are possible or impossible:

      (i) $(123, 132, 213)$, [1]

      (ii) $(321, 312, 213)$. [1]

   (b) Notice that a $(p+1)$-label has only two windows in its p-signature. For a given window ω_1, compute the number of windows ω_2 such that $S_p[L] = (\omega_1, \omega_2)$ for some $(p+1)$-label L. [1]

   (c) Justify your answer from part (b). [2]

7. (a) For a general n, determine the number of distinct possible p-signatures. [4]

   (b) If a randomly chosen p-signature is 575 times more likely of being impossible than possible, determine p and n. [1]

8. (a) Show that $(312, 231, 312, 132)$ is not a unique 3-signature. [1]

   (b) Show that $(231, 213, 123, 132)$ is a unique 3-signature. [2]

   (c) Find two 5-labels with unique 2-signatures. [1]

   (d) Find a 6-label with a unique 4-signature but which has the 3-signature from part (a). [2]

9. (a) For a general $n \geq 2$, compute all n-labels that have unique 2-signatures. [1]

   (b) Determine whether or not $S_5[495138627]$ is unique. [1]

   (c) Determine the smallest p for which the 20-label

   $$L = 3, 11, 8, 4, 17, 7, 15, 19, 6, 2, 14, 1, 10, 16, 5, 12, 20, 9, 13, 18$$

   has a unique p-signature. [3]

10. Show that for each $k \geq 2$, the number of unique 2^{k-1}-signatures on the set of 2^k-labels is at least 2^{2^k-3}. [5]

Solutions to 2009 Power Question

1. (a) $(312, 123, 231)$

   (b) There are three: $41352, 42351, 51342$.

   (c) There are four: $352146, 362145, 452136, 562134$.

2. (a) We will prove this by contradiction. Suppose for some other 4-label L we have $S_3[L] = S_3[1234] = (123, 123)$. Write out L as a_1, a_2, a_3, a_4. From the first window of $S_3[L]$, we have $a_1 < a_2 < a_3$. From the second window, we have $a_2 < a_3 < a_4$. Connecting these inequalities gives $a_1 < a_2 < a_3 < a_4$, which forces $L = 1234$, a contradiction. Therefore, the 3-signature above is unique.

   (b) There are 11 others (12 in all, if we include $S_3[1234]$):

$S_3[1234] = (123, 123)$	$S_3[1243] = (123, 132)$	$S_3[1324] = (132, 213)$
$S_3[1423] = (132, 213)$	$S_3[2134] = (213, 123)$	$S_3[2314] = (231, 213)$
$S_3[3241] = (213, 231)$	$S_3[3421] = (231, 321)$	$S_3[4132] = (312, 132)$
$S_3[4231] = (312, 231)$	$S_3[4312] = (321, 312)$	$S_3[4321] = (321, 321)$

   (c) If $S_3[a_1, a_2, a_3, a_4] = (123, 321)$, then the first window forces $a_2 < a_3$, whereas the second window forces $a_2 > a_3$. This is impossible, so the 3-signature $(123, 321)$ is impossible.

   (d) There are 18 impossible 3-signatures with two windows. In nine of these, the first window indicates that $a_2 < a_3$ (an increase), but the second window indicates that $a_2 > a_3$ (a decrease). In the other nine, the end of the first window indicates a decrease (that is, $a_2 > a_3$), but the beginning of the second window indicates an increase ($a_2 < a_3$). In general, for a 3-signature to be possible, the end of the first window and beginning of the second window must be consistent, indicating either an increase or a decrease. The impossible 3-signatures are:

$(123, 321)$	$(123, 312)$	$(123, 213)$	$(132, 231)$	$(132, 132)$	$(132, 123)$
$(213, 321)$	$(213, 312)$	$(213, 213)$	$(231, 231)$	$(231, 132)$	$(231, 123)$
$(312, 321)$	$(312, 312)$	$(312, 213)$	$(321, 231)$	$(321, 132)$	$(321, 123)$

3. (a) The first pair indicates an increase; the next three are decreases, and the last pair is an increase. So the 2-signature is $(12, 21, 21, 21, 12)$.

   (b) There are several:

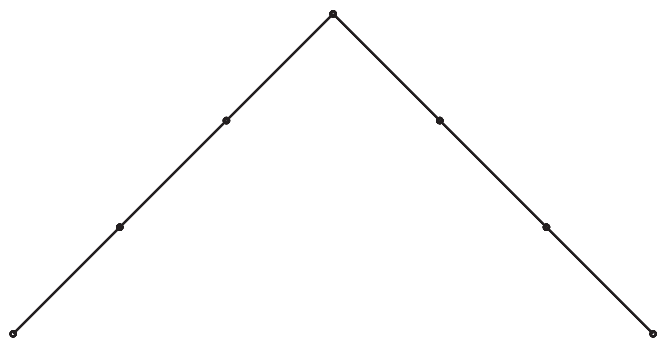
564312	564213	563214	465312	465213	463215
365412	365214	364215	265413	265314	264315
			165413	165314	164315
		453216	354216	254316	154326

4. In part (a), we can count by brute force, or use the formula from part (b) (with independent proof).

   (a) For the case of 5-labels, brute force counting is tractable.

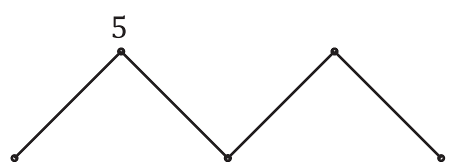
   $$\boxed{12543, 13542, 14532, 23541, 24531, 34521}.$$

   (b) The answer is $\boxed{\binom{2n}{n}}$. The shape of this signature is a wedge: n up steps followed by n down steps. The wedge for $n = 3$ is illustrated below:

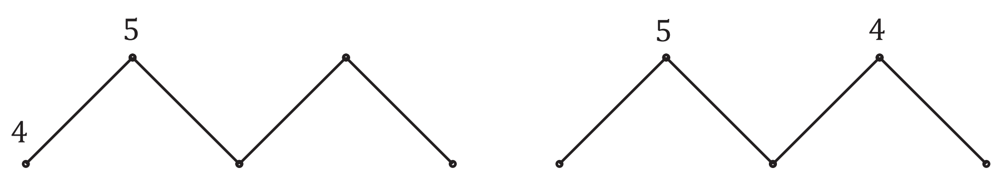


   The largest number in the label, $2n + 1$, must be placed at the peak in the center. If we choose the numbers to put in the first n spaces, then they must be placed in increasing order. Likewise, the remaining n numbers must be placed in decreasing order on the downward sloping piece of the shape. Thus there are exactly $\binom{2n}{n}$ such labels.

5. (a) The answer is $\boxed{16}$. We have a shape with two peaks and a valley in the middle. The 5 must go on one of the two peaks, so we place it on the first peak. By the shape's symmetry, we will double our answer at the end to account for the 5-labels where the 5 is on the other peak.

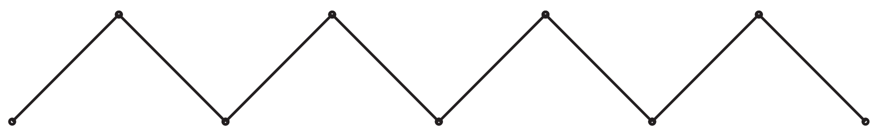


   The 4 can go to the left of the 5 or at the other peak. In the first case, shown below left, the 3 must go at the other peak and the 1 and 2 can go in either order. In the latter case, shown below right, the $1, 2$, and 3 can go in any of 3! arrangements.

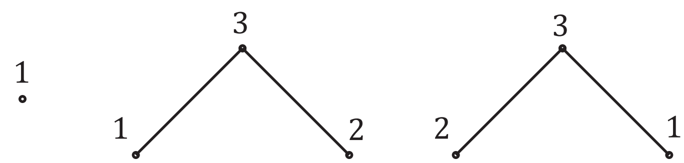


So there are $2! + 3! = 8$ possibilities. In all, there are 16 5-labels (including the ones where the 5 is at the other peak).

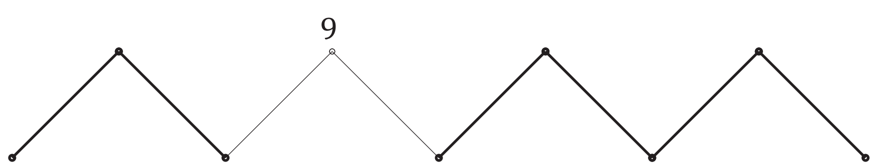
(b) The answer is $\boxed{7936}$. The shape of this 2-signature has four peaks and three intermediate valleys:



We will solve this problem by building up from smaller examples. Let f_n equal the number of $(2n+1)$-labels whose 2-signature consists of n peaks and $n-1$ intermediate valleys. In part (b) we showed that $f_2 = 16$. In the case where we have one peak, $f_1 = 2$. For the trivial case (no peaks), we get $f_0 = 1$. These cases are shown below.



Suppose we know the peak on which the largest number, $2n+1$, is placed. Then that splits our picture into two shapes with fewer peaks. Once we choose which numbers from $1, 2, \ldots, 2n$ to place each shape, we can compute the number of arrangements of the numbers on each shape, and then take the product. For example, if we place the 9 at the second peak, as shown below, we get a 1-peak shape on the left and a 2-peak shape on the right.



For the above shape, there are $\binom{8}{3}$ ways to pick the three numbers to place on the left-hand side, $f_1 = 2$ ways to place them, and $f_2 = 16$ ways to place the remaining five numbers on the right.

This argument works for any $n > 1$, so we have shown the following:

$$f_n = \sum_{k=1}^{n} \binom{2n}{2k-1} f_{k-1} f_{n-k}.$$

So we have:

$$f_1 = \binom{2}{1} f_0^2 = 2$$
$$f_2 = \binom{4}{1} f_0 f_1 + \binom{4}{3} f_1 f_0 = 16$$
$$f_3 = \binom{6}{1} f_0 f_2 + \binom{6}{3} f_1^2 + \binom{6}{5} f_2 f_0 = 272$$
$$f_4 = \binom{8}{1} f_0 f_3 + \binom{8}{3} f_1 f_2 + \binom{8}{5} f_2 f_1 + \binom{8}{7} f_3 f_0 = \boxed{7936}.$$

6. (a) Signature (i) is possible, because it is the 3-signature of 12435.
Signature (ii) is impossible. Let a 5-label be a_1, a_2, a_3, a_4, a_5. The second window of (ii) implies $a_3 < a_4$, whereas the third window implies $a_3 > a_4$, a contradiction.

(b) There are $\boxed{p}$ such windows, regardless of ω_1.

(c) Because the windows ω_1 and ω_2 overlap for $p-1$ numbers in L, the first $p-1$ integers in ω_2 must be in the same relative order as the last $p-1$ integers in ω_1. Therefore, by choosing the last integer in ω_2 (there are p choices), the placement of the remaining $p-1$ integers is determined. We only need to show that there exists some L such that $S_p[L] = (\omega_1, \omega_2)$.

To do so, set $\omega_1 = w_1, w_2, \ldots, w_p$. Provisionally, let $L^{(k)} = w_1, w_2, \ldots, w_p, k + 0.5$ for $k = 0, 1, \ldots, p$. For example, if $p = 4$ and $\omega_1 = 3124$, then $L^{(2)} = 3, 1, 2, 4, 2.5$. First we show that $L^{(k)}$ has the required p-signature; then we can renumber the entries in $L^{(k)}$ in consecutive order to make them all integers, so for example $3, 1, 2, 4, 2.5$ would become $4, 1, 2, 5, 3$.

Even though the last entry in $L^{(k)}$ is not an integer, we can still compute $S_p[L^{(k)}]$, since all we require is that the entries are all distinct. In each $S_p[L^{(k)}]$, the first window is ω_1. When we compare $L^{(k)}$ to $L^{(k+1)}$, the last integer in ω_2 can increase by at most 1, because the last integer in the label jumps over at most one integer in positions 2 through p (those jumps are in boldface):

$L^{(0)} = 3, 1, 2, 4, 0.5$	$S_4[L^{(0)}] = (3124, 234\underline{1})$	
$L^{(1)} = 3, \mathbf{1}, 2, 4, 1.5$	$S_4[L^{(1)}] = (3124, 134\underline{2})$	[jumps over the 1]
$L^{(2)} = 3, 1, \mathbf{2}, 4, 2.5$	$S_4[L^{(2)}] = (3124, 124\underline{3})$	[jumps over the 2]
$L^{(3)} = 3, 1, 2, 4, 3.5$	$S_4[L^{(3)}] = (3124, 124\underline{3})$	[jumps over nothing]
$L^{(4)} = 3, 1, 2, \mathbf{4}, 4.5$	$S_4[L^{(4)}] = (3124, 123\underline{4})$	[jumps over the 4]

The final entry of ω_2 in $S_p[L^{(0)}]$ is 1, because 0.5 is smaller than all other entries in $L^{(0)}$. Likewise, the final entry of ω_2 in $S_p[L^{(p)}]$ is p. Since the final entry increases in increments of 0 or 1 (as underlined above), we must see all p possibilities for ω_2.

By replacing the numbers in each $L^{(k)}$ with the integers 1 through $p+1$ (in the same relative order as the numbers in $L^{(k)}$), we have found the $(p+1)$-labels that yield all p possibilities.

7. (a) The answer is $\boxed{p! \cdot p^{n-p}}$.
Call two consecutive windows in a p-signature **compatible** if the last $p-1$ numbers in the first label and the first $p-1$ numbers in the second label (their "overlap") describe the same ordering. For example, in the p-signature $(\ldots, 2143, 2431, \ldots)$, 2143 and 2431 are compatible. Notice that the last three digits of 2143 and the first three digits of 2431 can be described by the same 3-label, 132.

Theorem: A signature σ is possible if and only if every pair of consecutive windows is compatible.

Proof: $(\Rightarrow)$ Consider a signature σ describing a p-label L. If some pair in σ is not compatible, then there is some string of $p-1$ numbers in our label L that has two different $(p-1)$-signatures. This is impossible, since the p-signature is well-defined.

$(\Leftarrow)$ Now suppose σ is a p-signature such that that every pair of consecutive windows is compatible. We need to show that there is at least one label L with $S_p[L] = \sigma$. We do so by induction on the number of windows in σ, using the results from 5(b).

Let $\sigma = \{\omega_1, \omega_2, \ldots, \omega_{k+1}\}$, and suppose $\omega_1 = a_1, a_2, \ldots, a_p$. Set $L_1 = \omega_1$.

Suppose that L_k is a $(p+k-1)$-label such that $S_p[L_k] = \{\omega_1, \ldots, \omega_k\}$. We will construct L_{k+1} for which $S_p[L_{k+1}] = \{\omega_1, \ldots, \omega_{k+1}\}$.

As in 5(b), denote by $L_k^{(j)}$ the label L_k with a $j+0.5$ appended; we will eventually renumber the elements in the label to make them all integers. Appending $j+0.5$ does not affect any of the non-terminal windows of $S_p[L_k]$, and as j varies from 0 to $p-k+1$ the final window of $S_p[L_k^{(j)}]$ varies over each of the p windows compatible with ω_k. Since ω_{k+1} is compatible with ω_k, there exists some j for which $S_p[L_k^{(j)}] = \{\omega_1, \ldots, \omega_{k+1}\}$. Now we renumber as follows: set $L_{k+1} = S_{k+p}[L_k^{(j)}]$, which replaces $L_k^{(j)}$ with the integers 1 through $k+p$ and preserves the relative order of all integers in the label.

By continuing this process, we conclude that the n-label L_{n-p+1} has p-signature σ, so σ is possible. $\square$

To count the number of possible p-signatures, we choose the first window ($p!$ choices), then choose each of the remaining $n-p$ compatible windows (p choices each). In all, there are $p! \cdot p^{n-p}$ possible p-signatures.

(b) The answer is $\boxed{n = 7, p = 5}$.
Let P denote the probability that a randomly chosen p-signature is possible. We are

given that $1 - P = 575$, so $P = \frac{1}{576}$. We want to find p and n for which

$$\begin{aligned} \frac{p! \cdot p^{n-p}}{(p!)^{n-p+1}} &= \frac{1}{576} \\ \frac{p^{n-p}}{(p!)^{n-p}} &= \frac{1}{576} \\ ((p-1)!)^{n-p} &= 576. \end{aligned}$$

The only factorial that has 576 as an integer power is $4! = \sqrt{576}$. Thus $p = 5$ and $n - p = 2 \Rightarrow n = 7$.

8. (a) The p-signature is not unique because it equals both $S_3[625143]$ and $S_3[635142]$.

   (b) Let $L = a_1, a_2, a_3, a_4, a_5, a_6$. We have $a_4 < a_6 < a_5$ (from window #4), $a_3 < a_1 < a_2$ (from window #1), and $a_2 < a_4$ (from window #2). Linking these inequalities, we get

   $$a_3 < a_1 < a_2 < a_4 < a_6 < a_5 \quad \Rightarrow \quad L = 231465,$$

   so $S_3[L]$ is unique.

   (c) 12345 and 54321 are the only ones.

   (d) $L = 645132$ will work. First, note that $S_3[645132] = (312, 231, 312, 132)$. Next, we need to show that $S_4[645132] = \{4231, 3412, 4132\}$ is unique. So let $L' = a_1, a_2, a_3, a_4, a_5, a_6$ be a 6-label such that $S_4[L'] = (4231, 3412, 4132)$.

   We get $a_4 < a_6 < a_5$ (from window #3), $a_5 < a_2 < a_3$ (from window #2), and $a_3 < a_1$ (from window #1). Linking these inequalities, we get

   $$a_4 < a_6 < a_5 < a_2 < a_3 < a_1 \quad \Rightarrow \quad L' = 645132,$$

   so $L' = L$, which means $S_4[L]$ is unique.

9. (a) The n-labels with unique 2-signatures are $1, 2, \ldots, n$ and $n, n-1, \ldots, 1$, and their respective 2-signatures are $(12, 12, \ldots, 12)$ and $(21, 21, \ldots, 21)$.

   Proof: (Not required for credit.) Let $L = a_1, a_2, \ldots, a_n$. The first signature above implies that $a_1 < a_2 < \cdots < a_n$, which forces $a_1 = 1, a_2 = 2$, and so on. Likewise, the second signature forces $a_1 = n, a_2 = n - 1$, and so on.

   To show that all other n-labels fail to have unique 2-signatures, we will show that in any other n-label L', there are two numbers k and $k + 1$ that are not adjacent. By switching k and $k + 1$ we get a label L'' for which $S_2[L'] = S_2[L'']$, since the differences between k and its neighbors in L' were at least 2 (and likewise for $k + 1$).

   To show that such a k and $k + 1$ exist, we proceed by contradiction. Suppose that all such pairs are adjacent in L'. Then 1 and n must be at the ends of L' (or else some intermediate number k will fail to be adjacent to $k + 1$ or $k - 1$). But if $a_1 = 1$, then that forces $a_2 = 2$, $a_3 = 3, \ldots, a_n = n$. That is, we get the two labels we already covered. (We get the other label if $a_n = 1$.)

Therefore, none of the $n! - 2$ remaining labels has a unique 2-signature.

(b) $S_5[495138627]$ is unique.
Let $L = a_1, \ldots, a_9$ and suppose $S_5[L] = S_5[495138627] = (\omega_1, \ldots, \omega_5)$. Then we get the following inequalities:

$$
\begin{array}{llll}
a_4 < a_8 & [\text{from } \omega_4] & a_8 < a_5 & [\text{from } \omega_4] \\
a_5 < a_1 & [\text{from } \omega_1] & a_1 < a_3 & [\text{from } \omega_1] \\
a_3 < a_7 & [\text{from } \omega_3] & a_7 < a_9 & [\text{from } \omega_5] \\
a_9 < a_6 & [\text{from } \omega_5] & a_6 < a_2 & [\text{from } \omega_2]
\end{array}
$$

Combining, we get $a_4 < a_8 < a_5 < a_1 < a_3 < a_7 < a_9 < a_6 < a_2$, which forces $a_4 = 1, a_8 = 2, \ldots, a_2 = 9$. So the label L is forced and $S_5[495138627]$ is therefore unique.

(c) The answer is $\boxed{p = 16}$. To show this fact we will need to extend the idea from part 8(b) about "linking" inequalities forced by the various windows:

Theorem: A p-signature for an n-label L is unique if and only if for every $k < n$, k and $k+1$ are in at least one window together. That is, the **distance** between them in the n-label is less than p.

Proof: Suppose that for some k, the distance between k and $k+1$ is p or greater. Then the label L' obtained by swapping k and $k+1$ has the same p-signature, because there are no numbers between k and $k+1$ in any window and because the two numbers never appear in the same window.

If the distance between all such pairs is less than k, we need to show that $S_p[L]$ is unique. For $i = 1, 2, \ldots, n$, let r_i denote the position where i appears in L. For example, if $L = 4123$, then $r_1 = 2, r_2 = 3, r_3 = 4$, and $r_4 = 1$.

Let $L = a_1, a_2, \ldots, a_n$. Since 1 and 2 are in some window together, $a_{r_1} < a_{r_2}$. Similarly, for any k, since k and $k+1$ are in some window together, $a_{r_k} < a_{r_{k+1}}$. We then get a linked inequality $a_{r_1} < a_{r_2} < \cdots < a_{r_n}$, which can only be satisfied if $a_{r_1} = 1, a_{r_2} = 2, \ldots, a_{r_n} = n$. Therefore, $S_p[L]$ is unique. □

From the proof above, we know that the signature is unique if and only if every pair of consecutive integers coexists in at least one window. Therefore, we seek the largest distance between consecutive integers in L. That distance is 15 (from 8 to 9, and from 17 to 18). Thus the smallest p is $\boxed{16}$.

10. Let s_k denote the number of such unique signatures. We proceed by induction with base case $k = 2$. From 8(c), a 2-signature for a label L is unique if and only if consecutive numbers in L appear together in some window. Because $k = 2$, the consecutive numbers must be adjacent

in the label. The 2^2-labels 1234 and 4321 satisfy this condition,[1] so their 2^1-signatures are unique. Thus we have shown that $s_2 \geq 2 = 2^{2^2-3}$, and the base case is established.

Now suppose $s_k \geq 2^{2^k-3}$ for some $k \geq 2$. Let L_k be a 2^k-label with a unique 2^{k-1}-signature. Write $L_k = (a_1, a_2, \ldots, a_{2^k})$. We will *expand* L_k to form a 2^{k+1} label by replacing each a_i above with the numbers $2a_i - 1$ and $2a_i$ (in some order). This process produces a valid 2^{k+1}-label, because the numbers produced are all the integers from 1 to 2^{k+1}. Furthermore, different L_k's will produce different labels: if the starting labels differ at place i, then the new labels will differ at places $2i-1$ and $2i$. Therefore, each starting label produces 2^{2^k} distinct 2^{k+1}-labels through this process. Summarizing, each valid 2^k-label can be expanded to produce 2^{2^k} distinct 2^{k+1}-labels, none of which could be obtained by expanding any other 2^k-label.

It remains to be shown that the new label has a unique 2^{k-1}-signature. Because L_k has a unique 2^{k-1}-signature, for all $i \leq 2^k - 1$, both i and $i+1$ appeared in some 2^{k-1}-window. Therefore, there were fewer than $2^{k-1} - 1$ numbers between i and $i+1$. When the label is expanded, $2i$ and $2i-1$ are adjacent, $2i+1$ and $2i+2$ are adjacent, and $2i$ and $2i+1$ are fewer than $2 \cdot (2^{k-1} - 1) + 2 = 2^k$ places apart. Thus, every pair of adjacent integers is within some 2^k-window.

Since each pair of consecutive integers in our new 2^{k+1}-label coexists in some 2^k-window for every possible such expansion of L_k, that means all 2^{2^k} ways of expanding L_k to a 2^{k+1}-label result in labels with unique 2^k-signatures. We then get

$$\begin{aligned} s_{k+1} &\geq 2^{2^k} \cdot s_k \\ &\geq 2^{2^k} \cdot 2^{2^k-3} \\ &= 2^{2^{k+1}-3}, \end{aligned}$$

which completes the induction.

[1]The proof in 8(a) establishes that these are the only 2^2-labels with unique 2-signatures, thus giving $s_2 = 2$. That proof was not required for credit and is not needed here, since the inequality above is good enough for this problem.

2009 Relay Problems

R1-1. A rectangular box has dimensions $8 \times 10 \times 12$. Compute the fraction of the box's volume that is *not* within 1 unit of any of the box's faces.

R1-2. Let $T = TNYWR$. Compute the largest real solution x to $(\log x)^2 - \log \sqrt{x} = T$.

R1-3. Let $T = TNYWR$. Kay has $T + 1$ different colors of fingernail polish. Compute the number of ways that Kay can paint the five fingernails on her left hand by using at least three colors and such that no two consecutive fingernails have the same color.

R2-1. Compute the number of ordered pairs (x, y) of positive integers satisfying $x^2 - 8x + y^2 + 4y = 5$.

R2-2. Let $T = TNYWR$ and let $k = 21 + 2T$. Compute the largest integer n such that $2n^2 - kn + 77$ is a positive prime number.

R2-3. Let $T = TNYWR$. In triangle ABC, $BC = T$ and $m\angle B = 30°$. Compute the number of integer values of AC for which there are two possible values for side length AB.

2009 Relay Answers

R1-1. $\frac{1}{2}$

R1-2. 10

R1-3. 109890

R2-1. 4

R2-2. 12

R2-3. 5

2009 Relay Solutions

R1-1. Let the box be defined by the product of the intervals on the x, y, and z axes as $[0,8] \times [0,10] \times [0,12]$ with volume $8 \times 10 \times 12$. The set of points inside the box that are not within 1 unit of any face is defined by the product of the intervals $[1,7] \times [1,9] \times [1,11]$ with volume $6 \times 8 \times 10$. This volume is $\frac{6\times8\times10}{8\times10\times12} = \mathbf{\frac{1}{2}}$ of the whole box.

R1-2. Let $u = \log x$. Then the given equation can be rewritten as $u^2 - \frac{1}{2}u - T = 0 \rightarrow 2u^2 - u - 2T = 0$. This quadratic has solutions $u = \frac{1\pm\sqrt{1+16T}}{4}$. As we are looking for the largest real solution for x (and therefore, for u), we want $u = \frac{1+\sqrt{1+16T}}{4} = 1$ when $T = \frac{1}{2}$. Therefore, $x = 10^1 = \mathbf{10}$.

R1-3. There are $T + 1$ possible colors for the first nail. Each remaining nail may be any color except that of the preceding nail, that is, there are T possible colors. Thus, using at least two colors, there are $(T+1)T^4$ possible colorings. The problem requires that at least three colors be used, so we must subtract the number of colorings that use only two colors. As before, there are $T + 1$ possible colors for the first nail and T colors for the second. With only two colors, there are no remaining choices; the colors simply alternate. The answer is therefore $(T+1)T^4 - (T+1)T$, and with $T = 10$, this expression is equal to $110000 - 110 = \mathbf{109890}$.

R2-1. Completing the square twice in x and y, we obtain the equivalent equation $(x-4)^2 + (y+2)^2 = 25$, which describes a circle centered at $(4, -2)$ with radius 5. The lattice points on this circle are points 5 units up, down, left, or right of the center, or points 3 units away on one axis and 4 units away on the other. Because the center is below the x-axis, we know that y must increase by at least 2 units; x cannot decrease by 4 or more units if it is to remain positive. Thus, we have:

$$\begin{aligned}
(x,y) &= (4,-2) + (-3,4) = (1,2) \\
(x,y) &= (4,-2) + (0,5) = (4,3) \\
(x,y) &= (4,-2) + (3,4) = (7,2) \\
(x,y) &= (4,-2) + (4,3) = (8,1).
\end{aligned}$$

There are **4** such ordered pairs.

R2-2. If k is positive, there are only four possible factorizations of $2n^2 - kn + 77$ over the integers, namely

$$\begin{aligned}
(2n-77)(n-1) &= 2n^2 - 79n + 77 \\
(2n-1)(n-77) &= 2n^2 - 145n + 77 \\
(2n-11)(n-7) &= 2n^2 - 25n + 77 \\
(2n-7)(n-11) &= 2n^2 - 29n + 77.
\end{aligned}$$

Because $T = 4$, $k = 29$, and so the last factorization is the correct one. Because $2n - 7$ and $n - 11$ are both integers, in order for their product to be prime, one factor must equal 1 or -1, so $n = 3, 4, 10$, or 12. Checking these possibilities from the greatest downward, $n = 12$ produces $17 \cdot 1 = 17$, which is prime. So the answer is **12**.

R2-3. By the Law of Cosines, $(AC)^2 = T^2 + (AB)^2 - 2T(AB)\cos 30^\circ \to (AB)^2 - 2T\cos 30^\circ(AB) + (T^2 - (AC)^2) = 0$. This quadratic in AB has two positive solutions when the discriminant and product of the roots are both positive. Thus $(2T\cos 30^\circ)^2 - 4(T^2 - (AC)^2) > 0$, and $(T^2 - (AC)^2) > 0$. The second inequality implies that $AC < T$. The first inequality simplifies to $4(AC)^2 - T^2 > 0$, so $T/2 < AC$. Since $T = 12$, we have that $6 < AC < 12$, giving **5** integral values for AC.

2009 Tiebreaker Problems

TB-1. In $\triangle ABC$, D is on $\overline{AC}$ so that $\overline{BD}$ is the angle bisector of $\angle B$. Point E is on $\overline{AB}$ and $\overline{CE}$ intersects $\overline{BD}$ at P. Quadrilateral $BCDE$ is cyclic, $BP = 12$ and $PE = 4$. Compute the ratio $\frac{AC}{AE}$.

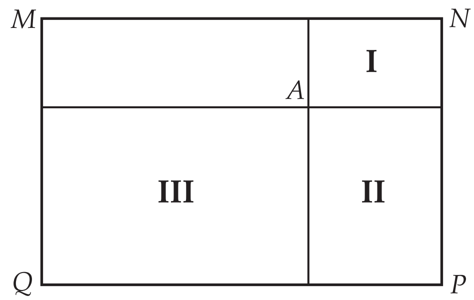
TB-2. Complete the following "cross-number puzzle", where each "Across" answer represents a four-digit number, and each "Down" answer represents a three-digit number. No answer begins with the digit 0.

Across:
1. $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is the cube of the sum of the digits in the answer to 1 Down.
5. From left to right, the digits in $\underline{E}\,\underline{F}\,\underline{G}\,\underline{H}$ are strictly decreasing.
6. From left to right, the digits in $\underline{I}\,\underline{J}\,\underline{K}\,\underline{L}$ are strictly decreasing.

Down:
1. $\underline{A}\,\underline{E}\,\underline{I}$ is a perfect fourth power.
2. $\underline{B}\,\underline{F}\,\underline{J}$ is a perfect square.
3. The digits in $\underline{C}\,\underline{G}\,\underline{K}$ form a geometric progression.
4. $\underline{D}\,\underline{H}\,\underline{L}$ has a two-digit prime factor.

1 A	2 B	3 C	4 D
5 E	F	G	H
6 I	J	K	L

TB-3. In rectangle $MNPQ$, point A lies on $\overline{QN}$. Segments parallel to the rectangle's sides are drawn through point A, dividing the rectangle into four regions. The areas of regions I, II, and III are *integers* in geometric progression. If the area of $MNPQ$ is 2009, compute the maximum possible area of region I.



2009 Tiebreaker Answers

TB-1. 3

TB-2.

[1] 2	[2] 1	[3] 9	[4] 7
[5] 5	4	3	1
[6] 6	4	1	0

TB-3. 1476

2009 Tiebreaker Solutions

TB-1. Let ω denote the circle that circumscribes quadrilateral $BCDE$. Draw in line segment $\overline{DE}$. Note that $\angle DPE$ and $\angle CPB$ are congruent, and $\angle DEC$ and $\angle DBC$ are congruent, since they cut off the same arc of ω. Therefore, $\triangle BCP$ and $\triangle EDP$ are similar. Thus $\frac{BC}{DE} = \frac{BP}{EP} = \frac{12}{4} = 3$.

Because $\angle BCE$ and $\angle BDE$ cut off the same arc of ω, these angles are congruent. Let α be the measure of these angles. Similarly, $\angle DCE$ and $\angle DBE$ cut off the same arc of ω. Let β be the measure of these angles. Since BD is an angle bisector, $\mathrm{m}\angle CBD = \beta$.

Note that $\mathrm{m}\angle ADE = 180° - \mathrm{m}\angle BDE - \mathrm{m}\angle BDC$. It follows that

$$\begin{aligned}
\mathrm{m}\angle ADE &= 180° - \mathrm{m}\angle BDE - (180° - \mathrm{m}\angle CBD - \mathrm{m}\angle BCD) \\
\Rightarrow \mathrm{m}\angle ADE &= 180° - \mathrm{m}\angle BDE - (180° - \mathrm{m}\angle CBD - \mathrm{m}\angle BCE - \mathrm{m}\angle DCE) \\
\Rightarrow \mathrm{m}\angle ADE &= 180° - \alpha - (180° - \beta - \alpha - \beta) \\
\Rightarrow \mathrm{m}\angle ADE &= 2\beta = \mathrm{m}\angle CBD.
\end{aligned}$$

Thus $\angle ADE$ is congruent to $\angle CBD$, and it follows that $\triangle ADE$ is similar to $\triangle ABC$. Hence $\frac{BC}{DE} = \frac{AC}{AE}$, and by substituting in given values, we have $\frac{AC}{AE} = \mathbf{3}$.

TB-2. From 1 Down, $\underline{\mathsf{A}}\,\underline{\mathsf{E}}\,\underline{\mathsf{I}} = \mathsf{256}$ or $\mathsf{625}$, either of which make $\underline{\mathsf{A}}\,\underline{\mathsf{B}}\,\underline{\mathsf{C}}\,\underline{\mathsf{D}} = \mathsf{2197}$, so $\underline{\mathsf{A}}\,\underline{\mathsf{E}}\,\underline{\mathsf{I}} = \mathsf{256}$.
From $\underline{\mathsf{B}} = \mathsf{1}$ together with 2 Down, $\underline{\mathsf{B}}\,\underline{\mathsf{F}}\,\underline{\mathsf{J}} = \mathsf{121}$ or $\mathsf{144}$. But $\underline{\mathsf{J}} = \mathsf{1}$ does not work because then 6 Across could not be satisfied. Therefore $\underline{\mathsf{B}}\,\underline{\mathsf{F}}\,\underline{\mathsf{J}} = \mathsf{144}$.
From $\underline{\mathsf{C}} = \mathsf{9}$ together with 5 Across and 3 Down, we have $\underline{\mathsf{C}}\,\underline{\mathsf{G}}\,\underline{\mathsf{K}} = \mathsf{931}$.
From $\underline{\mathsf{D}} = \mathsf{7}$ together with 5 and 6 Across, we get $\underline{\mathsf{D}}\,\underline{\mathsf{H}}\,\underline{\mathsf{L}} = \mathsf{720}$ or $\mathsf{710}$, but only 710 has a two-digit prime factor.

TB-3. Because A is on diagonal $\overline{NQ}$, rectangles $NXAB$ and $ACQY$ are similar. Thus $\frac{AB}{AX} = \frac{QY}{QC} = \frac{AC}{AY} \Rightarrow AB \cdot AY = AC \cdot AX$. Therefore, we have $2009 = [\mathrm{I}] + 2[\mathrm{II}] + [\mathrm{III}]$.

Let the common ratio of the geometric progression be $\frac{p}{q}$, where p and q are relatively prime positive integers (q may equal 1). Then $[\mathrm{I}]$ must be some integer multiple of q^2, which we will call aq^2. This gives $[\mathrm{II}] = apq$ and $[\mathrm{III}] = ap^2$. By factoring, we get

$$2009 = aq^2 + 2apq + ap^2 \Rightarrow 7^2 \cdot 41 = a(p+q)^2.$$

Thus we must have $p + q = 7$ and $a = 41$. Since $[\mathrm{I}] = aq^2$ and $p, q > 0$, the area is maximized when $\frac{p}{q} = \frac{1}{6}$, giving $[\mathrm{I}] = 41 \cdot 36 = \mathbf{1476}$. The areas of the other regions are $246, 246$, and 41.

2009 Super Relay Problems

1. Quadrilateral $ARML$ is a kite with $AR = RM = 5$, $AM = 8$, and $RL = 11$. Compute AL.

2. Let $T = TNYWR$. If $xy = \sqrt{5}$, $yz = 5$, and $xz = T$, compute the positive value of x.

3. Let $T = TNYWR$. In how many ways can T boys and $T + 1$ girls be arranged in a row if all the girls must be standing next to each other?

4. Let $T = TNYWR$. Let $T = TNYWR$. $\triangle ABC$ is on a coordinate plane such that $A = (3, 6)$, $B = (T, 0)$, and $C = (2T - 1, 1 - T)$. Let ℓ be the line containing the altitude to $\overline{BC}$. Compute the y-intercept of ℓ.

5. Let $T = TNYWR$. In triangle ABC, $AB = AC - 2 = T$, and $\mathrm{m}\angle A = 60°$. Compute BC^2.

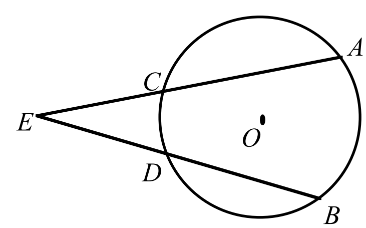
6. Let $T = TNYWR$. Let $\mathcal{S}_1$ denote the arithmetic sequence $0, \frac{1}{4}, \frac{1}{2}, \ldots$, and let $\mathcal{S}_2$ denote the arithmetic sequence $0, \frac{1}{6}, \frac{1}{3}, \ldots$. Compute the T^{th} smallest number that occurs in both sequences $\mathcal{S}_1$ and $\mathcal{S}_2$.

7. Let $T = TNYWR$. An integer n is randomly selected from the set $\{1, 2, 3, \ldots, 2T\}$. Compute the probability that the integer $|n^3 - 7n^2 + 13n - 6|$ is a prime number.

8. Let A be the number you will receive from position 7, and let B be the number you will receive from position 9. In $\frac{1}{A}$ minutes, 20 frogs can eat 1800 flies. At this rate, in $\frac{1}{B}$ minutes, how many flies will 15 frogs be able to eat?

9. Let $T = TNYWR$. If $|T| - 1 + 3i = \dfrac{1}{z}$, compute the sum of the real and imaginary parts of z.

10. Let $T = TNYWR$. In circle O, diagrammed at right, minor arc $\widehat{AB}$ measures $\frac{T}{4}$ degrees. If $\mathrm{m}\angle OAC = 10°$ and $\mathrm{m}\angle OBD = 5°$, compute the degree measure of $\angle AEB$. Just pass the number without the units.



11. Let $T = TNYWR$. Ann spends 80 seconds climbing up a T meter rope at a constant speed, and she spends 70 seconds climbing down the same rope at a constant speed (different from her upward speed). Ann begins climbing up and down the rope repeatedly, and she does not pause after climbing the length of the rope. After T minutes, how many meters will Ann have climbed in either direction?

12. Let $T = TNYWR$. Simplify $2^{\log_4 T}/2^{\log_{16} 64}$.

13. Let $T = TNYWR$. Let $P(x) = x^2 + Tx + 800$, and let r_1 and r_2 be the roots of $P(x)$. The polynomial $Q(x)$ is quadratic, it has leading coefficient 1, and it has roots $r_1 + 1$ and $r_2 + 1$. Find the sum of the coefficients of $Q(x)$.

14. Let $T = TNYWR$. Equilateral triangle ABC is given with side length T. Points D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Point F lies in space such that $\triangle DEF$ is equilateral and $\triangle DEF$ lies in a plane perpendicular to the plane containing $\triangle ABC$. Compute the volume of tetrahedron $ABCF$.

15. In triangle ABC, $AB = 5, AC = 6$, and $\tan \angle BAC = -\frac{4}{3}$. Compute the area of $\triangle ABC$.

2009 Super Relay Answers

1. $4\sqrt{5}$
2. 2
3. 36
4. 3
5. 19
6. 9
7. $\dfrac{1}{9}$
8. 3750
9. $\dfrac{1}{25}$
10. 5
11. 80
12. 10
13. 800
14. 108
15. 12

2009 Super Relay Solutions

1. Let K be the midpoint of $\overline{AM}$. Then $AK = KM = 8/2 = 4$, $RK = \sqrt{5^2 - 4^2} = 3$, and $KL = 11 - 3 = 8$. Thus $AL = \sqrt{AK^2 + KL^2} = \sqrt{4^2 + 8^2} = \mathbf{4\sqrt{5}}$.

2. Multiply the three given equations to obtain $x^2y^2z^2 = 5T\sqrt{5}$. Thus $xyz = \pm\sqrt[4]{125T^2}$, and the positive value of x is $x = xyz/yz = \sqrt[4]{125T^2}/5 = \sqrt[4]{T^2/5}$. With $T = 4\sqrt{5}$, we have $x = \mathbf{2}$.

3. First choose the position of the first girl, starting from the left. There are $T + 1$ possible positions, and then the positions for the girls are all determined. There are $(T + 1)!$ ways to arrange the girls, and there are $T!$ ways to arrange the boys, for a total of $(T+1)\cdot(T+1)!\cdot T! = ((T + 1)!)^2$ arrangements. With $T = 2$, the answer is **36**.

4. The slope of $\overleftrightarrow{BC}$ is $\dfrac{(1 - T) - 0}{(2T - 1) - T} = -1$, and since ℓ is perpendicular to $\overleftrightarrow{BC}$, the slope of ℓ is 1. Because ℓ passes through $A = (3, 6)$, the equation of ℓ is $y = x + 3$, and its y-intercept is **3** (independent of T).

5. By the Law of Cosines, $BC^2 = AB^2+AC^2-2\cdot AB\cdot AC\cdot\cos A = T^2+(T+2)^2-2\cdot T\cdot(T+2)\cdot\frac{1}{2} = T^2 + 2T + 4$. With $T = 3$, the answer is **19**.

6. $\mathcal{S}_1$ consists of all numbers of the form $\frac{n}{4}$, and $\mathcal{S}_2$ consists of all numbers of the form $\frac{n}{6}$, where n is a nonnegative integer. Since $\gcd(4, 6) = 2$, the numbers that are in both sequences are of the form $\frac{n}{2}$, and the T^{th} smallest such number is $\frac{T-1}{2}$. With $T = 19$, the answer is **9**.

7. Let $P(n) = n^3 - 7n^2 + 13n - 6$, and note that $P(n) = (n - 2)(n^2 - 5n + 3)$. Thus $|P(n)|$ is prime if either $|n - 2| = 1$ and $|n^2 - 5n + 3|$ is prime or if $|n^2 - 5n + 3| = 1$ and $|n - 2|$ is prime. Solving $|n - 2| = 1$ gives $n = 1$ or 3, and solving $|n^2 - 5n + 3| = 1$ gives $n = 1$ or 4 or $\frac{5\pm\sqrt{17}}{2}$. Note that $P(1) = 1, P(3) = -3$, and $P(4) = -2$. Thus $|P(n)|$ is prime only when n is 3 or 4, and if $T \geq 2$, then the desired probability is $\frac{2}{2T} = \frac{1}{T}$. With $T = 9$, the answer is $\mathbf{\frac{1}{9}}$.

8. In $\frac{1}{A}$ minutes, 1 frog can eat $1800/20 = 90$ flies; thus in $\frac{1}{B}$ minutes, 1 frog can eat $\frac{A}{B}\cdot 90$ flies. Thus in $\frac{1}{B}$ minutes, 15 frogs can eat $15 \cdot 90 \cdot \frac{A}{B}$ flies. With $A = \frac{1}{9}$ and $B = \frac{1}{25}$, this simplifies to $15 \cdot 250 = \mathbf{3750}$.

9. Let $t = |T|$. Note that $z = \dfrac{1}{t - 1 + 3i} = \dfrac{1}{t - 1 + 3i} \cdot \dfrac{t - 1 - 3i}{t - 1 - 3i} = \dfrac{t - 1 - 3i}{t^2 - 2t + 10}$. Thus the sum of the real and imaginary parts of z is $\dfrac{t - 1}{t^2 - 2t + 10} + \dfrac{-3}{t^2 - 2t + 10} = \dfrac{|T| - 4}{|T|^2 - 2|T| + 10}$. With $T = 5$, the answer is $\dfrac{1}{25}$.

10. Note that $\mathrm{m}\angle AEB = \frac{1}{2}(\mathrm{m}\overset{\frown}{AB} - m\,\overset{\frown}{CD}) = \frac{1}{2}(\mathrm{m}\overset{\frown}{AB} - \mathrm{m}\angle COD)$. Also note that $\mathrm{m}\angle COD = 360^\circ - (\mathrm{m}\angle AOC + \mathrm{m}\angle BOD + \mathrm{m}\angle AOB) = 360^\circ - (180^\circ - 2\mathrm{m}\angle OAC) - (180^\circ - 2\mathrm{m}\angle OBD) - \mathrm{m}\overset{\frown}{AB} = 2(\mathrm{m}\angle OAC + \mathrm{m}\angle OBD) - \mathrm{m}\overset{\frown}{AB}$. Thus $\mathrm{m}\angle AEB = \mathrm{m}\overset{\frown}{AB} - \mathrm{m}\angle OAC - \mathrm{m}\angle OBD = \frac{T}{4} - 10^\circ - 5^\circ$, and with $T = 80$, the answer is **5**.

11. In 150 seconds (or 2.5 minutes), Ann climbs up and down the entire rope. Thus in T minutes, she makes $\left\lfloor \frac{T}{2.5} \right\rfloor$ round trips, and therefore climbs $2T\left\lfloor \frac{T}{2.5} \right\rfloor$ meters. After making all her round trips, there are $t = 60\left(T - 2.5\left\lfloor \frac{T}{2.5} \right\rfloor\right)$ seconds remaining. If $t \le 80$, then the remaining distance climbed is $T \cdot \frac{t}{80}$ meters, and if $t > 80$, then the distance climbed is $T + T \cdot \left(\frac{t-80}{70}\right)$ meters. In general, the total distance in meters that Ann climbs is

$$2T\left\lfloor \frac{T}{2.5} \right\rfloor + T \cdot \min\left(1, \frac{60\left(T - 2.5\lfloor \frac{T}{2.5} \rfloor\right)}{80}\right) + T \cdot \max\left(0, \frac{60\left(T - 2.5\lfloor \frac{T}{2.5} \rfloor\right) - 80}{70}\right).$$

With $T = 10$, Ann makes exactly 4 round trips, and therefore climbs a total of $4 \cdot 2 \cdot 10 = \mathbf{80}$ meters.

12. Note that $2^{\log_4 T} = 4^{(\frac{1}{2}\log_4 T)} = 4^{\log_4 T^{\frac{1}{2}}} = \sqrt{T}$. Letting $\log_{16} 64 = x$, we see that $2^{4x} = 2^6$, thus $x = \frac{3}{2}$, and $2^x = \sqrt{8}$. Thus the given expression equals $\sqrt{\frac{T}{8}}$, and with $T = 800$, this is equal to **10**.

13. Let $Q(x) = x^2 + Ax + B$. Then $A = -(r_1 + 1 + r_2 + 1)$ and $B = (r_1 + 1)(r_2 + 1)$. Thus the sum of the coefficients of $Q(x)$ is $1 + (-r_1 - r_2 - 2) + (r_1r_2 + r_1 + r_2 + 1) = r_1r_2$. Note that $T = -(r_1 + r_2)$ and $800 = r_1r_2$, so the answer is **800** (independent of T). [Note: With $T = 108$, $\{r_1, r_2\} = \{-8, -100\}$.]

14. The volume of tetrahedron $ABCF$ is one-third the area of $\triangle ABC$ times the distance from F to $\triangle ABC$. Since D and E are midpoints, $DE = \frac{BC}{2} = \frac{T}{2}$, and the distance from F to $\triangle ABC$ is $\frac{T\sqrt{3}}{4}$. Thus the volume of $ABCF$ is $\frac{1}{3} \cdot \frac{T^2\sqrt{3}}{4} \cdot \frac{T\sqrt{3}}{4} = \frac{T^3}{16}$. With $T = 12$, the answer is **108**.

15. Let $s = \sin\angle BAC$. Then $s > 0$ and $\frac{s}{-\sqrt{1-s^2}} = -\frac{4}{3}$, which gives $s = \frac{4}{5}$. The area of triangle ABC is therefore $\frac{1}{2} \cdot AB \cdot AC \cdot \sin\angle BAC = \frac{1}{2} \cdot 5 \cdot 6 \cdot \frac{4}{5} = \mathbf{12}$.

2010 Contest

Contents

2010 Team Problems

T-1. Compute all ordered pairs of real numbers (x, y) that satisfy both of the equations:

$$x^2 + y^2 = 6y - 4x + 12 \qquad \text{and} \qquad 4y = x^2 + 4x + 12.$$

T-2. Define $\log^*(n)$ to be the smallest number of times the log function must be iteratively applied to n to get a result less than or equal to 1. For example, $\log^*(1000) = 2$ since $\log 1000 = 3$ and $\log(\log 1000) = \log 3 = 0.477\ldots \leq 1$. Let a be the smallest integer such that $\log^*(a) = 3$. Compute the number of zeros in the base 10 representation of a.

T-3. An integer N is worth 1 point for each pair of digits it contains that forms a prime in its original order. For example, 6733 is worth 3 points (for 67, 73, and 73 again), and 20304 is worth 2 points (for 23 and 03). Compute the smallest positive integer that is worth exactly 11 points. [Note: Leading zeros are not allowed in the original integer.]

T-4. The six sides of convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. Compute the number of colorings such that every triangle $A_iA_jA_k$ has at least one red side.

T-5. Compute the smallest positive integer n such that n^n has at least 1,000,000 positive divisors.

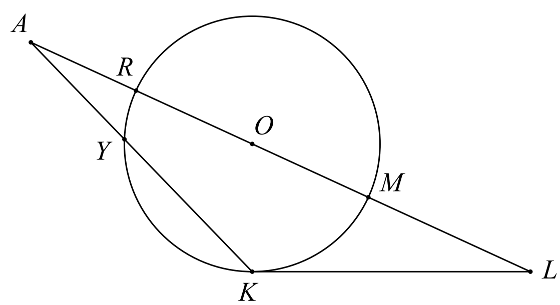
T-6. Given an arbitrary finite sequence of letters (represented as a word), a *subsequence* is a sequence of one or more letters that appear in the same order as in the original sequence. For example, N, CT, OTT, and $CONTEST$ are subsequences of the word $CONTEST$, but NOT, $ONSET$, and $TESS$ are not. Assuming the standard English alphabet $\{A, B, \ldots, Z\}$, compute the number of distinct four-letter "words" for which EE is a subsequence.

T-7. Six solid regular tetrahedra are placed on a flat surface so that their bases form a regular hexagon $\mathcal{H}$ with side length 1, and so that the vertices not lying in the plane of $\mathcal{H}$ (the "top" vertices) are themselves coplanar. A spherical ball of radius r is placed so that its center is directly above the center of the hexagon. The sphere rests on the tetrahedra so that it is tangent to one edge from each tetrahedron. If the ball's center is coplanar with the top vertices of the tetrahedra, compute r.

T-8. Derek starts at the point $(0, 0)$, facing the point $(0, 1)$, and he wants to get to the point $(1, 1)$. He takes unit steps parallel to the coordinate axes. A move consists of either a step forward, or a $90°$ right (clockwise) turn followed by a step forward, so that his path does not contain any left turns. His path is restricted to the square region defined by $0 \le x \le 17$ and $0 \le y \le 17$. Compute the number of ways he can get to $(1, 1)$ without returning to any previously visited point.

T-9. The equations $x^3 + Ax + 10 = 0$ and $x^3 + Bx^2 + 50 = 0$ have two roots in common. Compute the product of these common roots.

T-10. Points A and L lie outside circle ω, whose center is O, and $\overline{AL}$ contains diameter $\overline{RM}$, as shown below. Circle ω is tangent to $\overline{LK}$ at K. Also, $\overline{AK}$ intersects ω at Y, which is between A and K. If $KL = 3$, $ML = 2$, and $m\angle AKL - m\angle YMK = 90°$, compute $[AKM]$ (i.e., the area of $\triangle AKM$).



2010 Team Answers

T-1. $(-6, 6)$ and $(2, 6)$ [must have both answers, in either order]

T-2. 9

T-3. 100337

T-4. 392

T-5. 84

T-6. 3851

T-7. $\dfrac{\sqrt{2}}{3}$

T-8. 529

T-9. $5\sqrt[3]{4}$

T-10. $\dfrac{375}{182}$

2010 Team Solutions

T-1. Rearrange the terms in the first equation to yield $x^2 + 4x + 12 = 6y - y^2 + 24$, so that the two equations together yield $4y = 6y - y^2 + 24$, or $y^2 - 2y - 24 = 0$, from which $y = 6$ or $y = -4$. If $y = 6$, then $x^2 + 4x + 12 = 24$, from which $x = -6$ or $x = 2$. If $y = -4$, then $x^2 + 4x + 12 = -16$, which has no real solutions because $x^2 + 4x + 12 = (x + 2)^2 + 8 \geq 8$ for all real x. So there are two ordered pairs satisfying the system, namely $\mathbf{(-6, 6)}$ and $\mathbf{(2, 6)}$.

T-2. If $\log^*(a) = 3$, then $\log(\log(\log(a))) \leq 1$ and $\log(\log(a)) > 1$. If $\log(\log(a)) > 1$, then $\log(a) > 10$ and $a > 10^{10}$. Because the problem asks for the smallest such a that is an integer, choose $a = 10^{10} + 1 = 10{,}000{,}000{,}001$, which has **9** zeros.

T-3. If a number N has k base 10 digits, then its maximum point value is $(k-1)+(k-2)+\cdots+1 = \frac{1}{2}(k-1)(k)$. So if $k \leq 5$, the number N is worth at most 10 points. Therefore the desired number has at least six digits. If $100{,}000 < N < 101{,}000$, then N is of the form $100\underline{A}\,\underline{B}\,\underline{C}$, which could yield 12 possible primes, namely $1\underline{A}$, $1\underline{B}$, $1\underline{C}$, $0\underline{A}$ (twice), $0\underline{B}$ (twice), $0\underline{C}$ (twice), $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$. So search for N of the form $100\underline{A}\,\underline{B}\,\underline{C}$, starting with lowest values first. Notice that if any of A, B, or C is not a prime, at least two points are lost, and so all three numbers must be prime. Proceed by cases:

First consider the case $A = 2$. Then $1\underline{A}$ is composite, so all of $1\underline{B}$, $1\underline{C}$, $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$ must be prime. Considering now the values of $1\underline{B}$ and $1\underline{C}$, both B and C must be in the set $\{3, 7\}$. Because 27 is composite, $B = C = 3$, but then $\underline{B}\,\underline{C} = 33$ is composite. So A cannot equal 2.

If $A = 3$, then $B \neq 2$ because both 12 and 32 are composite. If $B = 3$, $1\underline{B}$ is prime but $\underline{A}\,\underline{B} = 33$ is composite, so all of C, $1\underline{C}$, and $3\underline{C}$ must be prime. These conditions are satisfied by $C = 7$ and no other value. So $A = B = 3$ and $C = 7$, yielding $N = \mathbf{100337}$.

T-4. Only two triangles have no sides that are sides of the original hexagon: $A_1A_3A_5$ and $A_2A_4A_6$. For each of these triangles, there are $2^3 - 1 = 7$ colorings in which at least one side is red, for a total of $7 \cdot 7 = 49$ colorings of those six diagonals. The colorings of the three central diagonals $\overline{A_1A_4}, \overline{A_2A_5}, \overline{A_3A_6}$ are irrelevant because the only triangles they can form include sides of the original hexagon, so they can be colored in $2^3 = 8$ ways, for a total of $8 \cdot 49 = \mathbf{392}$ colorings.

T-5. Let k denote the number of distinct prime divisors of n, so that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $a_i > 0$. Then if $d(x)$ denotes the number of positive divisors of x,

$$d\left(n^n\right) = \left(a_1 n + 1\right)\left(a_2 n + 1\right) \cdots \left(a_k n + 1\right) \geq (n+1)^k. \qquad (*)$$

Note that if $n \geq 99$ and $k \geq 3$, then $d(n^n) \geq 100^3 = 10^6$, so $102 = 2 \cdot 3 \cdot 17$ is an upper bound for the solution. Look for values less than 99, using two observations: (1) all $a_i \leq 6$

(because $p^7 > 99$ for all primes); and (2) $k \leq 3$ (because $2 \cdot 3 \cdot 5 \cdot 7 > 99$). These two facts rule out the cases $k = 1$ (because $(*)$ yields $d \leq (6n+1)^1 < 601$) and $k = 2$ (because $d(n^n) \leq (6n+1)^2 < 601^2$).

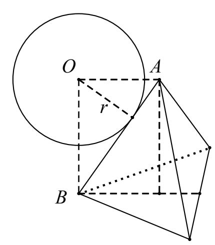
So $k = 3$. Note that if $a_1 = a_2 = a_3 = 1$, then from $(*)$, $d(n^n) = (n+1)^3 < 10^6$. So consider only $n < 99$ with exactly three prime divisors, and for which not all exponents are 1. The only candidates are 60, 84, and 90; of these, $n = 84$ is the smallest one that works:

$$\begin{aligned} d(60^{60}) &= d(2^{120} \cdot 3^{60} \cdot 5^{60}) = 121 \cdot 61 \cdot 61 < 125 \cdot 80 \cdot 80 = 800{,}000; \\ d(84^{84}) &= d(2^{168} \cdot 3^{84} \cdot 7^{84}) = 169 \cdot 85 \cdot 85 > 160 \cdot 80 \cdot 80 = 1{,}024{,}000. \end{aligned}$$

Therefore $n = \mathbf{84}$ is the least positive integer n such that $d(n^n) > 1{,}000{,}000$.

T-6. Divide into cases according to the number of E's in the word. If there are only two E's, then the word must have two non-E letters, represented by ?'s. There are $\binom{4}{2} = 6$ arrangements of two E's and two ?'s, and each of the ?'s can be any of 25 letters, so there are $6 \cdot 25^2 = 3750$ possible words. If there are three E's, then the word has exactly one non-E letter, and so there are 4 arrangements times 25 choices for the letter, or 100 possible words. There is one word with four E's, hence a total of **3851** words.

T-7. Let O be the center of the sphere, A be the top vertex of one tetrahedron, and B be the center of the hexagon.



Then BO equals the height of the tetrahedron, which is $\frac{\sqrt{6}}{3}$. Because A is directly above the centroid of the bottom face, AO is two-thirds the length of the median of one triangular face, so $AO = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{3}$. The radius of the sphere is the altitude to hypotenuse $\overline{AB}$ of $\triangle ABO$, so the area of $\triangle ABO$ can be represented in two ways: $[ABO] = \frac{1}{2}AO \cdot BO = \frac{1}{2}AB \cdot r$. Substitute given and computed values to obtain $\frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) = \frac{1}{2}(1)(r)$, from which $r = \frac{\sqrt{18}}{9} = \boldsymbol{\frac{\sqrt{2}}{3}}$.

T-8. Divide into cases according to the number of right turns Derek makes.

- There is one route involving only one turn: move first to $(0, 1)$ and then to $(1, 1)$.
- If he makes two turns, he could move up to $(0, a)$ then to $(1, a)$ and then down to $(1, 1)$. In order to do this, a must satisfy $1 < a \leq 17$, leading to 16 options.

– If Derek makes three turns, his path is entirely determined by the point at which he turns for the second time. If the coordinates of this second turn point are (a,b), then both a and b are between 2 and 17 inclusive, yielding $(17-1)^2$ possibilities.

– If Derek makes four turns, his last turn must be from facing in the $-x$-direction to the $+y$-direction. For this to be his last turn, it must occur at $(1,0)$. Then his next-to-last turn could be at any $(a,0)$, with $1 < a \leq 17$, depending on the location of his second turn as in the previous case. This adds another $(17-1)^2$ possibilities.

– It is impossible for Derek to make more than four turns and get to $(1,1)$ without crossing or overlapping his path.

Summing up the possibilities gives $1 + 16 + 16^2 + 16^2 = \mathbf{529}$ possibilities.

T-9. Let the roots of the first equation be p, q, r and the roots of the second equation be p, q, s. Then $pqr = -10$ and $pqs = -50$, so $\frac{s}{r} = 5$. Also $p + q + r = 0$ and $p + q + s = -B$, so $r - s = B$. Substituting yields $r - 5r = -4r = B$, so $r = -\frac{B}{4}$ and $s = -\frac{5B}{4}$. From the second given equation, $pq + ps + qs = pq + s(p+q) = 0$, so $pq - \frac{5B}{4}(p+q) = 0$, or $pq = \frac{5B}{4}(p+q)$. Because $p + q + r = 0$, $p + q = -r = \frac{B}{4}$, and so $pq = \frac{5B^2}{16}$. Because $pqr = -10$ and $r = -\frac{B}{4}$, conclude that $pq = \frac{40}{B}$. Thus $\frac{5B^2}{16} = \frac{40}{B}$, so $B^3 = 128$ and $B = 4\sqrt[3]{2}$. Then $pq = \frac{5B^2}{16}$ implies that $pq = \mathbf{5\sqrt[3]{4}}$ (and $r = -\sqrt[3]{2}$).

Alternate Solution: Let the common roots be p and q. Then the following polynomials (linear combinations of the originals) must also have p and q as common zeros:

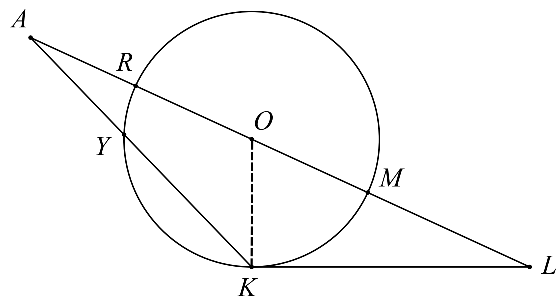
$$\begin{aligned}(x^3 + Bx^2 + 50) - (x^3 + Ax + 10) &= Bx^2 - Ax + 40 \\ -(x^3 + Bx^2 + 50) + 5(x^3 + Ax + 10) &= 4x^3 - Bx^2 + 5Ax.\end{aligned}$$

Because $pq \neq 0$, neither p nor q is zero, so the second polynomial has zeros p, q, and 0. Therefore p and q are zeros of $4x^2 - Bx + 5A$. [This result can also be obtained by using the Euclidean Algorithm on the original polynomials.]

Because the two quadratic equations have the same zeros, their coefficients are proportional: $\frac{4}{B} = \frac{5A}{40} \Rightarrow AB = 32$ and $\frac{4}{B} = \frac{-B}{-A} \Rightarrow 4A = B^2$. Hence $\frac{128}{B} = B^2$ and $B^3 = 128$, so $B = 4\sqrt[3]{2}$. Rewriting the first quadratic as $B\left(x^2 - \frac{A}{B}x + \frac{40}{B}\right)$ shows that the product $pq = \frac{40}{B} = \mathbf{5\sqrt[3]{4}}$.

Alternate Solution: Using the sum of roots formulas, notice that $pq+ps+qs = p+q+r = 0$. Therefore $0 = pq + ps + qs - (p+q+r)s = pq - rs$, and $pq = rs$. Hence $(pq)^3 = (pqr)(pqs) = (-10)(-50) = 500$, so $pq = \mathbf{5\sqrt[3]{4}}$.

T-10. Notice that $\overline{OK} \perp \overline{KL}$, and let r be the radius of ω.



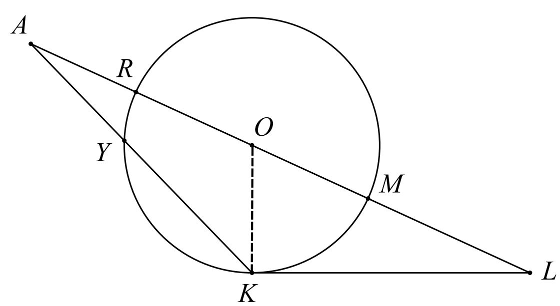
Then consider right triangle OKL. Because $ML = 2$, $OK = r$, and $OL = r + 2$, it follows that $r^2 + 3^2 = (r + 2)^2$, from which $r = \frac{5}{4}$.

Because $m\angle YKL = \frac{1}{2}m\widehat{YRK}$ and $m\angle YMK = \frac{1}{2}m\widehat{YK}$, it follows that $m\angle YKL + m\angle YMK = 180°$. By the given condition, $m\angle YKL - m\angle YMK = 90°$. It follows that $m\angle YMK = 45°$ and $m\angle YKL = 135°$. hence $m\widehat{YK} = 90°$. Thus,

$$\overline{YO} \perp \overline{OK} \quad \text{and} \quad \overline{YO} \parallel \overline{KL}. \tag{$*$}$$

From here there are several solutions:

First Solution: Compute $[AKM]$ as $\frac{1}{2}$base $\cdot$ height, using base $\overline{AM}$.



Because of $(*)$, $\triangle AYO \sim \triangle AKL$. To compute AM, notice that in $\triangle AYO$, $AO = AM - r$, while in $\triangle AKL$, the corresponding side $AL = AM + ML = AM + 2$. Therefore:

$$\begin{aligned} \frac{AO}{AL} &= \frac{YO}{KL} \\ \frac{AM - \frac{5}{4}}{AM + 2} &= \frac{5/4}{3}, \end{aligned}$$

from which $AM = \frac{25}{7}$. Draw the altitude of $\triangle AKM$ from vertex K, and let h be its length. In right triangle OKL, h is the altitude to the hypotenuse, so $\frac{h}{3} = \sin(\angle KLO) = \frac{r}{r+2}$. Hence $h = \frac{15}{13}$. Therefore $[AKM] = \frac{1}{2} \cdot \frac{25}{7} \cdot \frac{15}{13} = \mathbf{\frac{375}{182}}$.

Second Solution: By the Power of the Point Theorem, $LK^2 = LM \cdot LR$, so

$$\begin{aligned} LR &= \frac{9}{2}, \\ RM = LR - LM &= \frac{5}{2}, \\ OL = r + ML &= \frac{13}{4}. \quad (\dagger) \end{aligned}$$

From $(*)$, we know that $\triangle AYO \sim \triangle AKL$. Hence by $(\dagger)$,

$$\frac{AL}{AO} = \frac{AL}{AL - OL} = \frac{KL}{YO} = \frac{3}{5/4} = \frac{12}{5}, \quad \text{thus} \quad AL = \frac{12}{7} \cdot OL = \frac{12}{7} \cdot \frac{13}{4} = \frac{39}{7}.$$

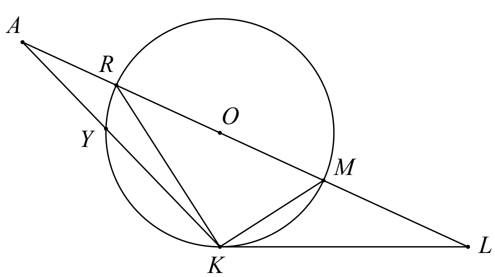
Hence $AM = AL - 2 = \frac{25}{7}$. The ratio between the areas of triangles AKM and RKM is equal to

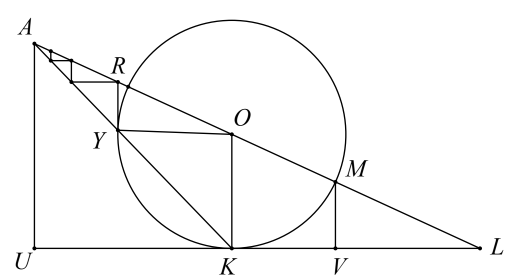
$$\frac{[AKM]}{[RKM]} = \frac{AM}{RM} = \frac{25/7}{5/2} = \frac{10}{7}.$$

Thus $[AKM] = \frac{10}{7} \cdot [RKM]$.

Because $\angle KRL$ and $\angle MKL$ both subtend $\widehat{KM}$, $\triangle KRL \sim \triangle MKL$. Therefore $\frac{KR}{MK} = \frac{LK}{LM} = \frac{3}{2}$. Thus let $KR = 3x$ and $MK = 2x$ for some positive real number x. Because RM is a diameter of ω (see left diagram below), $m\angle RKM = 90^\circ$. Thus triangle RKM is a right triangle with hypotenuse $\overline{RM}$. In particular, $13x^2 = KR^2 + MK^2 = RM^2 = \frac{25}{4}$, so $x^2 = \frac{25}{52}$ and $[RKM] = \frac{RK \cdot KM}{2} = 3x^2$. Therefore

$$[AKM] = \frac{10}{7} \cdot [RKM] = \frac{10}{7} \cdot 3 \cdot \frac{25}{52} = \mathbf{\frac{375}{182}}.$$





Third Solution: Let U and V be the respective feet of the perpendiculars dropped from A and M to $\overleftrightarrow{KL}$. From $(*)$, $\triangle AKL$ can be dissected into two infinite progressions of triangles: one progression of triangles similar to $\triangle OKL$ and the other similar to $\triangle YOK$, as shown in the right diagram above. In both progressions, the corresponding sides of the triangles have common ratio equal to

$$\frac{YO}{KL} = \frac{5/4}{3} = \frac{5}{12}.$$

Thus

$$AU = \frac{5}{4}\left(1 + \frac{5}{12} + \left(\frac{5}{12}\right)^2 + \cdots\right) = \frac{5}{4} \cdot \frac{12}{7} = \frac{15}{7}.$$

Because $\triangle LMV \sim \triangle LOK$, and because $LO = \frac{13}{4}$ by (†),

$$\frac{MV}{OK} = \frac{LM}{LO}, \quad \text{thus} \quad MV = \frac{OK \cdot LM}{LO} = \frac{\frac{5}{4} \cdot 2}{\frac{13}{4}} = \frac{10}{13}.$$

Finally, note that $[AKM] = [AKL] - [KLM]$. Because $\triangle AKL$ and $\triangle KLM$ share base $\overline{KL}$,

$$[AKM] = \frac{1}{2} \cdot 3 \cdot \left(\frac{15}{7} - \frac{10}{13}\right) = \frac{\mathbf{375}}{\mathbf{182}}.$$

2010 Individual Problems

I-1. Compute the number of positive integers less than 25 that cannot be written as the difference of two squares of integers.

I-2. For digits A, B, and C, $(\underline{A}\,\underline{B})^2 + (\underline{A}\,\underline{C})^2 = 1313$. Compute $A + B + C$.

I-3. Points P, Q, R, and S lie in the interior of square $ABCD$ such that triangles ABP, BCQ, CDR, and DAS are equilateral. If $AB = 1$, compute the area of quadrilateral $PQRS$.

I-4. For real numbers α, B, and C, the zeros of $T(x) = x^3 + x^2 + Bx + C$ are $\sin^2\alpha$, $\cos^2\alpha$, and $-\csc^2\alpha$. Compute $T(5)$.

I-5. Let $\mathcal{R}$ denote the circular region bounded by $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ partition $\mathcal{R}$ into four regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$. $[\mathcal{R}_i]$ denotes the area of region $\mathcal{R}_i$. If $[\mathcal{R}_1] > [\mathcal{R}_2] > [\mathcal{R}_3] > [\mathcal{R}_4]$, compute $[\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4]$.

I-6. Let x be a real number in the interval $[0, 360]$ such that the four expressions $\sin x^\circ$, $\cos x^\circ$, $\tan x^\circ$, $\cot x^\circ$ take on exactly three distinct (finite) real values. Compute the sum of all possible values of x.

I-7. Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence, and let $b_1, b_2, b_3, \ldots$ be a geometric sequence. The sequence $c_1, c_2, c_3, \ldots$ has $c_n = a_n + b_n$ for each positive integer n. If $c_1 = 1, c_2 = 4, c_3 = 15$, and $c_4 = 2$, compute c_5.

I-8. In square $ABCD$ with diagonal 1, E is on $\overline{AB}$ and F is on $\overline{BC}$ with $\mathrm{m}\angle BCE = \mathrm{m}\angle BAF = 30^\circ$. If $\overline{CE}$ and $\overline{AF}$ intersect at G, compute the distance between the incenters of triangles AGE and CGF.

I-9. Let a, b, m, n be positive integers with $am = bn = 120$ and $a \neq b$. In the coordinate plane, let $A = (a, m)$, $B = (b, n)$, and $O = (0, 0)$. If X is a point in the plane such that $AOBX$ is a parallelogram, compute the minimum area of $AOBX$.

I-10. Let $\mathcal{S}$ be the set of integers from 0 to 9999 inclusive whose base-2 and base-5 representations end in the same four digits. (Leading zeros are allowed, so $1 = 0001_2 = 0001_5$ is one such number.) Compute the remainder when the sum of the elements of $\mathcal{S}$ is divided by 10,000.

2010 Individual Answers

I-1. 6

I-2. 13

I-3. $2 - \sqrt{3}$

I-4. $\frac{567}{4}$ or equivalent $\left(141.75 \text{ or } 141\frac{3}{4}\right)$

I-5. 48

I-6. 990

I-7. 61

I-8. $4 - 2\sqrt{3}$

I-9. 44

I-10. 6248

2010 Individual Solutions

I-1. Suppose $n = a^2 - b^2 = (a+b)(a-b)$, where a and b are integers. Because $a+b$ and $a-b$ differ by an even number, they have the same parity. Thus n must be expressible as the product of two even integers or two odd integers. This condition is sufficient for n to be a difference of squares, because if n is odd, then $n = (k+1)^2 - k^2 = (2k+1) \cdot 1$ for some integer k, and if n is a multiple of 4, then $n = (k+1)^2 - (k-1)^2 = 2k \cdot 2$ for some integer k. Therefore any integer of the form $4k+2$ for integral k cannot be expressed as the difference of two squares of integers, hence the desired integers in the given range are $2, 6, 10, 14, 18$, and 22, for a total of **6** values.

Alternate Solution: Suppose that an integer n can be expressed as the difference of squares of two integers, and let the squares be a^2 and $(a+b)^2$, with $a, b \geq 0$. Then

$$\begin{aligned} n = (a+b)^2 - a^2 &= 2ab + b^2 \\ &= 2a + 1 \quad (b = 1) \\ &= 4a + 4 \quad (b = 2) \\ &= 6a + 9 \quad (b = 3) \\ &= 8a + 16 \quad (b = 4) \\ &= 10a + 25 \quad (b = 5). \end{aligned}$$

Setting $b = 1$ generates all odd integers. If $b = 3$ or $b = 5$, then the values of n are still odd, hence are already accounted for. If $b = 2$, then the values of $4a + 4 = 4(a+1)$ yield all multiples of 4; $b = 8$ yields multiples of 8 (hence are already accounted for). The remaining integers are even numbers that are not multiples of 4: 2, 6, 10, 14, 18, 22, for a total of **6** such numbers.

I-2. Because $10A \leq \underline{A}\,\underline{B} < 10(A+1)$, $200A^2 < (\underline{A}\,\underline{B})^2 + (\underline{A}\,\underline{C})^2 < 200(A+1)^2$. So $200A^2 < 1313 < 200(A+1)^2$, and $A = 2$. Note that B and C must have opposite parity, so without loss of generality, assume that B is even. Consider the numbers modulo 10: for any integer n, $n^2 \equiv 0, 1, 4, 5, 6,$ or $9 \bmod 10$. The only combination whose sum is congruent to 3 mod 10 is $4 + 9$. So $B = 2$ or 8 and $C = 3$ or 7. Checking cases shows that $28^2 + 23^2 = 1313$, so $B = 8, C = 3$, and $A + B + C = \mathbf{13}$.

Alternate Solution: Rewrite $1313 = 13 \cdot 101 = (3^2 + 2^2)(10^2 + 1^2)$. The two-square identity states:

$$\begin{aligned} (a^2 + b^2)(x^2 + y^2) &= (ax + by)^2 + (ay - bx)^2 \\ &= (ay + bx)^2 + (ax - by)^2. \end{aligned}$$

Therefore

$$\begin{aligned} 1313 = (30+2)^2 + (3-20)^2 &= 32^2 + 17^2 \\ &= (3+20)^2 + (30-2)^2 = 23^2 + 28^2. \end{aligned}$$

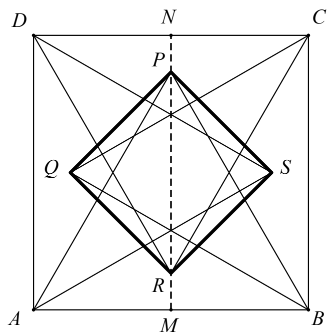
Hence $A = 2, B = 3, C = 8$, and $A + B + C = \mathbf{13}$.

Note: Factoring 1313 into the product of prime Gaussian integers

$$1313 = (3 + i)(3 - i)(10 + i)(10 - i)$$

helps show that this solution is unique. Because factorization in the Gaussian integers is unique (up to factors of i), there is only one way to write 1313 in the form $(a^2 + b^2)(x^2 + y^2)$. Therefore the values a, b, x, y given in the two-square identity above are uniquely determined (up to permutations).

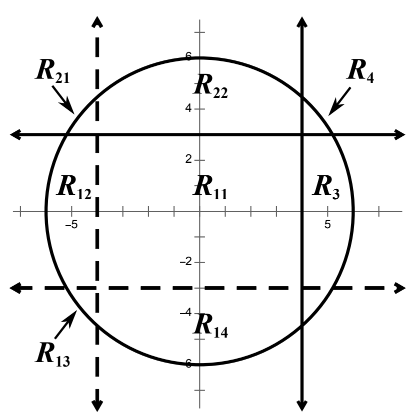
I-3. $PQRS$ is a square with diagonal $\overline{RP}$. Extend $\overline{RP}$ to intersect $\overline{AB}$ and $\overline{CD}$ at M and N respectively, as shown in the diagram below.



Then $\overline{MP}$ is an altitude of $\triangle ABP$ and $\overline{RN}$ is an altitude of $\triangle CDR$. Adding lengths, $MP + RN = MR + 2RP + PN = 1 + RP$, so $RP = \sqrt{3} - 1$. Therefore $[PQRS] = \frac{1}{2}(RP)^2 = \mathbf{2 - \sqrt{3}}$.

I-4. Use the sum of the roots formula to obtain $\sin^2\alpha + \cos^2\alpha + -\csc^2\alpha = -1$, so $\csc^2\alpha = 2$, and $\sin^2\alpha = \frac{1}{2}$. Therefore $\cos^2\alpha = \frac{1}{2}$. $T(x)$ has leading coefficient 1, so by the factor theorem, $T(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(x + 2)$. Then $T(5) = \left(5 - \frac{1}{2}\right)\left(5 - \frac{1}{2}\right)(5 + 2) = \mathbf{\frac{567}{4}}$.

I-5. Draw the lines $x = -4$ and $y = -3$, creating regions $\mathcal{R}_{21}, \mathcal{R}_{22}, \mathcal{R}_{11}, \mathcal{R}_{12}, \mathcal{R}_{13}, \mathcal{R}_{14}$ as shown below.



Then $[\mathcal{R}_{21}] = [\mathcal{R}_4] = [\mathcal{R}_{13}]$, $[\mathcal{R}_{22}] = [\mathcal{R}_{14}]$, and $[\mathcal{R}_3] = [\mathcal{R}_{12}] + [\mathcal{R}_{13}]$. Therefore

$$\begin{aligned} [\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4] &= ([\mathcal{R}_1] - [\mathcal{R}_2]) - ([\mathcal{R}_3] - [\mathcal{R}_4]) \\ &= ([\mathcal{R}_1] - [\mathcal{R}_{13}] - [\mathcal{R}_{14}]) - ([\mathcal{R}_{12}] + [\mathcal{R}_{13}] - [\mathcal{R}_{21}]) \\ &= ([\mathcal{R}_{11}] + [\mathcal{R}_{12}]) - [\mathcal{R}_{12}] \\ &= [\mathcal{R}_{11}]. \end{aligned}$$

This last region is simply a rectangle of height 6 and width 8, so its area is **48**.

I-6. If the four expressions take on three different values, exactly two of the expressions must have equal values. There are $\binom{4}{2} = 6$ cases to consider:

Case 1: $\sin x^\circ = \cos x^\circ$: Then $\tan x^\circ = \cot x^\circ = 1$, violating the condition that there be three distinct values.

Case 2: $\sin x^\circ = \tan x^\circ$: Because $\tan x^\circ = \dfrac{\sin x^\circ}{\cos x^\circ}$, either $\cos x^\circ = 1$ or $\sin x^\circ = 0$. However, in both of these cases, $\cot x^\circ$ is undefined, so it does not have a real value.

Case 3: $\sin x^\circ = \cot x^\circ$: Then $\sin x^\circ = \dfrac{\cos x^\circ}{\sin x^\circ}$, and so $\sin^2 x^\circ = \cos x^\circ$. Rewrite using the Pythagorean identity to obtain $\cos^2 x^\circ + \cos x^\circ - 1 = 0$, so $\cos x^\circ = \dfrac{-1+\sqrt{5}}{2}$ (the other root is outside the range of cos). Because $\cos x^\circ > 0$, this equation has two solutions in $[0, 360]$: an angle x_0° in the first quadrant and the angle $(360 - x_0)^\circ$ in the fourth quadrant. The sum of these two values is 360.

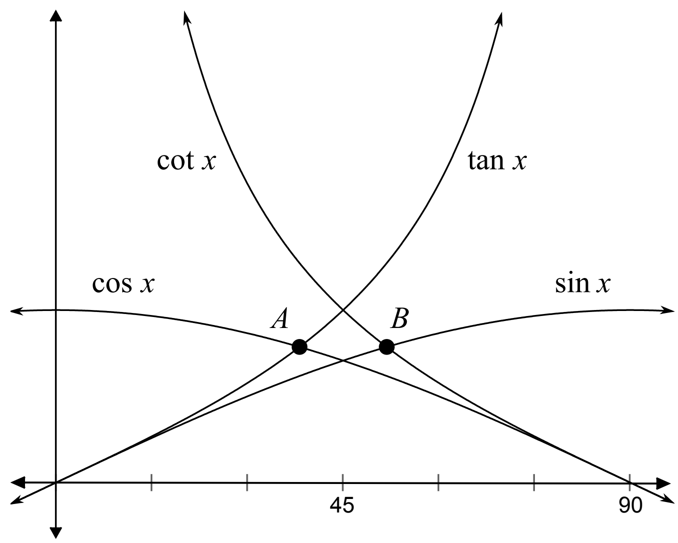
Case 4: $\cos x^\circ = \tan x^\circ$: Use similar logic as in the previous case to obtain the equation $\sin^2 x^\circ + \sin x^\circ - 1 = 0$, so now $\sin x^\circ = \dfrac{-1+\sqrt{5}}{2}$. Because $\sin x^\circ > 0$, this equation has two solutions, one an angle x_0° in the first quadrant, and the other its supplement $(180 - x_0)^\circ$ in the second quadrant. The sum of these two values is 180.

Case 5: $\cos x^\circ = \cot x^\circ$: In this case, $\tan x^\circ$ is undefined for reasons analogous to those in Case 2.

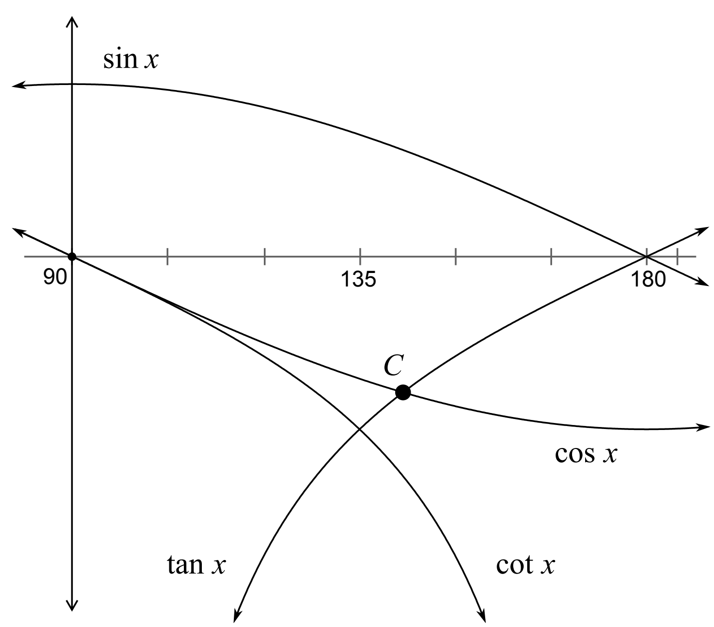
Case 6: $\tan x^\circ = \cot x^\circ$: Thus $\tan^2 x^\circ = 1$, hence $\tan x^\circ = \pm 1$. If $\tan x^\circ = 1$, then $\sin x^\circ = \cos x^\circ$, which yields only two distinct values. So $\tan x^\circ = -1$, which occurs at $x = 135$ and $x = 315$. The sum of these values is 450.

The answer is $360 + 180 + 450 = \mathbf{990}$.

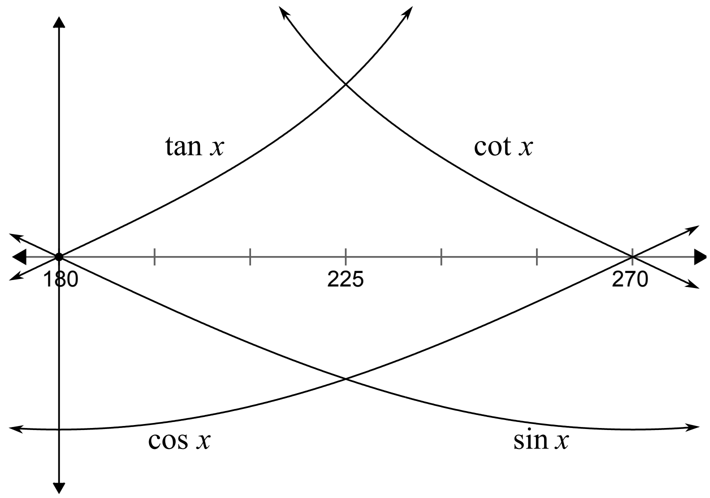
Alternate Solution: Consider the graphs of all four functions; notice first that $0, 90, 180, 270$ are not solutions because either $\tan x^\circ$ or $\cot x^\circ$ is undefined at each value.



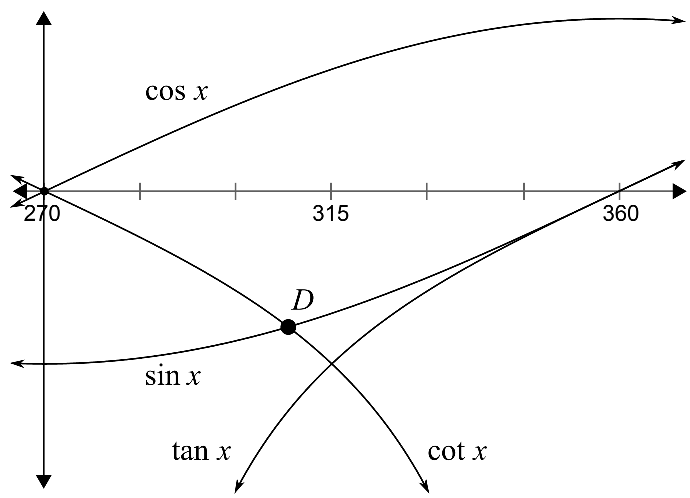
Start in the first quadrant. Let x_1 and x_2 be the values of x such that $\cos x^\circ = \tan x^\circ$ and $\sin x^\circ = \cot^\circ$, respectively, labeled A and B in the diagram. Because $\cos x^\circ = \sin(90 - x)^\circ$ and $\cot x^\circ = \tan(90 - x)^\circ$, $x_1 + x_2 = 90$. One can also see that the graphs of $y = \cot x^\circ$ and $y = \tan x^\circ$ cross at $x = 45$, but so do the graphs of $y = \sin x^\circ$ and $y = \cos x^\circ$. So at $x = 45$, there are only two distinct values, not three.



In the second quadrant, $\tan x^\circ = \cot x^\circ$ when $x = 135$. Also, because $\tan x^\circ$ increases from $-\infty$ to 0 while $\cos x^\circ$ decreases from 0 to -1, there exists a number x_3 such that $\tan x_3^\circ = \cos x_3^\circ$ (marked point C in the diagram above).



In the third quadrant, $\tan x^\circ$ and $\cot x^\circ$ are positive, while $\sin x^\circ$ and $\cos x^\circ$ are negative; the only place where graphs cross is at $x = 225$, but this value is not a solution because the four trigonometric functions have only two distinct values.



In the fourth quadrant, $\tan x^\circ = \cot x^\circ = -1$ when $x = 315$. Because $\sin x^\circ$ is increasing from -1 to 0 while $\cot x^\circ$ is decreasing from 0 to $-\infty$, there exists a number x_4 such that $\sin x_4^\circ = \cot x_4^\circ$ (marked D in the diagram above). Because $\cos x^\circ = \sin(90-x)^\circ = \sin(450-x)^\circ$ and $\cot x^\circ = \tan(90-x)^\circ = \tan(450-x)^\circ$, the values x_3 and x_4 are symmetrical around $x = 225$, that is, $x_3 + x_4 = 450$.

The sum is $(x_1 + x_2) + (135 + 315) + (x_3 + x_4) = 90 + 450 + 450 = \mathbf{990}$.

I-7. Let $a_2 - a_1 = d$ and $\frac{b_2}{b_1} = r$. Using $a = a_1$ and $b = b_1$, write the system of equations:

$$\begin{aligned} a + b &= 1 \\ (a+d) + br &= 4 \\ (a+2d) + br^2 &= 15 \\ (a+3d) + br^3 &= 2. \end{aligned}$$

Subtract the first equation from the second, the second from the third, and the third from the fourth to obtain three equations:

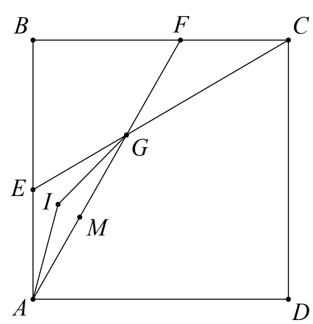
$$\begin{aligned} d + b(r-1) &= 3 \\ d + b(r^2 - r) &= 11 \\ d + b(r^3 - r^2) &= -13. \end{aligned}$$

Notice that the a terms have canceled. Repeat to find the second differences:

$$\begin{aligned} b(r^2 - 2r + 1) &= 8 \\ b(r^3 - 2r^2 + r) &= -24. \end{aligned}$$

Now divide the second equation by the first to obtain $r = -3$. Substituting back into either of these two last equations yields $b = \frac{1}{2}$. Continuing in the same vein yields $d = 5$ and $a = \frac{1}{2}$. Then $a_5 = \frac{41}{2}$ and $b_5 = \frac{81}{2}$, so $c_5 = \mathbf{61}$.

I-8. Let M be the midpoint of $\overline{AG}$, and I the incenter of $\triangle AGE$ as shown below.



Because $\dfrac{AB}{AC} = \sin 45^\circ$ and $\dfrac{EB}{AB} = \dfrac{EB}{BC} = \tan 30^\circ$,

$$\begin{aligned} AE &= AB - EB = AB(1 - \tan 30^\circ) \\ &= \sin 45^\circ (1 - \tan 30^\circ) \\ &= \frac{\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ}{\cos 30^\circ} \\ &= \frac{\sin(45^\circ - 30^\circ)}{\cos 30^\circ} \\ &= \frac{\sin 15^\circ}{\cos 30^\circ}. \end{aligned}$$

Note that $\dfrac{AM}{AE} = \cos 30^\circ$ and $\dfrac{AM}{AI} = \cos 15^\circ$. Therefore

$$\begin{aligned}\frac{AI}{AE} &= \frac{\cos 30^\circ}{\cos 15^\circ}\\ &= \frac{\sin 60^\circ}{\cos 15^\circ}\\ &= \frac{2\sin 30^\circ \cos 30^\circ}{\cos 15^\circ}\\ &= \frac{2(2\sin 15^\circ \cos 15^\circ)\cos 30^\circ}{\cos 15^\circ}\\ &= 4\sin 15^\circ \cos 30^\circ.\end{aligned}$$

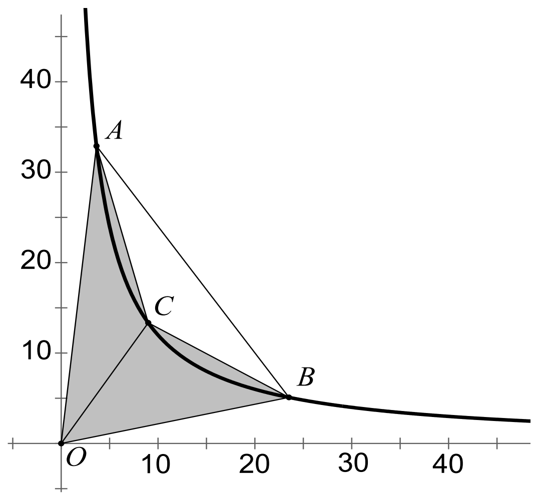
Thus $AI = (4\sin 15^\circ \cos 30^\circ)\left(\dfrac{\sin 15^\circ}{\cos 30^\circ}\right) = 4\sin^2 15^\circ = 4\cdot\left(\dfrac{1-\cos 30^\circ}{2}\right) = 2-\sqrt{3}$. Finally, the desired distance is $2IG = 2AI = \mathbf{4 - 2\sqrt{3}}$.

I-9. The area of parallelogram $AOBX$ is given by the absolute value of the cross product $|\langle a, m\rangle \times \langle b, n\rangle| = |an - mb|$. Because $m = \dfrac{120}{a}$ and $n = \dfrac{120}{b}$, the desired area of $AOBX$ equals $120\left|\dfrac{a}{b} - \dfrac{b}{a}\right|$. Note that the function $f(x) = x - \dfrac{1}{x}$ is monotone increasing for $x > 1$. (Proof: if $x_1 > x_2 > 0$, then $f(x_1) - f(x_2) = (x_1 - x_2) + \dfrac{x_1 - x_2}{x_1 x_2}$, where both terms are positive because $x_1 x_2 > 0$.) So the minimum value of $[AOBX]$ is attained when $\dfrac{a}{b}$ is as close as possible to 1, that is, when a and b are consecutive divisors of 120. By symmetry, consider only $a < b$; notice too that because $\dfrac{120/a}{120/b} = \dfrac{b}{a}$, only values with $b \leq \sqrt{120}$ need be considered. These observations can be used to generate the table below:

a, m	$1, 120$	$2, 60$	$3, 40$	$4, 30$	$5, 24$	$6, 20$	$8, 15$	$10, 12$
b, n	$2, 60$	$3, 40$	$4, 30$	$5, 24$	$6, 20$	$8, 15$	$10, 12$	$12, 10$
$[AOBX]$	180	100	70	54	44	70	54	44

The smallest value is **44**, achieved using $(5, 24)$ and $(6, 20)$, or using $(10, 12)$ and $(12, 10)$.

Note: The fact that a and b must be consecutive divisors of 120 can also be established by the following geometric argument. Notice that $[AOBX] = 2[AOB]$. Suppose C is a point on the hyperbola $y = 120/x$ between A and B, as shown in the diagram below.



Because the hyperbola is concave up, $[OAC] + [OCB] < [OAB]$, so in particular, $[OAC] < [OAB]$. Thus, if $[OAB]$ is minimal, there can be no point C with integer coordinates between A and B on the hyperbola.

I-10. The remainders of an integer N modulo $2^4 = 16$ and $5^4 = 625$ uniquely determine its remainder modulo 10000. There are only 16 strings of four 0's and 1's. In addition, because 16 and 625 are relatively prime, it will be shown below that for each such string s, there exists exactly one integer x_s in the range $0 \leq x_s < 10000$ such that the base-2 and base-5 representations of x_s end in the digits of s (e.g., x_{1001} is the unique positive integer less than 10000 such that x's base-5 representation and base-2 representation both end in 1001).

Here is a proof of the preceding claim: Let $p(s)$ be the number whose digits in base 5 are the string s, and $b(s)$ be the number whose digits in base 2 are the string s. Then the system $x \equiv p(s) \bmod 625$ and $x \equiv b(s) \bmod 16$ can be rewritten as $x = p(s)+625m$ and $x = b(s)+16n$ for integers m and n. These reduce to the Diophantine equation $16n - 625m = p(s) - b(s)$, which has solutions m, n in $\mathbb{Z}$, with at least one of $m, n \geq 0$. Assuming without loss of generality that $m > 0$ yields $x = p(s) + 625m \geq 0$. To show that there exists an $x_s < 10000$ and that it is unique, observe that the general form of the solution is $m' = m - 16t, n' = n + 625t$. Thus if $p(s)+625m > 10000$, an appropriate t can be found by writing $0 \leq p(s)+625(m-16t) < 10000$, which yields $p(s) + 625m - 10000 < 10000t \leq p(s) + 625m$. Because there are exactly 10000 integers in that interval, exactly one of them is divisible by 10000, so there is exactly one value of t satisfying $0 \leq p(s) + 625(m - 16t) < 10000$, and set $x_s = 625(m - 16t)$.

Therefore there will be 16 integers whose base-2 and base-5 representations end in the same four digits, possibly with leading 0's as in the example. Let $X = x_{0000} + \cdots + x_{1111}$. Then X is congruent modulo 16 to $0000_2 + \cdots + 1111_2 = 8 \cdot (1111_2) = 8 \cdot 15 \equiv 8$. Similarly, X is congruent modulo 625 to $0000_5 + \cdots + 1111_5 = 8 \cdot 1111_5 = 2 \cdot 4444_5 \equiv 2 \cdot (-1) = -2$.

So X must be 8 (mod 16) and -2 (mod 625). Noticing that $625 \equiv 1 \pmod{16}$, conclude that the answer is $-2 + 10 \cdot 625 = \mathbf{6248}$.

Power Question 2010: Power of Circular Subdivisions

A king strapped for cash is forced to sell off his kingdom $U = \{(x, y) : x^2 + y^2 \leq 1\}$. He sells the two circular plots C and C' centered at $\left(\pm\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$. The retained parts of the kingdom form two regions, each bordered by three arcs of circles; in what follows, we will call such regions *curvilinear triangles,* or *c-triangles* (c△) for short.

This sad day marks day 0 of a new fiscal era. Unfortunately, these drastic measures are not enough, and so each day thereafter, court geometers mark off the largest possible circle contained in each c-triangle in the remaining property. This circle is tangent to all three arcs of the c-triangle, and will be referred to as the *incircle* of the c-triangle. At the end of the day, all incircles demarcated that day are sold off, and the following day, the remaining c-triangles are partitioned in the same manner.

1. Without using Descartes' Circle Formula (see below):

   a. Show that the circles marked off and sold on day 1 are centered at $\left(0, \pm\frac{2}{3}\right)$ with radius $\frac{1}{3}$. [2]

   b. Find the combined area of the six remaining curvilinear territories. [2]

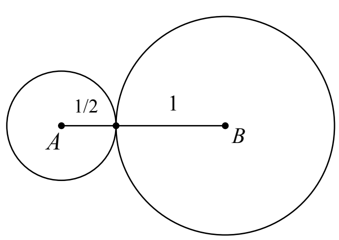
On day 2, the plots bounded by the incircles of the six remaining curvilinear territories are sold.

2a. Determine the number of curvilinear territories remaining at the *end* of day 3. [2]

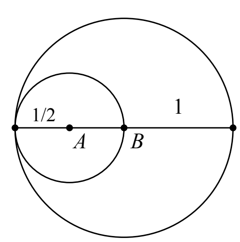
2b. Let X_n be the number of plots sold on day n. Find a formula for X_n in terms of n. [2]

2c. Determine the total number of plots sold up to and including day n. [2]

Some notation: when discussing mutually tangent circles (or arcs), it is convenient to refer to the curvature of a circle rather than its radius. We define *curvature* as follows. Suppose that circle A of radius r_a is externally tangent to circle B of radius r_b. Then the curvatures of the circles are simply the reciprocals of their radii, $\frac{1}{r_a}$ and $\frac{1}{r_b}$. If circle A is internally tangent to circle B, however, as in the right diagram below, the curvature of circle A is still $\frac{1}{r_a}$, while the curvature of circle B is $-\frac{1}{r_b}$, the opposite of the reciprocal of its radius.



Circle A has curvature 2; circle B has curvature 1.



Circle A has curvature 2; circle B has curvature -1.

Using these conventions allows us to express a beautiful theorem of Descartes: when four circles A, B, C, D are pairwise tangent, with respective curvatures a, b, c, d, then

$$(a + b + c + d)^2 = 2\left(a^2 + b^2 + c^2 + d^2\right),$$

where (as before) a is taken to be negative if B, C, D are internally tangent to A, and correspondingly for b, c, or d. This Power Question does not involve the proof of Descartes' Circle Formula, but the formula may be used for all problems below.

3a. Two unit circles and a circle of radius $\frac{2}{3}$ are mutually externally tangent. Compute all possible values of r such that a circle of radius r is tangent to all three circles. [2]

3b. Given three mutually tangent circles with curvatures $a, b, c > 0$, suppose that $(a, b, c, 0)$ does not satisfy Descartes' Circle Formula. Show that there are two distinct values of r such that there is a circle of radius r tangent to the given circles. [3]

3c. Algebraically, it is possible for a quadruple $(a, b, c, 0)$ to satisfy Descartes' Circle Formula, as occurs when $a = b = 1$ and $c = 4$. Find a geometric interpretation for this situation. [2]

4. Let $\phi = \dfrac{1+\sqrt{5}}{2}$, and let $\rho = \phi + \sqrt{\phi}$.

   a. Prove that $\rho^4 = 2\rho^3 + 2\rho^2 + 2\rho - 1$. [2]

   b. Show that four pairwise externally tangent circles with nonequal radii in geometric progression must have common ratio ρ. [2]

As shown in problem 3, given A, B, C, D as above with $s = a + b + c + d$, there is a second circle A' with curvature a' also tangent to B, C, and D. We can describe A and A' as *conjugate circles*.

5. Use Descartes' Circle Formula to show that $a' = 2s - 3a$ and therefore $s' = a' + b + c + d = 3s - 4a$. [4]

In the context of this problem, a *circle configuration* is a quadruple of real numbers (a, b, c, d) representing curvatures of mutually tangent circles A, B, C, D. In other words, a circle configuration is a quadruple (a, b, c, d) of real numbers satisfying Descartes' Circle Formula.

The result in problem 5 allows us to compute the curvatures of the six plots removed on day 2. In this case, $(a, b, c, d) = (-1, 2, 2, 3)$, and $s = 6$. For example, one such plot is tangent to both of the circles C and C' centered at $\left(\pm\frac{1}{2}, 0\right)$, and to one of the circles of radius $\frac{1}{3}$ removed on day 1; it is conjugate to the unit circle (the boundary of the original kingdom U). If this new plot's curvature is a, we can write $(-1, 2, 2, 3) \vdash (a, 2, 2, 3)$, and we say that the first circle configuration *yields* the second.

6a. Use the result of problem 5 to compute the curvatures of all circles removed on day 2, and the corresponding values of s'. [2]

6b. Show that by area, 12% of the kingdom is sold on day 2. [1]

6c. Find the areas of the circles removed on day 3. [2]

6d. Show that the plots sold on day 3 have mean curvature of 23. [1]

7. Prove that the curvature of each circular plot is an integer. [3]

Descartes' Circle Formula can be extended by interpreting the coordinates of points on the plane as complex numbers in the usual way: the point (x, y) represents the complex number $x + yi$. On the complex plane, let z_A, z_B, z_C, z_D be the centers of circles A, B, C, D respectively; as before, a, b, c, d are the curvatures of their respective circles. Then Descartes' Extended Circle Formula states

$$(a \cdot z_A + b \cdot z_B + c \cdot z_C + d \cdot z_D)^2 = 2\left(a^2 z_A^2 + b^2 z_B^2 + c^2 z_C^2 + d^2 z_D^2\right).$$

8a. Suppose that A' is a circle conjugate to A with center $z_{A'}$ and curvature a', and $\hat{s} = a \cdot z_A + b \cdot z_B + c \cdot z_C + d \cdot z_D$. Use Descartes' Extended Circle Formula to show that $a' \cdot z_{A'} = 2\hat{s} - 3a \cdot z_A$ and therefore $a' \cdot z_{A'} + b \cdot z_B + c \cdot z_C + d \cdot z_D = 3\hat{s} - 4a \cdot z_A$. [2]

8b. Prove that the center of each circular plot has coordinates $\left(\frac{u}{c}, \frac{v}{c}\right)$ where u and v are integers, and c is the curvature of the plot. [2]

Given a c-triangle T, let a, b, and c be the curvatures of the three arcs bounding T, with $a \leq b \leq c$, and let d be the curvature of the incircle of T. Define the *circle configuration associated with* T to be $\mathcal{C}(T) = (a, b, c, d)$. Define the c-triangle T to be *proper* if $c \leq d$. For example, circles of curvatures -1, 2, and 3 determine two c-triangles. The incircle of one has curvature 6, so it is proper; the incircle of the other has curvature 2, so it is not proper.

Let P and Q be two c-triangles, with associated configurations $\mathcal{C}(P) = (a, b, c, d)$ and $\mathcal{C}(Q) = (w, x, y, z)$. We say that P *dominates* Q if $a \leq w$, $b \leq x$, $c \leq y$, and $d \leq z$. (The term "dominates" refers to the fact that the radii of the arcs defining Q cannot be larger than the radii of the arcs defining P.)

Removing the incircle from T gives three c-triangles, $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, each bounded by the incircle of T and two of the arcs that bound T. These triangles have associated configurations

$$\begin{aligned}\mathcal{C}(T^{(1)}) &= (b, c, d, a'),\\ \mathcal{C}(T^{(2)}) &= (a, c, d, b'),\\ \mathcal{C}(T^{(3)}) &= (a, b, d, c'),\end{aligned}$$

where a', b', and c' are determined by the formula in problem 5.

9. Let P and Q be two proper c-triangles such that P dominates Q. Let $\mathcal{C}(P) = (a, b, c, d)$ and $\mathcal{C}(Q) = (w, x, y, z)$.

   a. Show that $P^{(3)}$ dominates $P^{(2)}$ and that $P^{(2)}$ dominates $P^{(1)}$. [2]

   b. Prove that $P^{(1)}$ dominates $Q^{(1)}$. [2]

   c. Prove that $P^{(3)}$ dominates $Q^{(3)}$. [2]

10a. Prove that the largest plot sold by the king on day n has curvature $n^2 + 2$. [3]

10b. If $\rho = \phi + \sqrt{\phi}$, as in problem 4, prove that the curvature of the smallest plot sold by the king on day n does not exceed $2\rho^n$. [3]

Solutions to 2010 Power Question

1a. By symmetry, P_1, P_2, the two plots sold on day 1, are centered on the y-axis, say at $(0, \pm y)$ with $y > 0$. Let these plots have radius r. Because P_1 is tangent to U, $y + r = 1$. Because P_1 is tangent to C, the distance from $(0, 1-r)$ to $\left(\frac{1}{2}, 0\right)$ is $r + \frac{1}{2}$. Therefore

$$\begin{aligned}
\left(\frac{1}{2}\right)^2 + (1-r)^2 &= \left(r + \frac{1}{2}\right)^2 \\
1 - 2r &= r \\
r &= \frac{1}{3} \\
y = 1 - r &= \frac{2}{3}.
\end{aligned}$$

Thus the plots are centered at $\left(0, \pm\frac{2}{3}\right)$ and have radius $\frac{1}{3}$.

1b. The four "removed" circles have radii $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}$ so the combined area of the six remaining curvilinear territories is:

$$\pi\left(1^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) = \frac{5\pi}{18}.$$

2a. At the beginning of day 2, there are six c-triangles, so six incircles are sold, dividing each of the six territories into three smaller curvilinear triangles. So a total of 18 curvilinear triangles exist at the start of day 3, each of which is itself divided into three pieces that day (by the sale of a total of 18 regions bounded by the territories' incircles). Therefore there are 54 regions at the end of day 3.

2b. Each day, every curvilinear territory is divided into three smaller curvilinear territories. Let R_n be the number of regions at the end of day n. Then $R_0 = 2$, and $R_{n+1} = 3 \cdot R_n$. Thus R_n is a geometric sequence, so $R_n = 2 \cdot 3^n$. For $n > 0$, the number of plots sold on day n equals the number of territories existing at the end of day $n - 1$, i.e., $X_n = R_{n-1}$, so $X_0 = X_1 = 2$, and for $n > 1, X_n = 2 \cdot 3^{n-1}$.

2c. The total number of plots sold up to and including day n is

$$\begin{aligned}
2 + \sum_{k=1}^{n} X_k &= 2 + 2\sum_{k=1}^{n} 3^{k-1} \\
&= 2 + 2 \cdot (1 + 3 + 3^2 + \ldots + 3^{n-1}) \\
&= 3^n + 1.
\end{aligned}$$

Alternatively, proceed by induction: on day 0, there are $2 = 3^0 + 1$ plots sold, and for $n \geq 0$,

$$\begin{aligned}
(3^n + 1) + X_{n+1} &= (3^n + 1) + 2 \cdot 3^n \\
&= 3 \cdot 3^n + 1 \\
&= 3^{n+1} + 1.
\end{aligned}$$

3a. Use Descartes' Circle Formula with $a = b = 1$ and $c = \frac{3}{2}$ to solve for d:

$$\begin{aligned} 2 \cdot \left(1^2 + 1^2 + \left(\frac{3}{2}\right)^2 + d^2\right) &= \left(1 + 1 + \frac{3}{2} + d\right)^2 \\ \frac{17}{2} + 2d^2 &= \frac{49}{4} + 7d + d^2 \\ d^2 - 7d - \frac{15}{4} &= 0, \end{aligned}$$

from which $d = \frac{15}{2}$ or $d = -\frac{1}{2}$. These values correspond to radii of $\frac{2}{15}$, a small circle nestled between the other three, or 2, a large circle enclosing the other three.

Alternatively, start by scaling the kingdom with the first four circles removed to match the situation given. Thus the three given circles are internally tangent to a circle of radius $r = 2$ and curvature $d = -\frac{1}{2}$. Descartes' Circle Formula gives a quadratic equation for d, and the sum of the roots is $2 \cdot (1 + 1 + \frac{3}{2}) = 7$, so the second root is $7 + \frac{1}{2} = \frac{15}{2}$, corresponding to a circle of radius $r = \frac{2}{15}$.

3b. Apply Descartes' Circle Formula to yield

$$(a + b + c + x)^2 = 2 \cdot (a^2 + b^2 + c^2 + x^2),$$

a quadratic equation in x. Expanding and rewriting in standard form yields the equation

$$x^2 - px + q = 0$$

where $p = 2(a + b + c)$ and $q = 2(a^2 + b^2 + c^2) - (a + b + c)^2$.

The discriminant of this quadratic is

$$\begin{aligned} p^2 - 4q &= 8(a + b + c)^2 - 8(a^2 + b^2 + c^2) \\ &= 16(ab + ac + bc). \end{aligned}$$

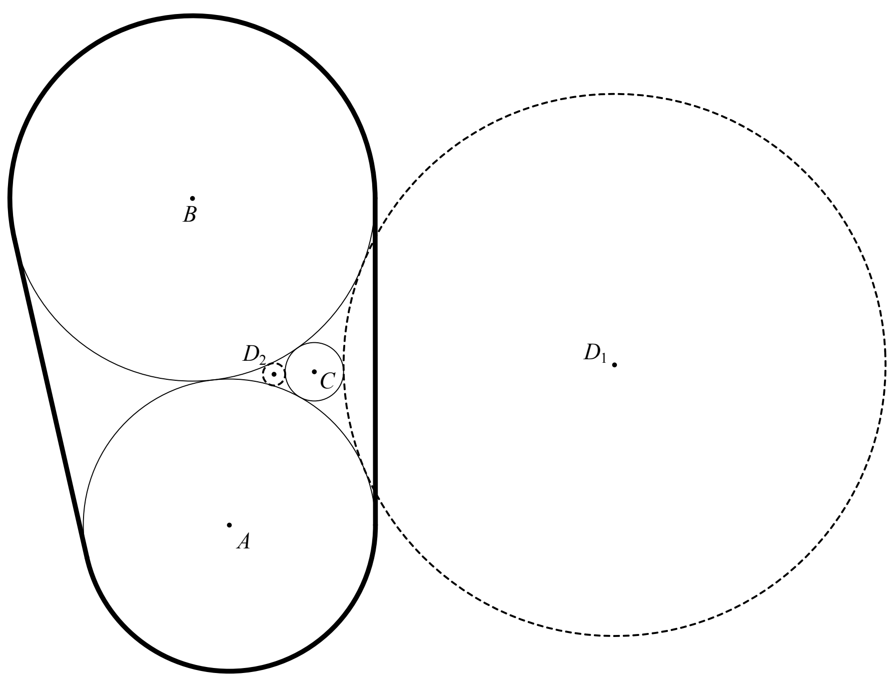
This last expression is positive because it is given that $a, b, c > 0$. Therefore the quadratic has two distinct real roots, say d_1 and d_2. These usually correspond to two distinct radii, $r_1 = \dfrac{1}{|d_1|}$ and $r_2 = \dfrac{1}{|d_2|}$.

There are two possible exceptions. If $(a, b, c, 0)$ satisfies Descartes' Circle Formula, then one of the radii is undefined. The other case to consider is if $r_2 = r_1$, which would occur if $d_2 = -d_1$. This case can be ruled out because $d_1 + d_2 = p = 2(a + b + c)$, which must be positive if $a, b, c > 0$. (Notice too that this inequality rules out the possibility that both circles have negative curvature, so that there cannot be two distinct circles to which the given circles are internally tangent.)

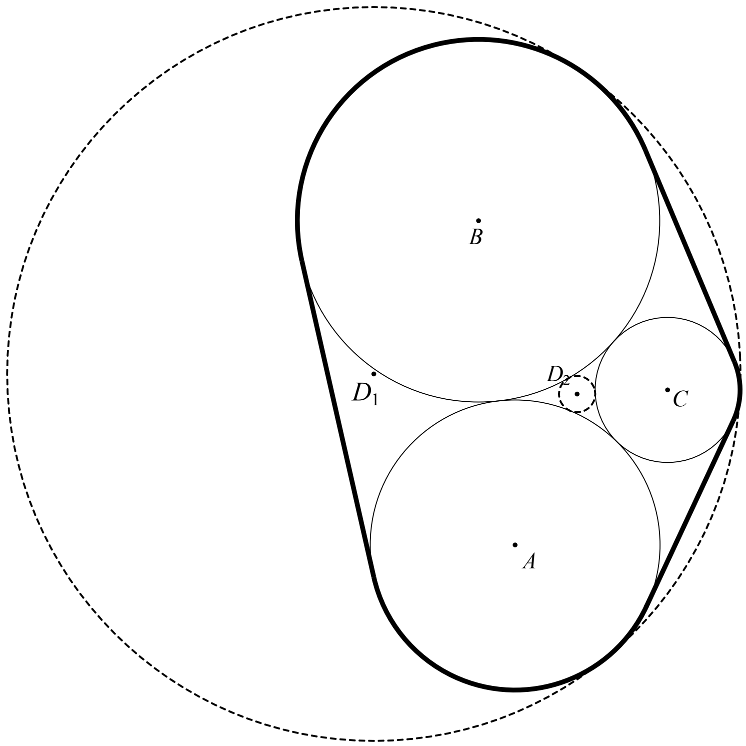
When both roots d_1 and d_2 are positive, the three given circles are externally tangent to both fourth circles. When one is positive and one is negative, the three given circles are internally tangent to one circle and externally tangent to the other.

While the foregoing answers the question posed, it is interesting to examine the result from a geometric perspective: why are there normally two possible fourth circles? Consider the case when one of a, b, and c is negative (i.e., two circles are internally tangent to a third). Let A and B be circles internally tangent to C. Then A and B partition the remaining area of C into two c-triangles, each of which has an incircle, providing the two solutions.

If, as in the given problem, $a, b, c > 0$, then all three circles A, B, and C, are mutually externally tangent. In this case, the given circles bound a c-triangle, which has an incircle, corresponding to one of the two roots. The complementary arcs of the given circles bound an infinite region, and this region normally contains a second circle tangent to the given circles. To demonstrate this fact geometrically, consider shrink-wrapping the circles: the shrink-wrap is the border of the smallest convex region containing all three circles. (This region is called the *convex hull* of the circles). There are two cases to address. If only two circles are touched by the shrink-wrap, then one circle is wedged between two larger ones and completely enclosed by their common tangents. In such a case, a circle can be drawn so that it is tangent to all three circles as shown in the diagram below (shrink-wrap in bold; locations of fourth circle marked at D_1 and D_2).



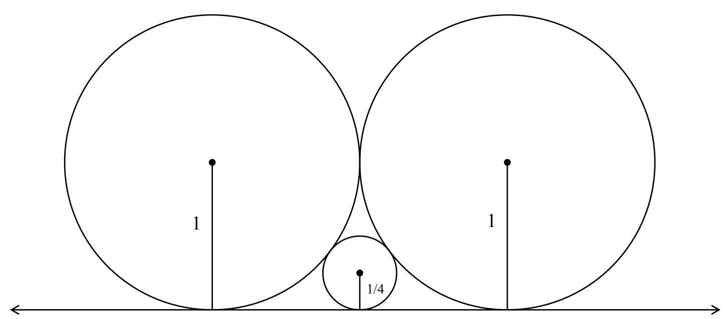
On the other hand, if the shrink-wrap touches all three circles, then it can be expanded to make a circle tangent to and containing A, B, and C, as shown below.



The degenerate case where $(a, b, c, 0)$ satisfies Descartes' Circle Formula is treated in 3c below.

One final question is left for the reader to investigate. Algebraically, it is possible that there is a double root if $p^2 - 4q = 0$. To what geometric situation does this correspond, and under what (geometric) conditions can it arise?

3c. In this case, the fourth "circle" is actually a line tangent to all three circles, as shown in the diagram below.



4a. Note that

$$\begin{aligned} \frac{1}{\rho} &= \frac{\phi^2 - \phi}{\phi + \sqrt{\phi}} \\ &= \phi - \sqrt{\phi}\,. \end{aligned}$$

Therefore

$$\begin{aligned}\left(\rho-\frac{1}{\rho}\right)^2 &= \left(2\sqrt{\phi}\right)^2 = 4\phi \\ &= 2\left(\rho+\frac{1}{\rho}\right).\end{aligned}$$

Multiplying both sides of the equation by ρ^2 gives $\left(\rho^2-1\right)^2 = 2\left(\rho^3+\rho\right)$. Expand and isolate ρ^4 to obtain $\rho^4 = 2\rho^3+2\rho^2+2\rho-1$.

Alternate Proof: Because $\phi^2 = \phi + 1$, any power of ρ can be expressed as an integer plus integer multiples of $\sqrt{\phi}$, ϕ, and $\phi\sqrt{\phi}$. In particular,

$$\begin{aligned}\rho^2 &= \phi^2+2\phi\sqrt{\phi}+\phi \\ &= 2\phi\sqrt{\phi}+2\phi+1, \\ \rho^3 &= 4\phi\sqrt{\phi}+5\phi+3\sqrt{\phi}+4, \text{ and} \\ \rho^4 &= 12\phi\sqrt{\phi}+16\phi+8\sqrt{\phi}+9.\end{aligned}$$

Therefore

$$\begin{aligned}2\rho^3+2\rho^2+2\rho-1 &= 2\left(4\phi\sqrt{\phi}+5\phi+3\sqrt{\phi}+4\right)+2\left(2\phi\sqrt{\phi}+2\phi+1\right)+2\rho-1 \\ &= 12\phi\sqrt{\phi}+16\phi+8\sqrt{\phi}+9 \\ &= \rho^4.\end{aligned}$$

4b. If the radii are in geometric progression, then so are their reciprocals (i.e., curvatures). Without loss of generality, let $(a,b,c,d) = (a, ar, ar^2, ar^3)$ for $r > 1$. By Descartes' Circle Formula,

$$\left(a+ar+ar^2+ar^3\right)^2 = 2\left(a^2+a^2r^2+a^2r^4+a^2r^6\right).$$

Cancel a^2 from both sides of the equation to obtain

$$\left(1+r+r^2+r^3\right)^2 = 2\left(1+r^2+r^4+r^6\right).$$

Because $1+r+r^2+r^3 = (1+r)(1+r^2)$ and $1+r^2+r^4+r^6 = (1+r^2)(1+r^4)$, the equation can be rewritten as follows:

$$\begin{aligned}(1+r)^2\left(1+r^2\right)^2 &= 2\left(1+r^2\right)\left(1+r^4\right) \\ (1+r)^2\left(1+r^2\right) &= 2\left(1+r^4\right) \\ r^4-2r^3-2r^2-2r+1 &= 0.\end{aligned}$$

Using the identity from 4a, $r = \rho$ is one solution; because the polynomial is palindromic, another real solution is $r = \rho^{-1} = \phi - \sqrt{\phi}$, but this value is less than 1. The product of the corresponding linear factors is $r^2 - 2\phi r + \phi^2 - \phi = r^2 - 2\phi r + 1$. Division verifies that the other quadratic factor of the polynomial is $r^2 + (2\phi - 2)r + 1$, which has no real roots because $(\sqrt{5}-1)^2 < 1$.

5. The equation $(x+b+c+d)^2 = 2\left(x^2+b^2+c^2+d^2\right)$ is quadratic with two solutions. Call them a and a'. These are the curvatures of the two circles which are tangent to circles with curvatures b, c, and d. Rewrite the equation in standard form to obtain $x^2 - 2(b+c+d)x + \ldots = 0$. Using the sum of the roots formula, $a + a' = 2(b+c+d) = 2(s-a)$. So $a' = 2s - 3a$, and therefore

$$\begin{aligned} s' &= a' + b + c + d \\ &= 2s - 3a + s - a \\ &= 3s - 4a. \end{aligned}$$

6. Day 1 starts with circles of curvature $-1, 2, 2$ bounding C and C'. The geometers mark off P, P' with curvature 3 yielding configurations $(-1, 2, 2, 3)$, $s = 6$. Then the king sells two plots of curvature 3.

   a. To find plots sold on day 2 start with the configuration $(-1, 2, 2, 3)$ and compute the three distinct curvatures of circles conjugate to one of the circles in this configuration. Because $s = 6$, the curvatures are $2 \cdot 6 - 3(-1) = 15$, $2 \cdot 6 - 3 \cdot 2 = 6$, and $2 \cdot 6 - 3 \cdot 3 = 3$. However, if P is the circle of curvature 3 included in the orientation, then P' is the new conjugate circle of curvature 3, which was also marked off on day 2. Thus the only options are 15 and 6. In the first case, $s' = 3s - 4a = 3 \cdot 6 - 4(-1) = 22$; in the second, $s' = 3s - 4a = 3 \cdot 6 - 4 \cdot 2 = 10$.

   b. On day 2, six plots are sold: two with curvature 15 from the configuration $(2, 2, 3, 15)$, and four with curvature 6 from the configuration $(-1, 2, 3, 6)$. The total area sold on day 2 is therefore
$$2 \cdot \frac{\pi}{15^2} + 4 \cdot \frac{\pi}{6^2} = \frac{3}{25}\pi,$$
   which is exactly 12% of the unit circle.

   c. Day 3 begins with two circles of curvature 15 from the configuration $(2, 2, 3, 15)$, and four circles of curvature 6 from the configuration $(-1, 2, 3, 6)$. Consider the following two cases:

Case 1: $(a, b, c, d) = (2, 2, 3, 15)$, $s = 22$

- $a = 2 : a' = 2s - 3a = \mathbf{38}$
- $b = 2 : b' = 2s - 3b = \mathbf{38}$
- $c = 3 : c' = 2s - 3c = \mathbf{35}$
- $d = 15 : d' = 2s - 3d = -1$, which is the configuration from day 1.

Case 2: $(a, b, c, d) = (-1, 2, 3, 6)$, $s = 10$

- $a = -1 : a' = 2s - 3a = \mathbf{23}$
- $b = 2 \;\; : b' = 2s - 3b = \mathbf{14}$
- $c = 3 \;\; : c' = 2s - 3c = \mathbf{11}$
- $d = 6 \;\; : d' = 2s - 3d = 2$, which is the configuration from day 1.

So the areas of the plots removed on day 3 are:

$$\frac{\pi}{38^2}, \frac{\pi}{35^2}, \frac{\pi}{23^2}, \frac{\pi}{14^2}, \text{ and } \frac{\pi}{11^2}.$$

There are two circles with area $\frac{\pi}{35^2}$, and four circles with each of the other areas, for a total of 18 plots.

d. Because 18 plots were sold on day 3, the mean curvature is

$$\frac{2(38+38+35)+4(23+14+11)}{18} = 23.$$

7. Proceed by induction. The base case, that all curvatures prior to day 2 are integers, was shown in problem 1a. Using the formula $a' = 2s - 3a$, if a, b, c, d, and s are integers on day n, then a', b', c', and d' are integer curvatures on day $n+1$, proving inductively that all curvatures are integers.

8a. Notice that substituting az_A, bz_B, cz_C, dz_D for a, b, c, d respectively in the derivation of the formula in problem 4 leaves the algebra unchanged, so in general, $a'z_{A'} = 2\hat{s} - 3az_A$, and similarly for $b'z_{B'}, c'z_{C'}, d'z_{D'}$.

8b. It suffices to show that for each circle C with curvature c and center z_C (in the complex plane), cz_C is of the form $u + iv$ where u and v are integers. If this is the case, then each center is of the form $\left(\frac{u}{c}, \frac{v}{c}\right)$.

Proceed by induction. To check the base case, check the original kingdom and the first four plots: U, C, C', P_1, and P_2. Circle U is centered at $(0,0)$, yielding $-1 \cdot z_U = 0 + 0i$. Circles C and C' are symmetric about the y-axis, so it suffices to check just one of them. Circle C has radius $\frac{1}{2}$ and therefore curvature 2. It is centered at $\left(\frac{1}{2}, \frac{0}{2}\right)$, yielding $2z_C = 2\left(\frac{1}{2} + 0i\right) = 1$. Circles P_1 and P_2 are symmetric about the x-axis, so it suffices to check just one of them. Circle P_1 has radius $\frac{1}{3}$ and therefore curvature 3. It is centered at $\left(\frac{0}{3}, \frac{2}{3}\right)$, yielding $3z_{P_3} = 0 + 2i$.

For the inductive step, suppose that $a\,z_A, b\,z_B, c\,z_C, d\,z_D$ have integer real and imaginary parts. Then by closure of addition and multiplication in the integers, $a'z_{A'} = 2\hat{s} - 3a\,z_A$ also has integer real and imaginary parts, and similarly for $b'z_{B'}, c'z_{C'}, d'z_{D'}$.

So for all plots A sold, $a\,z_A$ has integer real and imaginary parts, so each is centered at $\left(\frac{u}{c}, \frac{v}{c}\right)$ where u and v are integers, and c is the curvature.

9a. Let $s = a + b + c + d$. From problem 5, it follows that

$$\begin{aligned}
\mathcal{C}(P^{(1)}) &= (b, c, d, a'), \\
\mathcal{C}(P^{(2)}) &= (a, c, d, b'), \\
\mathcal{C}(P^{(3)}) &= (a, b, d, c'),
\end{aligned}$$

where $a' = 2s - 3a$, $b' = 2s - 3b$, and $c' = 2s - 3c$. Because $a \le b \le c$, it follows that $c' \le b' \le a'$. Therefore $P^{(3)}$ dominates $P^{(2)}$ and $P^{(2)}$ dominates $P^{(1)}$.

9b. Because $\mathcal{C}(P^{(1)}) = (b, c, d, a')$ and $\mathcal{C}(Q^{(1)}) = (x, y, z, w')$, it is enough to show that $a' \le w'$. As in the solution to problem 5, a and a' are the two roots of the quadratic given by Descartes' Circle Formula:

$$(X + b + c + d)^2 = 2(X^2 + b^2 + c^2 + d^2).$$

Solve by completing the square:

$$\begin{aligned} X^2 - 2(b+c+d)X + 2(b^2+c^2+d^2) &= (b+c+d)^2; \\ \big(X - (b+c+d)\big)^2 &= 2(b+c+d)^2 - 2(b^2+c^2+d^2) \\ &= 4(bc+bd+cd). \end{aligned}$$

Thus a, $a' = b + c + d \pm 2\sqrt{bc + bd + cd}$.

Because $a \le b \le c \le d$, and only a can be less than zero, a must get the minus sign, and a' gets the plus sign:

$$a' = b + c + d + 2\sqrt{bc + bd + cd}.$$

Similarly,

$$w' = x + y + z + 2\sqrt{xy + xz + yz}.$$

Because P dominates Q, each term in the expression for a' is less than or equal to the corresponding term in the expression for w', thus $a' \le w'$.

9c. Because $\mathcal{C}(P^{(3)}) = (a, b, d, c')$ and $\mathcal{C}(Q^{(3)}) = (w, x, z, y')$, it suffices to show that $c' \le y'$. If $a \ge 0$, then the argument is exactly the same as in problem 9a, but if $a < 0$, then there is more to be done.

Arguing as in 9b, c, $c' = a + b + d \pm 2\sqrt{ab + ad + bd}$. If $a < 0$, then the other three circles are *internally* tangent to the circle of curvature a, so this circle has the largest radius. In particular, $\frac{1}{|a|} > \frac{1}{b}$. Thus $b > |a| = -a$, which shows that $a + b > 0$. Therefore c must get the minus sign, and c' gets the plus sign. The same argument applies to y and y'.

When $a < 0$, it is also worth considering whether the square roots are defined (and real). In fact, they are. Consider the diameters of the circles with curvatures b and d along the line through the centers of these circles. These two diameters form a single segment inside the circle with curvature a, so the sum of the diameters is at most the diameter of that circle: $\frac{2}{b} + \frac{2}{d} \le \frac{2}{|a|}$. It follows that $-ad - ab = |a|\,d + |a|\,b \le bd$, or $ab + ad + bd \ge 0$. This is the argument of the square root in the expressions for c and c'. An analogous argument shows that the radicands are nonnegative in the expressions for b and b'.

The foregoing shows that

$$c' = a + b + d + 2\sqrt{ab + ad + bd}$$

and, by an analogous argument for $w < 0$,

$$y' = w + x + z + 2\sqrt{wx + wz + xz}.$$

It remains to prove that $c' \le y'$. Note that only a and w may be negative; b, c, d, x, y, and z are all positive. There are three cases.

(i) If $0 \le a \le w$, then $ab \le wx$, $ad \le wz$, and $bd \le xz$, so $c' \le y'$.

(ii) If $a < 0 \le w$, then $ab + ad + bd \le bd$, and $bd \le xz \le wx + wz + xz$, so $c' \le y'$. (As noted above, both radicands are nonnegative.)

(iii) If $a \le w < 0$, then it has already been established that $a + b$ is positive. Analogously, $a + d$, $w + x$, and $w + z$ are positive. Furthermore, $a^2 \ge w^2$. Thus $(a+b)(a+d) - a^2 \le (w+x)(w+z) - w^2$, which establishes that $ab + ad + bd \le wx + wz + xz$, so $c' \le y'$.

10a. First, show by induction that every c-triangle on every day in the kingdom is proper. For the base case, both c-triangles at the end of day 0 have configuration $(-1, 2, 2, 3)$, so they are proper. For the inductive step, let T be a proper c-triangle. If $\mathcal{C}(T) = (a, b, c, d)$, then the three c-triangles obtained from T on the next day have configurations $\mathcal{C}(T^{(1)}) = (b, c, d, a')$, $\mathcal{C}(T^{(2)}) = (a, c, d, b')$, and $\mathcal{C}(T^{(3)}) = (a, b, d, c')$. According to problem 5, $c' = 2a + 2b - c + 2d = 2(a + b) + (d - c) + d$. Arguing as in the proof of 9c, $a + b \ge 0$. By the inductive hypothesis, T is proper, $d - c \ge 0$; therefore $c' \ge d$. Because $a' \ge b' \ge c'$, all three c-triangles are proper.

If $n = 0$, then both circles sold on day n have curvature 2, and this fits the formula: $n^2 + 2 = 0^2 + 2 = 2$.

Let P_0 be one of the c-triangles left at the end of day 0. For $m > 0$, let $P_m = P_{m-1}^{(3)}$. Use induction to prove the following two claims:

(i) $\mathcal{C}(P_m) = (-1, 2, m^2 + 2, (m + 1)^2 + 2)$.

(ii) P_m dominates all c-triangles left at the end of day m.

For the moment, grant these two claims. Then (ii) implies that the incircle of P_{n-1} is at least as large as any plot sold on day n, and (i) shows that this incircle has curvature $n^2 + 2$.

For the base case, both c-triangles at the end of day 0 have associated circle configuration $\mathcal{C}(P_0) = (-1, 2, 2, 3) = (-1, 2, 0^2 + 2, 1^2 + 2)$, so either dominates the other.

For $m > 0$, assume inductively that $\mathcal{C}(P_{m-1}) = (-1, 2, (m - 1)^2 + 2, m^2 + 2) = (a, b, c, d)$. Because $P_m = P_{m-1}^{(3)}$, $\mathcal{C}(P_m) = (a, b, d, c') = (-1, 2, m^2 + 2, c')$. Use algebra and the result of problem 5 to obtain $c' = 2(a + b + c + d) - 3c = (m + 1)^2 + 2$. This completes the inductive step for (i).

Now let Q be any c-triangle left at the end of day $m - 1$, with $\mathcal{C}(Q) = (x, y, z, w)$. Any c-triangle left at the end of day m is of the form $Q^{(1)}$, $Q^{(2)}$, or $Q^{(3)}$ for some such Q. By the inductive hypothesis, P_{m-1} dominates Q. It has already been established that these c-triangles are both proper, so the results of problem 9 apply. By 9c, $P_m = P_{m-1}^{(3)}$ dominates $Q^{(3)}$, and by 9a, $Q^{(3)}$ dominates $Q^{(1)}$ and $Q^{(2)}$. This completes the inductive step for (ii).

10b. Let R_n be a c-triangle with configuration

$$\mathcal{C}(R_n) = (2\rho^{n-2}, 2\rho^{n-1}, 2\rho^n, 2\rho^{n+1}).$$

According to problem 4, these four numbers (a geometric progression with common ratio ρ) satisfy Descartes' Circle Formula, so there is such a c-triangle. The following inductive argument proves that for $n \ge 2$, each c-triangular plot remaining in the kingdom at the end of day n dominates R_n.

For the base case, problem 6 shows that the following are sufficient: $2 \leq 2\rho^0$, $3 \leq 2\rho^1$, $15 \leq 2\rho^2$, and $38 \leq 2\rho^3$. In fact, it is enough to calculate $\phi = \frac{1+\sqrt{5}}{2} > 1.6$, $\sqrt{\phi} > 1.2$, $\rho = \phi + \sqrt{\phi} > 2.8$ to conclude that $2\rho > 5.6$, $2\rho^2 > 15.68$, and $2\rho^3 > 43.9$. These same calculations show that the main result is true for $n \leq 2$.

For the inductive step, let T be a c-triangle in the kingdom remaining at the end of day n. The inductive hypothesis is that T dominates R_n. By problems 9a and 9b, all the c-triangles obtained from T on day $n+1$ dominate $R_n^{(1)}$. All that remains is to show that $R_n^{(1)}$ has the right configuration, so that $R_{n+1} = R_n^{(1)}$.

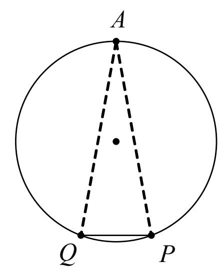
One approach is to use the formulas from problems 4a and 5 to show that $2(2\rho^{n-2} + 2\rho^{n-1} + 2\rho^n + 2\rho^{n+1}) - 3 \cdot 2\rho^{n-2} = 2\rho^{n+2}$. A method that avoids calculation is to note that $\mathcal{C}(R_n^{(1)}) = (2\rho^{n-1}, 2\rho^n, 2\rho^{n+1}, x)$ for some x. Because Descartes' Circle Formula is quadratic in x, there are at most two possibilities. According to 4b, two solutions are given by $x = 2\rho^{n-2}$ and $x = 2\rho^{n+2}$, because both of these give geometric progressions with common ratio ρ. The first corresponds to R_n, so the second must correspond to ${R_n}^{(1)}$. This completes the induction.

If $n \leq 2$, then (as already noted) the solution to problem 6 shows that the curvature of the smallest plot sold on day n does not exceed $2\rho^n$. If $n > 2$, then this smallest plot is the incircle of some c-triangle T that remains at the end of day $n-1$, with $n-1 \geq 2$. Because T dominates R_{n-1}, the curvature of its incircle does not exceed that of R_{n-1}, which is $2\rho^n$.

2010 Relay Problems

R1-1. If A, R, M, and L are positive integers such that $A^2 + R^2 = 20$ and $M^2 + L^2 = 10$, compute the product $A \cdot R \cdot M \cdot L$.

R1-2. Let $T = TNYWR$. A regular n-gon is inscribed in a circle; P and Q are consecutive vertices of the polygon, and A is another vertex of the polygon as shown. If $m\angle APQ = m\angle AQP = T \cdot m\angle QAP$, compute the value of n.



R1-3. Let $T = TNYWR$. Compute the last digit, in base 10, of the sum

$$T^2 + (2T)^2 + (3T)^2 + \ldots + \left(T^2\right)^2.$$

R2-1. A fair coin is flipped n times. Compute the smallest positive integer n for which the probability that the coin has the same result every time is less than 10%.

R2-2. Let $T = TNYWR$. Compute the smallest positive integer n such that there are at least T positive integers in the domain of $f(x) = \sqrt{-x^2 - 2x + n}$.

R2-3. Let $T = TNYWR$. Compute the smallest positive real number x such that $\dfrac{\lfloor x \rfloor}{x - \lfloor x \rfloor} = T$.

2010 Relay Answers

R1-1. 24

R1-2. 49

R1-3. 5

R2-1. 5

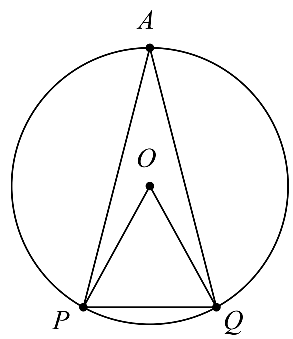
R2-2. 35

R2-3. $\dfrac{36}{35}$

2010 Relay Solutions

R1-1. The only positive integers whose squares sum to 20 are 2 and 4. The only positive integers whose squares sum to 10 are 1 and 3. Thus $A \cdot R = 8$ and $M \cdot L = 3$, so $A \cdot R \cdot M \cdot L = \mathbf{24}$.

R1-2. Let $\mathrm{m}\angle A = x$. Then $\mathrm{m}\angle P = \mathrm{m}\angle Q = Tx$, and $(2T+1)x = 180^\circ$, so $x = \dfrac{180^\circ}{2T+1}$. Let O be the center of the circle, as shown below.



Then $\mathrm{m}\angle POQ = 2\mathrm{m}\angle PAQ = 2\left(\dfrac{180^\circ}{2T+1}\right) = \dfrac{360^\circ}{2T+1}$. Because $\mathrm{m}\angle POQ = \dfrac{360^\circ}{n}$, the denominators must be equal: $n = 2T+1$. Substitute $T = 24$ to find $n = \mathbf{49}$.

R1-3. Let S be the required sum. Factoring T^2 from the sum yields

$$\begin{aligned} S &= T^2\left(1+4+9+\ldots+T^2\right) \\ &= T^2\left(\frac{T(T+1)(2T+1)}{6}\right) \\ &= \frac{T^3(T+1)(2T+1)}{6}. \end{aligned}$$

Further analysis makes the final computation simpler. If $T \equiv 0, 2,$ or 3 mod 4, then S is even. Otherwise, S is odd. And if $T \equiv 0, 2,$ or 4 mod 5, then $S \equiv 0$ mod 5; otherwise, $S \equiv 1$ mod 5. These observations yield the following table:

T mod 4	T mod 5	S mod 10
$0, 2, 3$	$0, 2, 4$	0
$0, 2, 3$	$1, 3$	6
1	$0, 2, 4$	5
1	$1, 3$	1

Because $T = 49$, the value corresponds to the third case above; the last digit is **5**.

R2-1. After the first throw, the probability that the succeeding $n-1$ throws have the same result is $\frac{1}{2^{n-1}}$. Thus $\frac{1}{2^{n-1}} < \frac{1}{10} \Rightarrow 2^{n-1} > 10 \Rightarrow n-1 \geq 4$, so $n = \mathbf{5}$ is the smallest possible value.

R2-2. Completing the square under the radical yields $\sqrt{n+1-(x+1)^2}$. The larger zero of the radicand is $-1+\sqrt{n+1}$, and the smaller zero is negative because $-1-\sqrt{n+1} < 0$, so the T positive integers in the domain of f must be $1, 2, 3, \ldots, T$. Therefore $-1+\sqrt{n+1} \geq T$. Hence $\sqrt{n+1} \geq T+1$, and $n+1 \geq (T+1)^2$. Therefore $n \geq T^2+2T$, and substituting $T = 5$ yields $n \geq 35$. So $n = \mathbf{35}$ is the smallest such value.

R2-3. If $\frac{\lfloor x \rfloor}{x - \lfloor x \rfloor} = T$, the equation can be rewritten as follows:

$$\begin{aligned}
\frac{x - \lfloor x \rfloor}{\lfloor x \rfloor} &= \frac{1}{T} \\
\frac{x}{\lfloor x \rfloor} - 1 &= \frac{1}{T} \\
\frac{x}{\lfloor x \rfloor} &= \frac{T+1}{T}.
\end{aligned}$$

Now $0 < x < 1$ is impossible because it makes the numerator of the original expression 0. To make x as small as possible, place it in the interval $1 < x < 2$, so that $\lfloor x \rfloor = 1$. Then $x = \frac{T+1}{T}$. When $T = 35$, $x = \frac{36}{35}$.

2010 Tiebreaker Problems

TB-1. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set T be the set of all subsets of S (including the empty set and S itself). Let $t_1,\ t_2,\ t_3$ be elements of T, not necessarily distinct. The ordered triple (t_1, t_2, t_3) is called *satisfactory* if either

(a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, or

(b) $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

Compute the number of satisfactory ordered triples (t_1, t_2, t_3).

TB-2. Let $ABCD$ be a parallelogram with $\angle ABC$ obtuse. Let $\overline{BE}$ be the altitude to side $\overline{AD}$ of $\triangle ABD$. Let X be the point of intersection of $\overline{AC}$ and $\overline{BE}$, and let F be the point of intersection of $\overline{AB}$ and $\overleftrightarrow{DX}$. If $BC = 30$, $CD = 13$, and $BE = 12$, compute the ratio $\dfrac{AC}{AF}$.

TB-3. Compute the sum of all positive two-digit factors of $2^{32} - 1$.

2010 Tiebreaker Answers

TB-1. 31186

TB-2. $\dfrac{222}{13}$

TB-3. 168

2010 Tiebreaker Solutions

TB-1. Let $T_1 = \{(t_1, t_2, t_3) \mid t_1 \subseteq t_3 \text{ and } t_2 \subseteq t_3\}$ and let $T_2 = \{(t_1, t_2, t_3) \mid t_3 \subseteq t_1 \text{ and } t_3 \subseteq t_2\}$. Notice that if $(t_1, t_2, t_3) \in T_1$, then $(S \setminus t_1, S \setminus t_2, S \setminus t_3) \in T_2$, so that $|T_1| = |T_2|$. To count T_1, note that if $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, then $t_1 \cup t_2 \subseteq t_3$. Now each set t_3 has $2^{|t_3|}$ subsets; t_1 and t_2 could be any of these, for a total of $\left(2^{|t_3|}\right)^2 = 4^{|t_3|}$ possibilities given a particular subset t_3. For $n = 0, 1, \ldots, 6$, if $|t_3| = n$, there are $\binom{6}{n}$ choices for the elements of t_3. So the total number of elements in T_1 is

$$\begin{aligned} |T_1| &= \sum_{k=0}^{6} \binom{6}{k} 4^k \\ &= (4+1)^6 = 15625, \end{aligned}$$

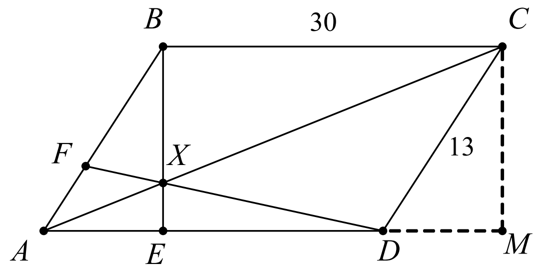
by the Binomial Theorem. However, $T_1 \cap T_2 \neq \emptyset$, because if $t_1 = t_2 = t_3$, the triple (t_1, t_2, t_3) satisfies both conditions and is in both sets. Therefore there are 64 triples that are counted in both sets. So $|T_1 \cup T_2| = 2 \cdot 15625 - 64 = \mathbf{31186}$.

Alternate Solution: Let T_1 and T_2 be defined as above. Then count $|T_1|$ based on the number n of elements in $t_1 \cup t_2$. There are $\binom{6}{n}$ ways to choose those n elements. For each element a in $t_1 \cup t_2$, there are three possibilities: $a \in t_1$ but not t_2, or $a \in t_2$ but not t_1, or $a \in t_1 \cap t_2$. Then for each element b in $S \setminus (t_1 \cup t_2)$, there are two possibilities: either $b \in t_3$, or $b \notin t_3$. Combine these observations in the table below:

$\|t_1 \cup t_2\|$	Choices for $t_1 \cup t_2$	Ways of dividing between t_1 and t_2	$\|S \setminus (t_1 \cup t_2)\|$	Choices for t_3	Total
0	1	1	6	2^6	64
1	6	3	5	2^5	576
2	15	3^2	4	2^4	2160
3	20	3^3	3	2^3	4320
4	15	3^4	2	2^2	4860
5	6	3^5	1	2^1	2916
6	1	3^6	0	2^0	729

The total is 15625, so $|T_1| = |T_2| = 15625$. As noted in the first solution, there are 64 triples that are counted in both T_1 and T_2, so $|T_1 \cup T_2| = 2 \cdot 15625 - 64 = \mathbf{31186}$.

TB-2. Extend $\overline{AD}$ to a point M such that $\overline{CM} \parallel \overline{BE}$ as shown below.



Because $CD = AB = 13$ and $BE = 12 = CM$, $AE = DM = 5$. Then $AC = \sqrt{35^2 + 12^2} = \sqrt{1369} = 37$. Because $\overline{EX} \parallel \overline{CM}$, $XE/CM = AE/AM = \frac{1}{7}$. Thus $EX = \frac{12}{7}$ and $XB = \frac{72}{7}$, from which $EX/XB = \frac{1}{6}$. Apply Menelaus's Theorem to $\triangle AEB$ and Menelaus line $\overline{FD}$:

$$\begin{aligned}
\frac{AD}{ED} \cdot \frac{EX}{XB} \cdot \frac{BF}{FA} &= 1 \\
\frac{30}{25} \cdot \frac{1}{6} \cdot \frac{13 - FA}{FA} &= 1 \\
\frac{13 - FA}{FA} &= 5.
\end{aligned}$$

Thus $FA = \dfrac{13}{6}$. The desired ratio is:

$$\frac{37}{13/6} = \mathbf{\frac{222}{13}}.$$

Alternate Solution: After calculating AC as above, draw $\overline{BD}$, intersecting $\overline{AC}$ at Y. Because the diagonals of a parallelogram bisect each other, $DY = YB$. Then apply Ceva's Theorem to $\triangle ABD$ and concurrent cevians $\overline{AY}, \overline{BE}, \overline{DF}$:

$$\begin{aligned}
\frac{AE}{ED} \cdot \frac{DY}{YB} \cdot \frac{BF}{FA} &= 1 \\
\frac{5}{25} \cdot 1 \cdot \frac{13 - FA}{FA} &= 1.
\end{aligned}$$

Thus $FA = \dfrac{13}{6}$, and the desired ratio is $\mathbf{\dfrac{222}{13}}$.

Alternate Solution: By AA similarity, note that $\triangle AFX \sim \triangle CDX$ and $\triangle AEX \sim \triangle CBX$. Thus $\frac{AF}{CD} = \frac{AX}{XC} = \frac{AE}{CB}$. Thus $\frac{AF}{13} = \frac{AE}{CB} = \frac{5}{30}$, so $AF = \frac{13}{6}$, and the answer follows after calculating AC, as in the first solution.

TB-3. Using the difference of squares, $2^{32} - 1 = (2^{16} - 1)(2^{16} + 1)$. The second factor, $2^{16} + 1$, is the Fermat prime 65537, so continue with the first factor:

$$\begin{aligned}
2^{16} - 1 &= \left(2^8 + 1\right)\left(2^8 - 1\right) \\
2^8 - 1 &= \left(2^4 + 1\right)\left(2^4 - 1\right) \\
2^4 - 1 &= 15 = 3 \cdot 5.
\end{aligned}$$

Because the problem does not specify that the two-digit factors must be prime, the possible two-digit factors are 17, $3{\cdot}17 = 51$, $5{\cdot}17 = 85$ and $3{\cdot}5 = 15$, for a sum of $17+51+85+15 =$ **168**.

2010 Super Relay Problems

1. Let N be a perfect square between 100 and 400, inclusive. What is the only digit that <u>cannot</u> appear in N?

2. Let $T = TNYWR$. Let A and B be distinct digits in base T, and let N be the largest number of the form $\underline{A}\,\underline{B}\,\underline{A}_T$. Compute the value of N in base 10.

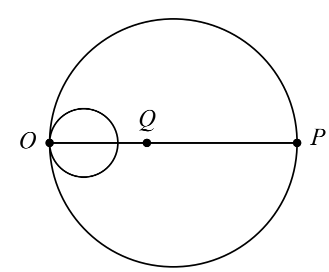
3. Let $T = TNYWR$. Given a nonzero integer n, let $f(n)$ denote the sum of all numbers of the form i^d, where $i = \sqrt{-1}$, and d is a divisor (positive or negative) of n. Compute $f(2T + 1)$.

4. Let $T = TNYWR$. Compute the real value of x for which there exists a solution to the system of equations

$$\begin{aligned} x + y &= 0 \\ x^3 - y^3 &= 54 + T. \end{aligned}$$

5. Let $T = TNYWR$. In $\triangle ABC$, $AC = T^2$, $\mathrm{m}\angle ABC = 45^\circ$, and $\sin \angle ACB = \frac{8}{9}$. Compute AB.

6. Let $T = TNYWR$. In the diagram at right, the smaller circle is internally tangent to the larger circle at point O, and $\overline{OP}$ is a diameter of the larger circle. Point Q lies on $\overline{OP}$ such that $PQ = T$, and $\overline{PQ}$ does <u>not</u> intersect the smaller circle. If the larger circle's radius is three times the smaller circle's radius, find the least possible <u>integral</u> radius of the larger circle.



7. Let $T = TNYWR$. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic progression, d is the common difference, $a_T = 10$, and $a_K = 2010$, where $K > T$. If d is an integer, compute the value of K such that $|K - d|$ is minimal.

8. Let A be the number you will receive from position 7, and let B be the number you will receive from position 9. There are exactly two ordered pairs of real numbers $(x_1, y_1), (x_2, y_2)$ that satisfy both $|x + y| = 6(\sqrt{A} - 5)$ and $x^2 + y^2 = B^2$. Compute $|x_1| + |y_1| + |x_2| + |y_2|$.

9. Let $T = TNYWR$. In triangle ABC, the altitude from A to $\overline{BC}$ has length $\sqrt{T}$, $AB = AC$, and $BC = T - K$, where K is the real root of the equation $x^3 - 8x^2 - 8x - 9 = 0$. Compute the length AB.

10. Let $T = TNYWR$. A cube has volume $T - 2$. The cube's surface area equals one-eighth the surface area of a $2 \times 2 \times n$ rectangular prism. Compute n.

11. Let $T = TNYWR$, and let K be the sum of the digits of T. Let A_n be the number of ways to tile a $1 \times n$ rectangle using 1×3 and 1×1 tiles that do not overlap. Tiles of both types need not be used; for example, $A_3 = 2$ because a 1×3 rectangle can be tiled with three 1×1 tiles or one 1×3 tile. Compute the smallest value of n such that $A_n \geq K$.

12. Let $T = TNYWR$, and let $K = T + 2$. Compute the largest K-digit number which has distinct digits and is a multiple of 63.

13. Let $T = TNYWR$. Suppose that a, b, c, and d are real numbers so that $\log_a c = \log_b d = T$. Compute

$$\frac{\log_{\sqrt{ab}}(cd)^3}{\log_a c + \log_b d}.$$

14. Let $T = TNYWR + 2000$. Given that $\mathsf{A}, \mathsf{D}, \mathsf{E}, \mathsf{H}, \mathsf{S}$, and W are distinct digits, and that $\underline{\mathsf{W}}\,\underline{\mathsf{A}}\,\underline{\mathsf{D}}\,\underline{\mathsf{E}} + \underline{\mathsf{A}}\,\underline{\mathsf{S}}\,\underline{\mathsf{H}} = T$, what is the largest possible value of $\mathsf{D} + \mathsf{E}$?

15. Let $f(x) = 2^x + x^2$. Compute the smallest integer $n > 10$ such that $f(n)$ and $f(10)$ have the same units digit.

2010 Super Relay Answers

1. 7
2. 335
3. 0
4. 3
5. $8\sqrt{2}$
6. 9
7. 49
8. 24
9. $6\sqrt{2}$
10. 23
11. 10
12. 98721
13. 3
14. 9
15. 30

2010 Super Relay Solutions

1. When the perfect squares between 100 and 400 inclusive are listed out, every digit except **7** is used. Note that the perfect squares 100, 256, 289, 324 use each of the other digits.

2. To maximize $\underline{A}\,\underline{B}\,\underline{A}_T$ with $A \neq B$, let $A = T - 1$ and $B = T - 2$. Then $\underline{A}\,\underline{B}\,\underline{A}_T = (T-1)\cdot T^2 + (T-2)\cdot T^1 + (T-1)\cdot T^0 = T^3 - T - 1$. With $T = 7$, the answer is **335**.

3. Let $n = 2^m r$, where r is odd. If $m = 0$, then n is odd, and for each d that divides n, $i^d + i^{-d} = i^d + \dfrac{i^d}{(i^2)^d} = 0$, hence $f(n) = 0$ when n is odd. If $m = 1$, then for each d that divides n, $i^d + i^{-d}$ equals 0 if d is odd, and -2 if d is even. Thus when n is a multiple of 2 but not 4, $f(n) = -2P$, where P is the number of positive odd divisors of n. Similarly, if $m = 2$, then $f(n) = 0$, and in general, $f(n) = 2(m-2)P$ for $m \geq 1$. Because T is an integer, $2T + 1$ is odd, hence the answer is **0**. [Note: If $r = p_1^{a_1} p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$, where the p_i are distinct odd primes, it is well known that $P = (a_1+1)(a_2+1)\ldots(a_k+1)$.]

4. Plug $y = -x$ into the second equation to obtain $x = \sqrt[3]{\dfrac{54+T}{2}}$. With $T = 0, x = \sqrt[3]{27} = \mathbf{3}$.

5. From the Law of Sines, $\dfrac{AB}{\sin\angle ACB} = \dfrac{AC}{\sin\angle ABC}$. Thus $AB = \dfrac{8}{9}\cdot\dfrac{T^2}{1/\sqrt{2}} = \dfrac{8\sqrt{2}}{9}\cdot T^2$. With $T = 3$, $AB = \mathbf{8\sqrt{2}}$.

6. Let r be the radius of the smaller circle. Then the conditions defining Q imply that $PQ = T < 4r$. With $T = 8\sqrt{2}$, note that $r > 2\sqrt{2} \rightarrow 3r > 6\sqrt{2} = \sqrt{72}$. The least integer greater than $\sqrt{72}$ is **9**.

7. Note that $a_T = a_1 + (T-1)d$ and $a_K = a_1 + (K-1)d$, hence $a_K - a_T = (K-T)d = 2010 - 10 = 2000$. Thus $K = \frac{2000}{d} + T$, and to minimize $|T + \frac{2000}{d} - d|$, choose a positive integer d such that $\frac{2000}{d}$ is also an integer and $\frac{2000}{d} - d$ is as close as possible to $-T$. Note that $T > 0$, so $\frac{2000}{d} - d$ should be negative, i.e., $d^2 > 2000$ or $d > 44$. The value of T determines how far apart $\frac{2000}{d}$ and d need to be. For example, if T is close to zero, then choose d such that $\frac{2000}{d}$ and d are close to each other. With $T = 9$, take $d = 50$ so that $\frac{2000}{d} = 40$ and $|K - d| = |49 - 50| = 1$. Thus $K = \mathbf{49}$.

8. Note that the graph of $x^2 + y^2 = B^2$ is a circle of radius $|B|$ centered at $(0,0)$ (as long as $B^2 > 0$). Also note that the graph of $|x+y| = 6(\sqrt{A} - 5)$ is either the line $y = -x$ if $A = 25$, or the graph consists of two parallel lines with slope -1 if $A > 25$. In the former case, the

line $y = -x$ intersects the circle at the points $\left(\pm\frac{|B|}{\sqrt{2}}, \mp\frac{|B|}{\sqrt{2}}\right)$. In the latter case, the graph is symmetric about the origin, and in order to have exactly two intersection points, each line must be tangent to the circle, and the tangency points are $\left(\frac{|B|}{\sqrt{2}}, \frac{|B|}{\sqrt{2}}\right)$ and $\left(-\frac{|B|}{\sqrt{2}}, -\frac{|B|}{\sqrt{2}}\right)$. In either case, $|x_1|+|y_1|+|x_2|+|y_2| = 2\sqrt{2}\cdot|B|$, and in the case where the graph is two lines, this is also equal to $12(\sqrt{A}-5)$. Thus if $A \neq 25$, then only one of A or B is needed to determine the answer. With $A = 49$ and $B = 6\sqrt{2}$, the answer is $2\sqrt{2}\cdot 6\sqrt{2} = 12(\sqrt{49}-5) = \mathbf{24}$.

9. Rewrite the equation as $x^3 - 1 = 8(x^2+x+1)$, so that $(x-1)(x^2+x+1) = 8(x^2+x+1)$. Because x^2+x+1 has no real zeros, it can be canceled from both sides of the equation to obtain $x - 1 = 8$ or $x = 9$. Hence $BC = T - 9$, and $AB^2 = (\sqrt{T})^2 + \left(\frac{T-9}{2}\right)^2 = T + \left(\frac{T-9}{2}\right)^2$. Substitute $T = 23$ to obtain $AB = \sqrt{72} = \mathbf{6\sqrt{2}}$.

10. The cube's side length is $\sqrt[3]{T}$, so its surface area is $6\sqrt[3]{T^2}$. The rectangular prism has surface area $2(2\cdot 2+2\cdot n+2\cdot n) = 8+8n$, thus $6\sqrt[3]{T^2} = 1+n$. With $T = 8$, $n = 6\sqrt[3]{64}-1 = \mathbf{23}$.

11. Consider the rightmost tile of the rectangle. If it's a 1×1 tile, then there are A_{n-1} ways to tile the remaining $1\times(n-1)$ rectangle, and if it's a 1×3 tile, then there are A_{n-3} ways to tile the remaining $1\times(n-3)$ rectangle. Hence $A_n = A_{n-1}+A_{n-3}$ for $n > 3$, and $A_1 = A_2 = 1, A_3 = 2$. Continuing the sequence gives the following values:

n	1	2	3	4	5	6	7	8	9	10
A_n	1	1	2	3	4	6	9	13	19	28

With $T = 98721, K = 27$, hence the answer is **10**.

12. Let N_K be the largest K-digit number which has distinct digits and is a multiple of 63. It can readily be verified that $N_1 = 0, N_2 = 63$, and $N_3 = 945$. For $K > 3$, compute N_K using the following strategy: start with the number $M_0 = \underline{9}\,\underline{8}\,\underline{7}\ldots\underline{(10-K)}$; let M_1 be the largest multiple of 63 not exceeding M_0. That is, to compute M_1, divide M_0 by 63 and discard the remainder: $M_0 = 1587\cdot 63+44$, so $M_1 = M_0 - 44 = 1587\cdot 63$. If M_1 has distinct digits, then $N_K = M_1$. Otherwise, let $M_2 = M_1 - 63, M_3 = M_2 - 63$, and so on; then N_K is the first term of the sequence $M_1, M_2, M_3, \ldots$ that has distinct digits. Applying this strategy gives $N_4 = 9765, N_5 = 98721, N_6 = 987651$, and $N_7 = 9876510$. With $T = 3$, $K = 5$, and the answer is **98721**.

13. Note that $a^T = c$ and $b^T = d$, thus $(ab)^T = cd$. Further note that $(ab)^{3T} = (\sqrt{ab})^{6T} = (cd)^3$, thus $\log_{\sqrt{ab}}(cd)^3 = 6T$. Thus the given expression simplifies to $\frac{6T}{2T} = \mathbf{3}$ (as long as $T \neq 0$).

14. First note that if $T \geq 10000$, then $\mathsf{W} = 9$ and $\mathsf{A} \geq 5$. If $T < 10000$ and x is the leading digit of T, then either $\mathsf{W} = x$ and $\mathsf{A} \leq 4$ or $\mathsf{W} = x-1$ and $\mathsf{A} \geq 5$. With $T = 2030$, either $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 20$

or $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 15$. In either case, $\underline{\mathsf{D}}\,\underline{\mathsf{E}} + \underline{\mathsf{S}}\,\underline{\mathsf{H}} = 30$. Considering values of $\mathsf{D} + \mathsf{E}$, there are three possibilities to consider:

$\mathsf{D} + \mathsf{E} = 11$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 29$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 01$, which duplicates digits;
$\mathsf{D} + \mathsf{E} = 10$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 28$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 02$ or $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 19$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 11$, both of which duplicate digits;
$\mathsf{D} + \mathsf{E} = 9$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 27$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 03$, in which no digits are duplicated if $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 15$.

Therefore the answer is **9**.

15. The units digit of $f(10)$ is the same as the units digit of 2^{10}. Because the units digits of powers of 2 cycle in groups of four, the units digit of 2^{10} is 4, so the units digit of $f(10)$ is 4. Note that n must be even, otherwise, the units digit of $f(n)$ is odd. If n is a multiple of 4, then 2^n has 6 as its units digit, which means that n^2 would need to have a units digit of 8, which is impossible. Thus n is even, but is not a multiple of 4. This implies that the units digit of 2^n is 4, and so n^2 must have a units digit of 0. The smallest possible value of n is therefore **30**.

2011 Contest

Contents

2011 Team Problems

T-1. If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

T-2. Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - {a_k}^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

T-3. Compute the base b for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}\,_b$, for some base-$b$ digits X, Y, Z.

T-4. Some portions of the line $y = 4x$ lie below the curve $y = 10\pi \sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi \sin^2 x$.

T-5. In equilateral hexagon $ABCDEF$, $\mathrm{m}\angle A = 2\mathrm{m}\angle C = 2\mathrm{m}\angle E = 5\mathrm{m}\angle D = 10\mathrm{m}\angle B = 10\mathrm{m}\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

T-6. The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

T-7. The function f satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers n, and $f(n) > 0$ for all positive integers n. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

T-8. Frank Narf accidentally read a degree n polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1} x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \leq n \leq 7$, compute the number of such polynomials such that $\mathrm{GCD}(a_0, a_1, \ldots, a_n) = 1$.

T-9. Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

T-10. Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle θ. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.

2011 Team Answers

T-1. $\dfrac{15}{4}$

T-2. 253

T-3. 20

T-4. $\dfrac{5\pi}{4}\sqrt{17}$

T-5. $\dfrac{9}{2}$

T-6. 96

T-7. 4096

T-8. 70

T-9. 11

T-10. $\dfrac{5}{4}$

2011 Team Solutions

T-1. The common ratio in the geometric sequence $1, x, y$ is $\frac{x}{1} = x$, so $y = x^2$. The arithmetic sequence $x, y, 3$ has a common difference, so $y - x = 3 - y$. Substituting $y = x^2$ in the equation yields

$$\begin{aligned} x^2 - x &= 3 - x^2 \\ 2x^2 - x - 3 &= 0, \end{aligned}$$

from which $x = \frac{3}{2}$ or -1. The respective values of y are $y = x^2 = \frac{9}{4}$ or 1. Thus the possible values of $x + y$ are $\frac{15}{4}$ and 0, so the answer is $\mathbf{\frac{15}{4}}$.

T-2. The requirement that a_n be the smallest positive value of $n - {a_k}^2$ for $k < n$ is equivalent to determining the largest value of a_k such that ${a_k}^2 < n$. For $n = 3$, use either $a_1 = a_2 = 1$ to find $a_3 = 3 - 1^2 = 2$. For $n = 4$, the strict inequality eliminates a_3, so $a_4 = 4 - 1^2 = 3$, but a_3 can be used to compute $a_5 = 5 - 2^2 = 1$. In fact, until $n = 10$, the largest allowable prior value of a_k is $a_3 = 2$, yielding the values $a_6 = 2$, $a_7 = 3$, $a_8 = 4$, $a_9 = 5$. In general, this pattern continues: from $n = m^2 + 1$ until $n = (m + 1)^2$, the values of a_n increase from 1 to $2m + 1$. Let $S_m = 1 + 2 + \cdots + (2m + 1)$. Then the problem reduces to computing $S_0 + S_1 + \cdots + S_6 + 1$, because $a_{49} = 49 - 6^2$ while $a_{50} = 50 - 7^2 = 1$. $S_m = \frac{(2m+1)(2m+2)}{2} = 2m^2 + 3m + 1$, so

$$\begin{aligned} S_0 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 &= 1 + 6 + 15 + 28 + 45 + 66 + 91 \\ &= 252. \end{aligned}$$

Therefore the desired sum is $252 + 1 = \mathbf{253}$.

T-3. Write $253_b \cdot 341_b = (2b^2 + 5b + 3)(3b^2 + 4b + 1) = 6b^4 + 23b^3 + 31b^2 + 17b + 3$. Compare the coefficients in this polynomial to the digits in the numeral $\underline{7}\ \underline{4}\ \underline{X}\ \underline{Y}\ \underline{Z}$. In the polynomial, the coefficient of b^4 is 6, so there must be a carry from the b^3 place to get the $7b^4$ in the numeral. After the carry, there should be no more than 4 left for the coefficient of b^3 as only one b is carried. Therefore $23 - b \leq 4$ or $b \geq 19$. By comparing digits, note that $Z = 3$. Then

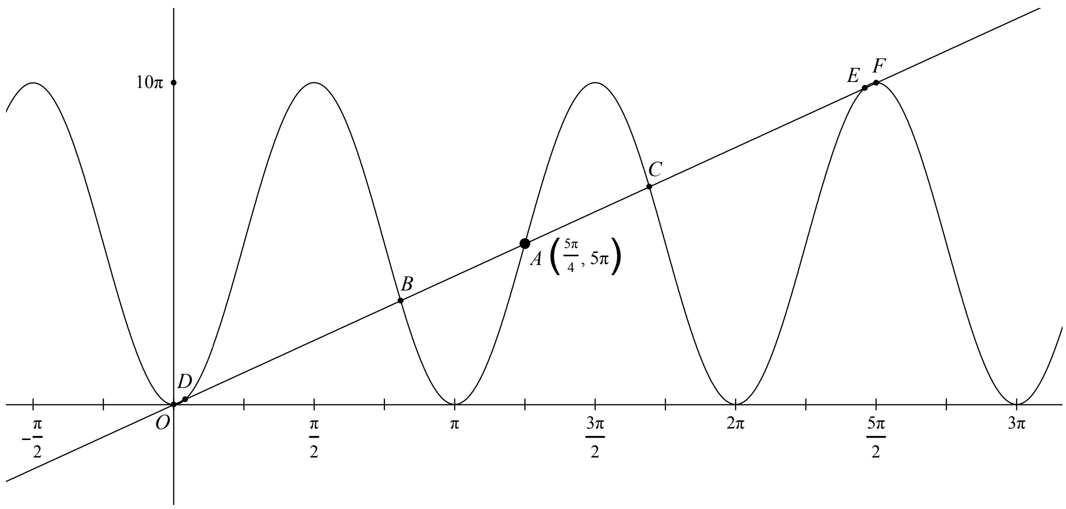
$$\begin{aligned} 6b^4 + 23b^3 + 31b^2 + 17b &= \underline{7}\ \underline{4}\ \underline{X}\ \underline{Y}\ \underline{0} \\ &= 7b^4 + 4b^3 + X \cdot b^2 + Y \cdot b. \end{aligned}$$

Because $b > 0$, this equation can be simplified to

$$b^3 + X \cdot b + Y = 19b^2 + 31b + 17.$$

Thus $Y = 17$ and $b^2 + X = 19b + 31$, from which $b(b - 19) = 31 - X$. The expression on the left side is positive (because $b > 19$) and the expression on the right side is at most 31 (because $X > 0$), so the only possible solution is $b = 20$, $X = 11$. The answer is **20**.

T-4. Notice first that all intersections of the two graphs occur in the interval $0 \le x \le \frac{5\pi}{2}$, because the maximum value of $10\pi\sin^2 x$ is 10π (at odd multiples of $\frac{\pi}{2}$), and $4x > 10\pi$ when $x > \frac{5\pi}{2}$. The graphs are shown below.



Within that interval, both graphs are symmetric about the point $A = \left(\frac{5\pi}{4}, 5\pi\right)$. For the case of $y = 10\pi\sin^2 x$, this symmetry can be seen by using the power-reducing identity $\sin^2 x = \frac{1-\cos 2x}{2}$. Then the equation becomes $y = 5\pi - 5\pi\cos 2x$, which has amplitude 5π about the line $y = 5\pi$, and which crosses the line $y = 5\pi$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$. Label the points of intersection A, B, C, D, E, F, and O as shown. Then $\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{CE}$, and $\overline{OD} \cong \overline{EF}$. Thus

$$\begin{aligned} BD + AC + EF &= OD + DB + BA \\ &= OA. \end{aligned}$$

By the Pythagorean Theorem,

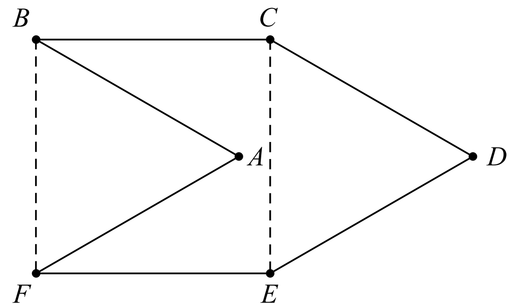
$$\begin{aligned} OA &= \sqrt{\left(\frac{5\pi}{4}\right)^2 + (5\pi)^2} \\ &= \frac{5\pi}{4}\sqrt{1^2 + 4^2} \\ &= \boldsymbol{\frac{5\pi}{4}\sqrt{17}}. \end{aligned}$$

T-5. Let $\mathrm{m}\angle B = \alpha$. Then the sum of the measures of the angles in the hexagon is:

$$\begin{aligned} 720^\circ &= \mathrm{m}\angle A + \mathrm{m}\angle C + \mathrm{m}\angle E + \mathrm{m}\angle D + \mathrm{m}\angle B + \mathrm{m}\angle F \\ &= 10\alpha + 5\alpha + 5\alpha + 2\alpha + \alpha + \alpha = 24\alpha. \end{aligned}$$

Thus $30° = \alpha$ and $\text{m}\angle A = 300°$, so the exterior angle at A has measure $60° = \text{m}\angle D$. Further, because $AB = CD$ and $DE = AF$, it follows that $\triangle CDE \cong \triangle BAF$. Thus

$$[ABCDEF] = [ABCEF] + [CDE] = [ABCEF] + [ABF] = [BCEF].$$



To compute $[BCEF]$, notice that because $\text{m}\angle D = 60°$, $\triangle CDE$ is equilateral. In addition,

$$\begin{aligned} 150° &= \text{m}\angle BCD \\ &= \text{m}\angle BCE + \text{m}\angle DCE = \text{m}\angle BCE + 60°. \end{aligned}$$

Therefore $\text{m}\angle BCE = 90°$. Similarly, because the hexagon is symmetric, $\text{m}\angle CEF = 90°$, so quadrilateral $BCEF$ is actually a square with side length 3. Thus $CE = \frac{BE}{\sqrt{2}} = \frac{3}{\sqrt{2}}$, and $[ABCDEF] = [BCEF] = \mathbf{\frac{9}{2}}$.

Alternate Solution: Calculate the angles of the hexagon as in the first solution. Then proceed as follows.

First, $ABCDEF$ can be partitioned into four congruent triangles. Because the hexagon is equilateral and $\text{m}\angle ABC = \text{m}\angle AFE = 30°$, it follows that $\triangle ABC$ and $\triangle AFE$ are congruent isosceles triangles whose base angles measure 75°. Next, $\text{m}\angle ABC + \text{m}\angle BCD = 30° + 150° = 180°$, so $\overline{AB} \parallel \overline{CD}$. Because these two segments are also congruent, quadrilateral $ABCD$ is a parallelogram. In particular, $\triangle CDA \cong \triangle ABC$. Similarly, $\triangle EDA \cong \triangle AFE$.

Now let $a = AC = AE$ be the length of the base of these isosceles triangles, and let $b = AB$ be the length of the other sides (or of the equilateral hexagon). Because the four triangles are congruent, $[ABCDEF] = [ABC] + [ACD] + [ADE] + [AEF] = 4[ABC] = 4 \cdot \frac{1}{2} b^2 \sin 30° = b^2$. Applying the Law of Cosines to $\triangle ABC$ gives $a^2 = b^2 + b^2 - 2b^2 \cos 30° = (2 - \sqrt{3})b^2$. Because $4 - 2\sqrt{3} = (\sqrt{3} - 1)^2$, this gives $a = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) b$. Using the given length $BE = 3$ and applying the Law of Cosines to $\triangle ABE$ gives

$$\begin{aligned} 9 &= a^2 + b^2 - 2ab \cos 135° \\ &= a^2 + b^2 + \sqrt{2}ab \\ &= (2 - \sqrt{3})b^2 + b^2 + (\sqrt{3} - 1)b^2 \\ &= 2b^2. \end{aligned}$$

Thus $[ABCDEF] = b^2 = \mathbf{\frac{9}{2}}$.

T-6. Let $A = (0, 5)$ and $B = (12, 0)$, and let $C = (1, -1)$. First compute the distance sum: $d(A, C) + d(B, C) = 19$. Notice that if $P = (x, y)$ is on the segment from $(0, -1)$ to $(12, -1)$, then $d(A, P) + d(B, P)$ is constant. This is because if $0 < x < 12$,

$$\begin{aligned} d(A, P) + d(B, P) &= |0 - x| + |5 - (-1)| + |12 - x| + |0 - (-1)| \\ &= x + 6 + (12 - x) + 1 \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(P, B) = 19$ whenever P is on the segment from $(0, 6)$ to $(12, 6)$. If P is on the segment from $(13, 0)$ to $(13, 5)$, then P's coordinates are $(13, y)$, with $0 \leq y \leq 5$, and thus

$$\begin{aligned} d(A, P) + d(B, P) &= |0 - 13| + |5 - y| + |12 - 13| + |0 - y| \\ &= 13 + (5 - y) + 1 + y \\ &= 19. \end{aligned}$$

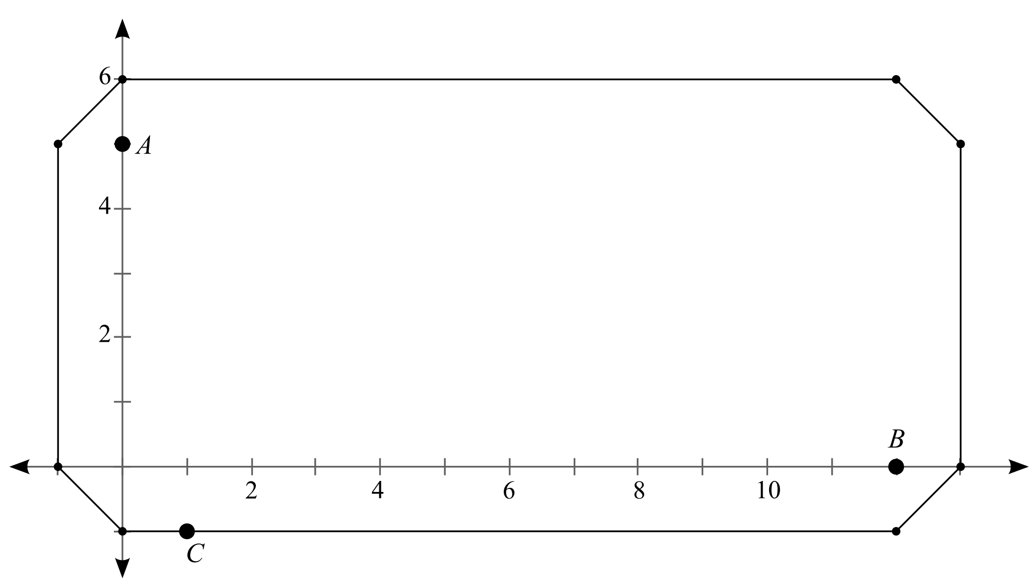
Similarly, $d(A, P) + d(P, B) = 19$ whenever P is on the segment from $(-1, 0)$ to $(-1, 5)$. Finally, if P is on the segment from $(12, -1)$ to $(13, 0)$, then $d(A, P) + d(B, P)$ is constant:

$$\begin{aligned} d(A, P) + d(B, P) &= |0 - x| + |5 - y| + |12 - x| + |0 - y| \\ &= x + (5 - y) + (x - 12) + (-y) \\ &= 2x - 2y - 7, \end{aligned}$$

and because the line segment has equation $x - y = 13$, this expression reduces to

$$\begin{aligned} d(A, P) + d(B, P) &= 2(x - y) - 7 \\ &= 2(13) - 7 \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(B, P) = 19$ on the segments joining $(13, 5)$ and $(12, 6)$, $(0, 6)$ and $(-1, 5)$, and $(-1, 0)$ to $(0, -1)$. The shape of the "ellipse" is given below.



The simplest way to compute the polygon's area is to subtract the areas of the four corner triangles from that of the enclosing rectangle. The enclosing rectangle's area is $14 \cdot 7 = 98$, while each triangle has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus the area is $98 - 4 \cdot \frac{1}{2} = \mathbf{96}$.

T-7. Substituting yields $\frac{512}{f(2)} = 2f(2) \Rightarrow (f(2))^2 = 256 \Rightarrow f(2) = 16$. Therefore $f(1) = \frac{1}{32}$. Using the recursion, $f(3) = \frac{1}{2}$ and $f(4) = 8$. So $f(f(4)) = f(8)$. Continue to apply the recursion:

$$f(5) = 4, \quad f(6) = 32, \quad f(7) = 128, \quad f(8) = \mathbf{4096}.$$

Alternate Solution: Let $g(n) = \log_2 f(n)$. Then $g(n) = g(n-1) + g(n-2)$, with initial conditions $g(1) = g(2) - 9$ and $-g(1) = 1 + g(2)$. From this, $g(1) = -5$ and $g(2) = 4$, and from the recursion,

$$g(3) = -1, \quad g(4) = 3,$$

so $f(4) = 2^{g(4)} = 8$. Continue to apply the recursion:

$$g(5) = 2, \quad g(6) = 5, \quad g(7) = 7, \quad g(8) = 12.$$

Because $g(f(4)) = 12$, it follows that $f(f(4)) = 2^{12} = \mathbf{4096}$.

T-8. When the coefficients of a polynomial f are reversed to form a new polynomial g, the zeros of g are the reciprocals of the zeros of f: r is a zero of f if and only if r^{-1} is a zero of g. In this case, the two polynomials have the *same* zeros; that is, whenever r is a zero of either, so must be r^{-1}. Furthermore, both r and r^{-1} must be real as well as integers, so $r = \pm 1$. As the only zeros are ± 1, and the greatest common divisor of all the coefficients is 1, the polynomial must have leading coefficient 1 or -1. Thus

$$\begin{aligned} f(x) &= \pm(x \pm 1)(x \pm 1) \cdots (x \pm 1) \\ &= \pm(x+1)^k (x-1)^{n-k}. \end{aligned}$$

If A_n is the number of such degree n polynomials, then there are $n+1$ choices for k, $0 \leq k \leq n$. Thus $A_n = 2(n+1)$. The number of such degree n polynomials for $1 \leq n \leq 7$ is the sum:

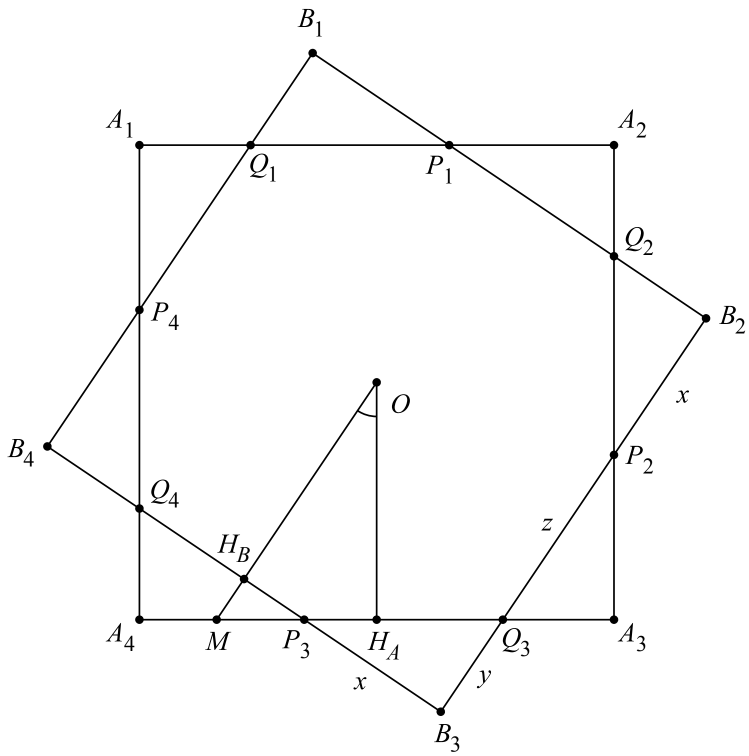
$$A_1 + A_2 + \ldots + A_7 = 2(2 + 3 + \ldots + 8) = 2 \cdot 35 = \mathbf{70}.$$

T-9. Label the sides of the polygon, in order, $s_0, s_1, \ldots, s_{15}$. First note that two sides of the polygon intersect at a vertex if and only if the sides are adjacent. So the sides chosen must be nonconsecutive. Second, if nonparallel sides s_i and s_j are extended, the angle of intersection is determined by $|i-j|$, as are the lengths of the extended portions of the segments. In other words, the *spacing* of the extended sides completely determines the shape of the triangle. So the problem reduces to selecting appropriate spacings, that is, finding integers $a, b, c \geq 2$ whose sum is 16. However, diametrically opposite sides are parallel, so (for example) the sides s_3 and s_{11} cannot both be used. Thus none of a, b, c may equal 8. Taking s_0 as the first side, the second side would be $s_{0+a} = s_a$, and the third side would be s_{a+b}, with c sides between s_{a+b} and s_0. To eliminate reflections and rotations, specify additionally that $a \geq b \geq c$. The allowable partitions are in the table below.

a	b	c	triangle
12	2	2	$s_0 s_{12} s_{14}$
11	3	2	$s_0 s_{11} s_{14}$
10	4	2	$s_0 s_{10} s_{14}$
10	3	3	$s_0 s_{10} s_{13}$
9	5	2	$s_0 s_9 s_{14}$
9	4	3	$s_0 s_9 s_{13}$
7	7	2	$s_0 s_7 s_{14}$
7	6	3	$s_0 s_7 s_{13}$
7	5	4	$s_0 s_7 s_{12}$
6	6	4	$s_0 s_6 s_{12}$
6	5	5	$s_0 s_6 s_{11}$

Thus there are **11** distinct such triangles.

T-10. In the diagram below, O is the center of both squares $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be the intersections of the sides of the squares as shown. Let H_A be on $\overline{A_3A_4}$ so that $\angle A_3H_AO$ is right. Similarly, let H_B be on $\overline{B_3B_4}$ such that $\angle B_3H_BO$ is right. Then the angle by which $B_1B_2B_3B_4$ was rotated is $\angle H_AOH_B$. Extend $\overline{OH_B}$ to meet $\overline{A_3A_4}$ at M.



Both $\triangle H_AOM$ and $\triangle H_BP_3M$ are right triangles sharing acute $\angle M$, so $\triangle H_AOM \sim \triangle H_BP_3M$. By an analogous argument, both triangles are similar to $\triangle B_3P_3Q_3$. Thus m$\angle Q_3P_3B_3 = \theta$. Now let $B_3P_3 = x, B_3Q_3 = y$, and $P_3Q_3 = z$. By symmetry, notice that $B_3P_3 = B_2P_2$ and that $P_3Q_3 = P_2Q_3$. Thus

$$x + y + z = B_3Q_3 + Q_3P_2 + P_2B_2 = B_2B_3 = 3.$$

By the Pythagorean Theorem, $x^2 + y^2 = z^2$. Therefore

$$\begin{aligned} x + y &= 3 - z \\ x^2 + y^2 + 2xy &= 9 - 6z + z^2 \\ 2xy &= 9 - 6z. \end{aligned}$$

The value of xy can be determined from the areas of the four triangles $\triangle B_iP_iQ_i$. By symmetry, these four triangles are congruent to each other. Their total area is the area not in both squares, i.e., $9 - 8 = 1$. Thus $\frac{xy}{2} = \frac{1}{4}$, so $2xy = 1$. Applying this result to the above equation,

$$\begin{aligned} 1 &= 9 - 6z \\ z &= \frac{4}{3}. \end{aligned}$$

The desired quantity is $\sin\theta + \cos\theta = \frac{x}{z} + \frac{y}{z}$, and

$$\begin{aligned}
\frac{x}{z} + \frac{y}{z} &= \frac{x+y+z}{z} - \frac{z}{z} \\
&= \frac{3}{z} - 1 \\
&= \boldsymbol{\frac{5}{4}}.
\end{aligned}$$

2011 Individual Problems

I-1. Compute the 2011^{th} smallest positive integer N that gains an extra digit when doubled.

I-2. In triangle ABC, C is a right angle and M is on $\overline{AC}$. A circle with radius r is centered at M, is tangent to $\overline{AB}$, and is tangent to $\overline{BC}$ at C. If $AC = 5$ and $BC = 12$, compute r.

I-3. The product of the first five terms of a geometric progression is 32. If the fourth term is 17, compute the second term.

I-4. Polygon $A_1A_2 \dots A_n$ is a regular n-gon. For some integer $k < n$, quadrilateral $A_1A_2A_kA_{k+1}$ is a rectangle of area 6. If the area of $A_1A_2 \dots A_n$ is 60, compute n.

I-5. A bag contains 20 lavender marbles, 12 emerald marbles, and some number of orange marbles. If the probability of drawing an orange marble in one try is $\frac{1}{y}$, compute the sum of all possible integer values of y.

I-6. Compute the number of ordered quadruples of integers (a, b, c, d) satisfying the following system of equations:

$$\begin{cases} abc &= 12{,}000 \\ bcd &= 24{,}000 \\ cda &= 36{,}000. \end{cases}$$

I-7. Let n be a positive integer such that $\dfrac{3+4+\cdots+3n}{5+6+\cdots+5n} = \dfrac{4}{11}$. Compute $\dfrac{2+3+\cdots+2n}{4+5+\cdots+4n}$.

I-8. The quadratic polynomial $f(x)$ has a zero at $x = 2$. The polynomial $f(f(x))$ has only one real zero, at $x = 5$. Compute $f(0)$.

I-9. The Local Area Inspirational Math Exam comprises 15 questions. All answers are integers ranging from 000 to 999, inclusive. If the 15 answers form an arithmetic progression with the largest possible difference, compute the largest possible sum of those 15 answers.

I-10. Circle ω_1 has center O, which is on circle ω_2. The circles intersect at points A and C. Point B lies on ω_2 such that $BA = 37$, $BO = 17$, and $BC = 7$. Compute the area of ω_1.

2011 Individual Answers

I-1. 6455

I-2. $\dfrac{12}{5}$

I-3. $\dfrac{4}{17}$

I-4. 40

I-5. 69

I-6. 12

I-7. $\dfrac{27}{106}$

I-8. $-\dfrac{32}{9}$

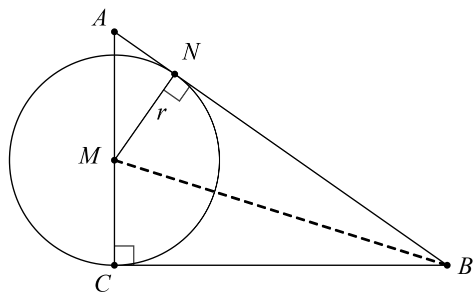
I-9. 7530

I-10. 548π

2011 Individual Solutions

I-1. Let S be the set of numbers that gain an extra digit when doubled. First notice that the numbers in S are precisely those whose first digit is at least 5. Thus there are five one-digit numbers in S, 50 two-digit numbers in S, and 500 three-digit numbers in S. Therefore 5000 is the 556$^{\text{th}}$ smallest number in S, and because all four-digit numbers greater than 5000 are in S, the 2011$^{\text{th}}$ smallest number in S is $5000 + (2011 - 556) = \mathbf{6455}$.

I-2. Let N be the point of tangency of the circle with $\overline{AB}$ and draw $\overline{MB}$, as shown below.



Because $\triangle BMC$ and $\triangle BMN$ are right triangles sharing a hypotenuse, and $\overline{MN}$ and $\overline{MC}$ are radii, $\triangle BMC \cong \triangle BMN$. Thus $BN = 12$ and $AN = 1$. Also $\triangle ANM \sim \triangle ACB$ because the right triangles share $\angle A$, so $\frac{NM}{AN} = \frac{CB}{AC}$. Therefore $\frac{r}{1} = \frac{12}{5}$, so $r = \mathbf{\frac{12}{5}}$.

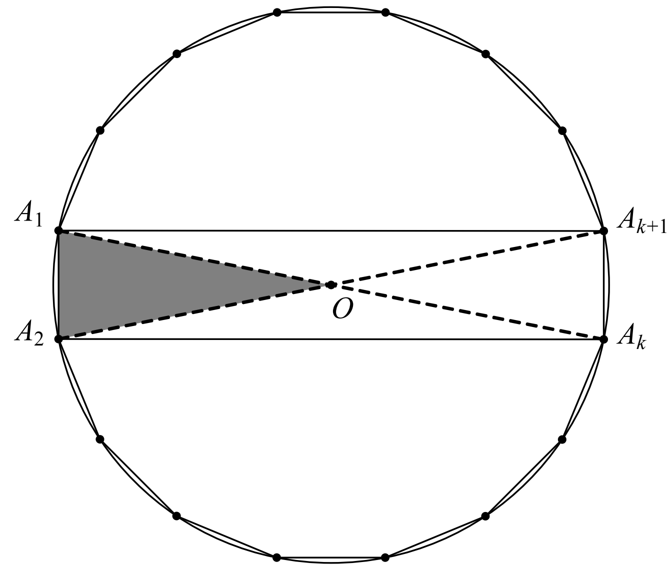
Alternate Solution: Let r denote the radius of the circle, and let D be the foot of the perpendicular from O to $\overline{AB}$. Note that $\triangle ABC \sim \triangle AOD$. Thus $\frac{AB}{AO} = \frac{BC}{DO} \Longrightarrow \frac{13}{5-r} = \frac{12}{r}$, and $r = \mathbf{\frac{12}{5}}$.

I-3. Let a be the third term of the geometric progression, and let r be the common ratio. Then the product of the first five terms is

$$(ar^{-2})(ar^{-1})(a)(ar)(ar^2) = a^5 = 32,$$

so $a = 2$. Because the fourth term is 17, $r = \frac{17}{a} = \frac{17}{2}$. The second term is $ar^{-1} = \frac{2}{17/2} = \mathbf{\frac{4}{17}}$.

I-4. Because $A_1A_2A_kA_{k+1}$ is a rectangle, n must be even, and moreover, $k = \frac{n}{2}$. Also, the rectangle's diagonals meet at the center O of the circumscribing circle. O is also the center of the n-gon. The diagram below shows the case $n = 16$.



Then $[A_1A_2O] = \frac{1}{4}[A_1A_2A_kA_{k+1}] = \frac{1}{n}[A_1A_2\dots A_n] = 60$. So $\frac{1}{4}(6) = \frac{1}{n}(60)$, and $n = \mathbf{40}$.

I-5. Let x be the number of orange marbles. Then the probability of drawing an orange marble is $\frac{x}{x+20+12} = \frac{x}{x+32}$. If this probability equals $\frac{1}{y}$, then $y = \frac{x+32}{x} = 1+\frac{32}{x}$. This expression represents an integer only when x is a factor of 32, thus $x \in \{1, 2, 4, 8, 16, 32\}$. The corresponding y-values are $33, 17, 9, 5, 3,$ and 2, and their sum is **69**.

I-6. From the first two equations, conclude that $d = 2a$. From the last two, $3b = 2a$. Thus all solutions to the system will be of the form $(3K, 2K, c, 6K)$ for some integer K. Substituting these expressions into the system, each equation now becomes $cK^2 = 2000 = 2^4 \cdot 5^3$. So K^2 is of the form $2^{2m}5^{2n}$. There are 3 choices for m and 2 for n, so there are 6 values for K^2, which means there are **12** solutions overall, including negative values for K.

Although the problem does not require finding them, the twelve values of K are ±1, ±2, ±4, ±5, ±10, ±20. These values yield the following quadruples (a, b, c, d):

$$\begin{gathered}
(3,2,2000,6),(-3,-2,2000,-6),\\
(6,4,500,12),(-6,-4,500,-12),\\
(12,8,125,24),(-12,-8,125,-24),\\
(15,10,80,30),(-15,-10,80,-30),\\
(30,20,20,60),(-30,-20,20,-60),\\
(60,40,5,120),(-60,-40,5,-120).
\end{gathered}$$

I-7. In simplifying the numerator and denominator of the left side of the equation, notice that

$$\begin{aligned} k + (k+1) + \cdots + kn &= \frac{1}{2}(kn(kn+1) - k(k-1)) \\ &= \frac{1}{2}(k(n+1)(kn-k+1)). \end{aligned}$$

This identity allows the given equation to be transformed:

$$\begin{aligned} \frac{3(n+1)(3n-3+1)}{5(n+1)(5n-5+1)} &= \frac{4}{11} \\ \frac{3(n+1)(3n-2)}{5(n+1)(5n-4)} &= \frac{4}{11} \\ \frac{3n-2}{5n-4} &= \frac{20}{33}. \end{aligned}$$

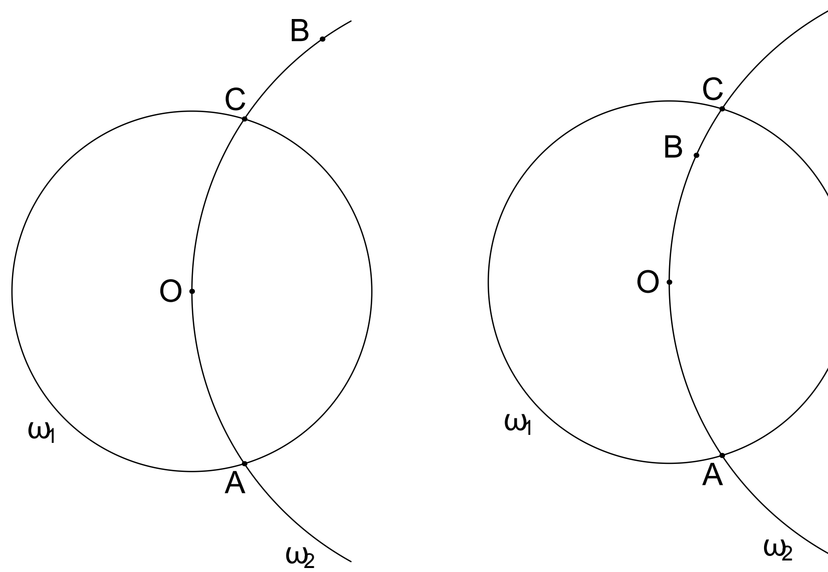
Solving this last equation yields $n = 14$. Using the same identity twice more, for $n = 14$ and $k = 2$ and $k = 4$, the desired quantity is $\frac{2(2n-1)}{4(4n-3)} = \mathbf{\frac{27}{106}}$.

I-8. Let $f(x) = a(x-b)^2 + c$. The graph of f is symmetric about $x = b$, so the graph of $y = f(f(x))$ is also symmetric about $x = b$. If $b \neq 5$, then $2b - 5$, the reflection of 5 across b, must be a zero of $f(f(x))$. Because $f(f(x))$ has exactly one zero, $b = 5$.

Because $f(2) = 0$ and f is symmetric about $x = 5$, the other zero of f is $x = 8$. Because the zeros of f are at 2 and 8 and $f(5)$ is a zero of f, either $f(5) = 2$ or $f(5) = 8$. The following argument shows that $f(5) = 8$ is impossible. Because f is continuous, if $f(5) = 8$, then $f(x_0) = 2$ for some x_0 in the interval $2 < x_0 < 5$. In that case, $f(f(x_0)) = 0$, so 5 would not be a unique zero of $f(f(x))$. Therefore $f(5) = 2$ and $c = 2$. Setting $f(2) = 0$ yields the equation $a(2-5)^2 + 2 = 0$, so $a = -\frac{2}{9}$, and $f(0) = \mathbf{-\frac{32}{9}}$.

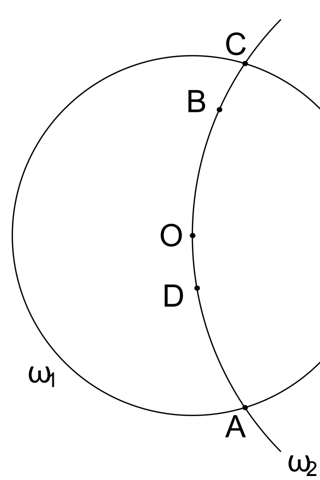
I-9. Let a represent the middle (8^{th}) term of the sequence, and let d be the difference. Then the terms of the sequence are $a - 7d, a - 6d, \ldots, a + 6d, a + 7d$, their sum is $15a$, and the difference between the largest and the smallest terms is $14d$. The largest d such that $14d \leq 999$ is $d = 71$. Thus the largest possible value for a is $999 - 7 \cdot 71 = 502$. The maximal sum of the sequence is therefore $15a = \mathbf{7530}$.

I-10. The points O, A, B, C all lie on ω_2 in some order. There are two possible cases to consider: either B is outside circle ω_1, or it is inside the circle, as shown below.

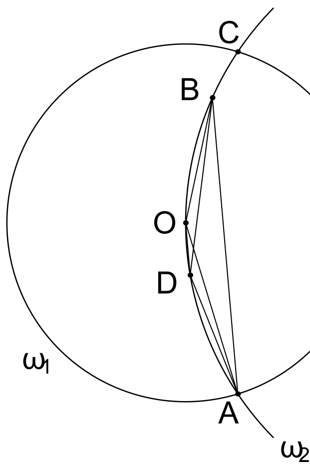


The following argument shows that the first case is impossible. By the Triangle Inequality on $\triangle ABO$, the radius r_1 of circle ω_1 must be at least 20. But because B is outside ω_1, $BO > r_1$, which is impossible, because $BO = 17$. So B must be inside the circle.

Construct point D on minor arc AO of circle ω_2, so that $AD = OB$ (and therefore $DO = BC$).



Because A, D, O, B all lie on ω_2, Ptolemy's Theorem applies to quadrilateral $ADOB$.



Therefore $AD \cdot OB + OD \cdot AB = AO \cdot DB = r_1^2$. Substituting $AD = OB = 17$, $DO = BC = 7$, and $AB = 37$ yields $r_1^2 = 37 \cdot 7 + 17^2 = 548$. Thus the area of ω_1 is $\boldsymbol{548\pi}$.

Power Question 2011: Power of Triangles

The arrangement of numbers known as Pascal's Triangle has fascinated mathematicians for centuries. In fact, about 700 years before Pascal, the Indian mathematician Halayudha wrote about it in his commentaries to a then-1000-year-old treatise on verse structure by the Indian poet and mathematician Pingala, who called it the *Meruprastāra*, or "Mountain of Gems". In this Power Question, we'll explore some properties of Pingala's/Pascal's Triangle ("PT") and its variants.

Unless otherwise specified, the **only** definition, notation, and formulas you may use for PT are the definition, notation, and formulas given below.

PT consists of an infinite number of rows, numbered from 0 onwards. The n^{th} row contains $n+1$ numbers, identified as $\mathrm{Pa}(n,k)$, where $0 \leq k \leq n$. For all n, define $\mathrm{Pa}(n,0) = \mathrm{Pa}(n,n) = 1$. Then for $n > 1$ and $1 \leq k \leq n-1$, define $\mathrm{Pa}(n,k) = \mathrm{Pa}(n-1,k-1) + \mathrm{Pa}(n-1,k)$. It is convenient to define $\mathrm{Pa}(n,k) = 0$ when $k < 0$ or $k > n$. We write the nonzero values of PT in the familiar pyramid shown below.

			1 $\mathrm{Pa}(0,0)$			
		1 $\mathrm{Pa}(1,0)$		**1** $\mathrm{Pa}(1,1)$		
	1 $\mathrm{Pa}(2,0)$		**2** $\mathrm{Pa}(2,1)$		**1** $\mathrm{Pa}(2,2)$	
1 $\mathrm{Pa}(3,0)$		**3** $\mathrm{Pa}(3,1)$		**3** $\mathrm{Pa}(3,2)$		**1** $\mathrm{Pa}(3,3)$

As is well known, $\mathrm{Pa}(n,k)$ gives the number of ways of choosing a committee of k people from a set of n people, so a simple formula for $\mathrm{Pa}(n,k)$ is $\mathrm{Pa}(n,k) = \frac{n!}{k!(n-k)!}$. You may use this formula or the recursive definition above throughout this Power Question.

1a. For $n = 1, 2, 3, 4$, and $k = 4$, find $\mathrm{Pa}(n,n) + \mathrm{Pa}(n+1,n) + \cdots + \mathrm{Pa}(n+k,n)$. [2]

1b. If $\mathrm{Pa}(n,n) + \mathrm{Pa}(n+1,n) + \cdots + \mathrm{Pa}(n+k,n) = \mathrm{Pa}(m,j)$, find and justify formulas for m and j in terms of n and k. [2]

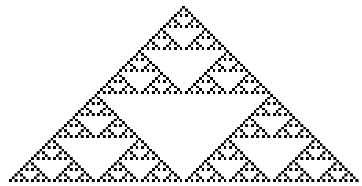
Consider the *parity* of each entry: define

$$\mathrm{PaP}(n,k) = \begin{cases} 1 & \text{if } \mathrm{Pa}(n,k) \text{ is odd,} \\ 0 & \text{if } \mathrm{Pa}(n,k) \text{ is even.} \end{cases}$$

2a. Prove that $\mathrm{PaP}(n,0) = \mathrm{PaP}(n,n) = 1$ for all nonnegative integers n. [1]

2b. Compute rows $n = 0$ to $n = 8$ of PaP. [2]

You may have learned that the array of parities of Pascal's Triangle forms the famous fractal known as Sierpinski's Triangle. The first 128 rows are shown below, with a dot marking each entry where $\mathrm{PaP}(n,k) = 1$.



The next problem helps explain this surprising connection.

3a. If $n = 2^j$ for some nonnegative integer j, and $0 < k < n$, show that $\mathrm{PaP}(n,k) = 0$. [2]

3b. Let $j \geq 0$, and suppose $n \geq 2^j$. Prove that $\mathrm{Pa}(n,k)$ has the same parity as the sum $\mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k)$, i.e., either both $\mathrm{Pa}(n,k)$ and the given sum are even, or both are odd. [2]

3c. If j is an integer such that $2^j \leq n < 2^{j+1}$, and $k < 2^j$, prove that

$$\mathrm{PaP}(n,k) = \mathrm{PaP}(n-2^j, k).$$ [2]

Clark's Triangle: If the left side of PT is replaced with consecutive multiples of 6, starting with 0, but the right entries (except the first) and the generating rule are left unchanged, the result is called *Clark's Triangle*. If the k^{th} entry of the n^{th} row is denoted by $\mathrm{Cl}(n,k)$, then the formal rule is:

$$\begin{cases} \mathrm{Cl}(n,0) = 6n & \text{for all } n, \\ \mathrm{Cl}(n,n) = 1 & \text{for } n \geq 1, \\ \mathrm{Cl}(n,k) = \mathrm{Cl}(n-1,k-1) + \mathrm{Cl}(n-1,k) & \text{for } n \geq 1 \text{ and } 1 \leq k \leq n-1. \end{cases}$$

The first four rows of Clark's Triangle are given below.

$$\begin{array}{ccccccc}
 & & & \mathbf{0} & & & \\
 & & & \mathrm{Cl}(0,0) & & & \\
 & & \mathbf{6} & & \mathbf{1} & & \\
 & & \mathrm{Cl}(1,0) & & \mathrm{Cl}(1,1) & & \\
 & \mathbf{12} & & \mathbf{7} & & \mathbf{1} & \\
 & \mathrm{Cl}(2,0) & & \mathrm{Cl}(2,1) & & \mathrm{Cl}(2,2) & \\
\mathbf{18} & & \mathbf{19} & & \mathbf{8} & & \mathbf{1} \\
\mathrm{Cl}(3,0) & & \mathrm{Cl}(3,1) & & \mathrm{Cl}(3,2) & & \mathrm{Cl}(3,3)
\end{array}$$

4a. Compute the next three rows of Clark's Triangle. [2]

4b. If $\mathrm{Cl}(n,1) = an^2 + bn + c$, determine the values of a, b, and c. [2]

4c. Prove the formula you found in 4b. [2]

5a. Compute $\mathrm{Cl}(11,2)$. [1]

5b. Find and justify a formula for $\mathrm{Cl}(n, 2)$ in terms of n. [2]

5c. Compute $\mathrm{Cl}(11, 3)$. [1]

5d. Find and justify a formula for $\mathrm{Cl}(n, 3)$ in terms of n. [2]

6. Find and prove a closed formula (that is, a formula with a fixed number of terms and no "…") for $\mathrm{Cl}(n, k)$ in terms of n, k, and the Pa function. [3]

Leibniz's Harmonic Triangle: Consider the triangle formed by the rule

$$\begin{cases} \mathrm{Le}(n, 0) = \frac{1}{n+1} & \text{for all } n, \\ \mathrm{Le}(n, n) = \frac{1}{n+1} & \text{for all } n, \\ \mathrm{Le}(n, k) = \mathrm{Le}(n+1, k) + \mathrm{Le}(n+1, k+1) & \text{for all } n \text{ and } 0 \leq k \leq n. \end{cases}$$

This triangle, discovered first by Leibniz, consists of reciprocals of integers as shown below.

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & & \mathrm{Le}(0,0) & & & \\
 & & \mathbf{\frac{1}{2}} & & \mathbf{\frac{1}{2}} & & \\
 & & \mathrm{Le}(1,0) & & \mathrm{Le}(1,1) & & \\
 & \mathbf{\frac{1}{3}} & & \mathbf{\frac{1}{6}} & & \mathbf{\frac{1}{3}} & \\
 & \mathrm{Le}(2,0) & & \mathrm{Le}(2,1) & & \mathrm{Le}(2,2) & \\
\mathbf{\frac{1}{4}} & & \mathbf{\frac{1}{12}} & & \mathbf{\frac{1}{12}} & & \mathbf{\frac{1}{4}} \\
\mathrm{Le}(3,0) & & \mathrm{Le}(3,1) & & \mathrm{Le}(3,2) & & \mathrm{Le}(3,3)
\end{array}$$

For this contest, you may assume that $\mathrm{Le}(n, k) > 0$ whenever $0 \leq k \leq n$, and that $\mathrm{Le}(n, k)$ is undefined if $k < 0$ or $k > n$.

7a. Compute the entries in the next two rows of Leibniz's Triangle. [1]

7b. Compute $\mathrm{Le}(17, 1)$. [1]

7c. Compute $\mathrm{Le}(17, 2)$. [1]

8a. Find and justify a formula for $\mathrm{Le}(n, 1)$ in terms of n. [2]

8b. Compute $\displaystyle\sum_{n=1}^{2011} \mathrm{Le}(n, 1)$. [2]

8c. Find and justify a formula for $\mathrm{Le}(n, 2)$ in terms of n. [2]

9a. If $\displaystyle\sum_{i=1}^{\infty} \mathrm{Le}(i, 1) = \mathrm{Le}(n, k)$, determine the values of n and k. [2]

9b. If $\sum_{i=m}^{\infty} \operatorname{Le}(i, m) = \operatorname{Le}(n, k)$, compute expressions for n and k in terms of m. [2]

9c. Justify your result in 9b. [2]

10. Find three distinct sets of positive integers $\{a, b, c, d\}$ with $a < b < c < d$ such that

$$\frac{1}{3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$ [3]

11. Find and prove a closed formula (that is, a formula with a fixed number of terms and no "...") for $\operatorname{Le}(n, k)$ in terms of n, k, and the Pa function. [4]

Solutions to 2011 Power Question

1a.

$$\begin{array}{rclcl}
\mathrm{Pa}(1,1)+\mathrm{Pa}(2,1)+\mathrm{Pa}(3,1)+\mathrm{Pa}(4,1)+\mathrm{Pa}(5,1) &=& 1+2+3+4+5 &=& \mathbf{15} \\
\mathrm{Pa}(2,2)+\mathrm{Pa}(3,2)+\mathrm{Pa}(4,2)+\mathrm{Pa}(5,2)+\mathrm{Pa}(6,2) &=& 1+3+6+10+15 &=& \mathbf{35} \\
\mathrm{Pa}(3,3)+\mathrm{Pa}(4,3)+\mathrm{Pa}(5,3)+\mathrm{Pa}(6,3)+\mathrm{Pa}(7,3) &=& 1+4+10+20+35 &=& \mathbf{70} \\
\mathrm{Pa}(4,4)+\mathrm{Pa}(5,4)+\mathrm{Pa}(6,4)+\mathrm{Pa}(7,4)+\mathrm{Pa}(8,4) &=& 1+5+15+35+70 &=& \mathbf{126}.
\end{array}$$

1b. Notice that $\mathrm{Pa}(n,n)+\mathrm{Pa}(n+1,n)+\cdots+\mathrm{Pa}(n+k,n)=\mathrm{Pa}(n+k+1,n+1)$, so $m=n+k+1$ and $j=n+1$. (By symmetry, $j=k$ is also correct.) The equation is true for all n when $k=0$, because the sum is simply $\mathrm{Pa}(n,n)$ and the right side is $\mathrm{Pa}(n+1,n+1)$, both of which are 1. Proceed by induction on k. If $\mathrm{Pa}(n,n)+\mathrm{Pa}(n+1,n)+\cdots+\mathrm{Pa}(n+k,n)=\mathrm{Pa}(n+k+1,n+1)$, then adding $\mathrm{Pa}(n+k+1,n)$ to both sides yields $\mathrm{Pa}(n+k+1,n)+\mathrm{Pa}(n+k+1,n+1)=\mathrm{Pa}(n+k+2,n+1)$ by the recursive rule for Pa.

2a. By definition of Pa, $\mathrm{Pa}(n,0)=\mathrm{Pa}(n,n)=1$ for all nonnegative integers n, and this value is odd, so $\mathrm{PaP}(n,0)=\mathrm{PaP}(n,n)=1$ by definition.

2b.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&0&&1&&&&&& \\
&&&&&1&&1&&1&&1&&&&& \\
&&&&1&&0&&0&&0&&1&&&& \\
&&&1&&1&&0&&0&&1&&1&&& \\
&&1&&0&&1&&0&&1&&0&&1&& \\
&1&&1&&1&&1&&1&&1&&1&&1& \\
1&&0&&0&&0&&0&&0&&0&&0&&1
\end{array}$$

3a. Notice that

$$\begin{aligned}
\mathrm{Pa}(n,k) &= \frac{n!}{k!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= \frac{n}{k}\cdot \mathrm{Pa}(n-1,k-1).
\end{aligned}$$

Examining the right hand side in the case where $n=2^j$ and $0<k<n$, the second factor, $\mathrm{Pa}(n-1,k-1)$ is an integer, and the first factor has an even numerator, so $\mathrm{Pa}(n,k)$ is even. Therefore $\mathrm{PaP}(n,k)=0$.

3b. Proceed by induction on j. It is easier to follow the logic when the computations are expressed modulo 2. The claim is that $\mathrm{Pa}(n,k)\equiv \mathrm{Pa}(n-2^j,k-2^j)+\mathrm{Pa}(n-2^j,k) \bmod 2$. If $j=0$, so that

$2^j = 1$, then this congruence is exactly the recursive definition of $\mathrm{Pa}(n,k)$ when $0 < k < n$. For $k = 0$ and $k = n$, the left-hand side is 1 and the right-hand side is either $0+1$ or $1+0$. For other values of k, all three terms are zero.

Now, assume $\mathrm{Pa}(n,k) \equiv \mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k) \bmod 2$ for some $j \geq 0$, and let $n \geq 2^{j+1}$. For any k, apply the inductive hypothesis three times:

$$\begin{aligned}
\mathrm{Pa}(n,k) &\equiv \mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k) \bmod 2 \\
&\equiv \mathrm{Pa}(n-2^j-2^j, k-2^j-2^j) + \mathrm{Pa}(n-2^j-2^j, k-2^j) \\
&\qquad + \mathrm{Pa}(n-2^j-2^j, k-2^j) + \mathrm{Pa}(n-2^j-2^j, k).
\end{aligned}$$

But $u + u \equiv 0 \bmod 2$ for any integer u. Thus the two occurrences of $\mathrm{Pa}(n-2^j-2^j, k-2^j)$ cancel each other out, and

$$\begin{aligned}
\mathrm{Pa}(n,k) &\equiv \mathrm{Pa}(n-2^j-2^j, k-2^j-2^j) + \mathrm{Pa}(n-2^j-2^j, k) \\
&\equiv \mathrm{Pa}(n-2^{j+1}, k-2^{j+1}) + \mathrm{Pa}(n-2^{j+1}, k) \bmod 2.
\end{aligned}$$

3c. By part (b), $\mathrm{Pa}(n,k) \equiv \mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k) \bmod 2$ when $j \geq 0$ and $n \geq 2^j$. If $2^j \leq n < 2^{j+1}$ and $0 \leq k < 2^j$, then $\mathrm{Pa}(n-2^j, k-2^j) = 0$, so

$$\mathrm{Pa}(n,k) \equiv \mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k) \equiv \mathrm{Pa}(n-2^j, k) \bmod 2.$$

Thus $\mathrm{PaP}(n,k) = \mathrm{PaP}(n-2^j, k)$.

Alternate Solution: Problem 3a establishes the statement when $n = 2^j$. For $2^j < n < 2^{j+1}$, proceed by induction on n. Then $\mathrm{PaP}(n,k) \equiv \mathrm{PaP}(n-1,k-1) + \mathrm{PaP}(n-1,k) \bmod 2$, while $\mathrm{PaP}(n-1,k-1) = \mathrm{PaP}(n-1-2^j, k-1)$ and $\mathrm{PaP}(n-1,k) = \mathrm{PaP}(n-1-2^j, k)$. But $\mathrm{PaP}(n-1-2^j, k-1) + \mathrm{PaP}(n-1-2^j, k) \equiv \mathrm{PaP}(n-2^j, k) \bmod 2$, establishing the statement.

4a.

$$\begin{array}{ccccccccccccc}
 & & 24 & & 37 & & 27 & & 9 & & 1 & & \\
 & 30 & & 61 & & 64 & & 36 & & 10 & & 1 & \\
36 & & 91 & & 125 & & 100 & & 46 & & 11 & & 1
\end{array}$$

4b. Using the given values yields the system of equations below.

$$\left\{\begin{array}{rcrcl}
\mathrm{Cl}(1,1) & = & 1 & = & a(1)^2 + b(1) + c \\
\mathrm{Cl}(2,1) & = & 7 & = & a(2)^2 + b(2) + c \\
\mathrm{Cl}(3,1) & = & 19 & = & a(3)^2 + b(3) + c
\end{array}\right.$$

Solving this system, $a = 3, b = -3, c = 1$.

4c. Use induction on n. For $n = 1,2,3$, the values above demonstrate the theorem. If $\mathrm{Cl}(n,1) = 3n^2 - 3n + 1$, then $\mathrm{Cl}(n+1,1) = \mathrm{Cl}(n,0) + \mathrm{Cl}(n,1) = 6n + (3n^2 - 3n + 1) = (3n^2 + 6n + 3) - (3n+3) + 1 = 3(n+1)^2 - 3(n+1) + 1$.

5a. $\mathrm{Cl}(11,2) = 1000$.

5b. $\mathrm{Cl}(n,2)=(n-1)^3$. Use induction on n. First, rewrite $\mathrm{Cl}(n,1)=3n^2-3n+1=n^3-(n-1)^3$, and notice that $\mathrm{Cl}(2,2)=1=(2-1)^3$. Then if $\mathrm{Cl}(n,2)=(n-1)^3$, using the recursive definition, $\mathrm{Cl}(n+1,2)=\mathrm{Cl}(n,2)+\mathrm{Cl}(n,1)=(n-1)^3+(n^3-(n-1)^3)=n^3$.

5c. $\mathrm{Cl}(11,3)=2025$.

5d. Notice that $\mathrm{Cl}(3,3)=1=1^3$, and then for $n>3$, $\mathrm{Cl}(n,3)=\mathrm{Cl}(n-1,2)+\mathrm{Cl}(n-1,3)$; replacing $\mathrm{Cl}(n-1,3)$ analogously on the right side yields the summation

$$\mathrm{Cl}(n,3)=\mathrm{Cl}(n-1,2)+\mathrm{Cl}(n-2,2)+\ldots+\mathrm{Cl}(3,2)+1.$$

By 5b, $\mathrm{Cl}(n-1,2)=(n-2)^3$, so this formula is equivalent to

$$\mathrm{Cl}(n,3)=(n-2)^3+(n-3)^3+\ldots+2^3+1.$$

Use the identity $1^3+2^3+\cdots+m^3=\frac{m^2(m+1)^2}{4}$ and substitute $n-2$ for m to obtain $\mathrm{Cl}(n,3)=\frac{(n-2)^2(n-1)^2}{4}$.

6. Notice that $\mathrm{Cl}(n,0)=6\cdot\mathrm{Pa}(n,1)$, and that for $n>0$, $\mathrm{Cl}(n,n)=\mathrm{Pa}(n,n)$. From problem 4c, $\mathrm{Cl}(n,1)=6\mathrm{Pa}(n,2)+1$ (where $\mathrm{Pa}(n,k)=0$ if $k>n$).

For $k>1$, repeated application of the formula $\mathrm{Cl}(m,k)=\mathrm{Cl}(m-1,k-1)+\mathrm{Cl}(m-1,k)$ allows each value of $\mathrm{Cl}(n,k)$ to be written as a sum:

$$\begin{aligned}\mathrm{Cl}(n,k) &= \mathrm{Cl}(n-1,k-1)+\mathrm{Cl}(n-1,k)\\ &= \mathrm{Cl}(n-1,k-1)+\mathrm{Cl}(n-2,k-1)+\mathrm{Cl}(n-2,k),\ \text{and eventually:}\\ &= \mathrm{Cl}(n-1,k-1)+\mathrm{Cl}(n-2,k-1)+\cdots+\mathrm{Cl}(k,k-1)+\mathrm{Cl}(k,k)\\ &= \mathrm{Cl}(n-1,k-1)+\mathrm{Cl}(n-2,k-1)+\cdots+\mathrm{Cl}(k-1,k-1),\end{aligned}$$

because $\mathrm{Cl}(k,k)=\mathrm{Cl}(k-1,k-1)=1$. Then

$$\begin{aligned}\mathrm{Cl}(n,2) &= \mathrm{Cl}(n-1,1)+\mathrm{Cl}(n-2,1)+\cdots+\mathrm{Cl}(1,1)\\ &= (6\mathrm{Pa}(n-1,2)+1)+(6\mathrm{Pa}(n-2,2)+1)+\cdots+(6\mathrm{Pa}(2,2)+1)+1\\ &= 6(\mathrm{Pa}(n-1,2)+\mathrm{Pa}(n-2,2)+\cdots+\mathrm{Pa}(2,2))+(n-1).\end{aligned}$$

By the identity from 1b, $\mathrm{Pa}(n-1,2)+\mathrm{Pa}(n-2,2)+\cdots+\mathrm{Pa}(2,2)=\mathrm{Pa}(n,3)$. Therefore $\mathrm{Cl}(n,2)=6\mathrm{Pa}(n,3)+n-1=6\mathrm{Pa}(n,3)+\mathrm{Pa}(n-1,1)$.

The general formula is $\mathrm{Cl}(n,k)=6\mathrm{Pa}(n,k+1)+\mathrm{Pa}(n-1,k-1)$. This formula follows by induction on n. If $n=k$, then $\mathrm{Cl}(n,k)=1$, and $6\mathrm{Pa}(n,k+1)+\mathrm{Pa}(n-1,k-1)=6\cdot 0+1=1$. Then suppose for some $n\geq k$, $\mathrm{Cl}(n,k)=6\mathrm{Pa}(n,k+1)+\mathrm{Pa}(n-1,k-1)$. It follows that

$$\begin{aligned}\mathrm{Cl}(n+1,k) &= \mathrm{Cl}(n,k-1)+\mathrm{Cl}(n,k)\\ &= (6\mathrm{Pa}(n,k)+\mathrm{Pa}(n-1,k-2))+(6\mathrm{Pa}(n,k+1)+\mathrm{Pa}(n-1,k-1))\\ &= 6(\mathrm{Pa}(n,k)+\mathrm{Pa}(n,k+1))+(\mathrm{Pa}(n-1,k-2)+\mathrm{Pa}(n-1,k-1))\\ &= 6\mathrm{Pa}(n+1,k+1)+\mathrm{Pa}(n,k-1).\end{aligned}$$

7a.

$$\begin{array}{ccccccccccc} & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\ \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} \end{array}$$

7b. $\mathrm{Le}(17,1) = \mathrm{Le}(16,0) - \mathrm{Le}(17,0) = \frac{1}{17} - \frac{1}{18} = \frac{1}{306}$.

7c. $\mathrm{Le}(17,2) = \mathrm{Le}(16,1) - \mathrm{Le}(17,1) = \mathrm{Le}(15,0) - \mathrm{Le}(16,0) - \mathrm{Le}(17,1) = \frac{1}{2448}$.

8a. $\mathrm{Le}(n,1) = \mathrm{Le}(n-1,0) - \mathrm{Le}(n,0) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

8b. Because $\mathrm{Le}(n,1) = \frac{1}{n} - \frac{1}{n+1}$,

$$\begin{aligned} \sum_{i=1}^{2011} \mathrm{Le}(i,1) &= \sum_{i=1}^{2011} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2010} - \frac{1}{2011}\right) + \left(\frac{1}{2011} - \frac{1}{2012}\right) \\ &= 1 - \frac{1}{2012} \\ &= \frac{2011}{2012}. \end{aligned}$$

8c. $\mathrm{Le}(n,2) = \mathrm{Le}(n-1,1) - \mathrm{Le}(n,1) = \frac{1}{n(n-1)} - \frac{1}{n(n+1)} = \frac{2}{(n-1)(n)(n+1)}$. Note that this result appears in the table as a unit fraction because at least one of the integers $n-1, n, n+1$ is even.

9a. Extending the result of 8b gives

$$\sum_{i=1}^{n} \mathrm{Le}(i,1) = \frac{1}{1} - \frac{1}{n},$$

so as $n \to \infty$, $\sum_{i=1}^{n} \mathrm{Le}(i,1) \to 1$. This value appears as $\mathrm{Le}(0,0)$, so $n = k = 0$.

9b. $n = k = m - 1$.

9c. Because in general $\mathrm{Le}(i,m) = \mathrm{Le}(i-1,m-1) - \mathrm{Le}(i,m-1)$, a partial sum can be rewritten as follows:

$$\begin{aligned} \sum_{i=m}^{n} \mathrm{Le}(i,m) &= \sum_{i=m}^{n} (\mathrm{Le}(i-1,m-1) - \mathrm{Le}(i,m-1)) \\ &= (\mathrm{Le}(m-1,m-1) - \mathrm{Le}(m,m-1)) + (\mathrm{Le}(m,m-1) - \mathrm{Le}(m+1,m-1)) + \\ &\quad \cdots + (\mathrm{Le}(n-1,m-1) - \mathrm{Le}(n,m-1)) \\ &= \mathrm{Le}(m-1,m-1) - \mathrm{Le}(n,m-1). \end{aligned}$$

Because the values of $\mathrm{Le}(n,m-1)$ get arbitrarily small as n increases (proof: $\mathrm{Le}(i,j) < \mathrm{Le}(i-1,j-1)$ by construction, so $\mathrm{Le}(n,m-1) < \mathrm{Le}(n-m+1,0) = \frac{1}{n-m+1}$), the limit of these partial sums is $\mathrm{Le}(m-1,m-1)$. So $n = k = m - 1$.

Note: This result can be extended even further. In fact, for every value of $k < n$,

$$\operatorname{Le}(n, k) = \sum_{i=n+1}^{\infty} \operatorname{Le}(i, k+1).$$

In other words, each entry in Leibniz's triangle is an infinite sum of the entries in the diagonal directly to its right, beginning with the entry below and to the right of the given one.

10. Note that $\frac{1}{3} = \operatorname{Le}(2,0) = \operatorname{Le}(3,0) + \operatorname{Le}(3,1)$. Also $\operatorname{Le}(3,0) = \operatorname{Le}(4,0) + \operatorname{Le}(4,1)$ and $\operatorname{Le}(4,0) = \operatorname{Le}(5,0) + \operatorname{Le}(5,1)$. So

$$\begin{aligned}\frac{1}{3} &= \operatorname{Le}(5,0) + \operatorname{Le}(5,1) + \operatorname{Le}(4,1) + \operatorname{Le}(3,1)\\ &= \frac{1}{6} + \frac{1}{30} + \frac{1}{20} + \frac{1}{12}.\end{aligned}$$

Therefore $a = 6, b = 12, c = 20, d = 30$ is a solution.

On the other hand, a similar analysis yields

$$\begin{aligned}\frac{1}{3} &= \operatorname{Le}(4,0) + \operatorname{Le}(5,1) + \operatorname{Le}(5,2) + \operatorname{Le}(3,1)\\ &= \frac{1}{5} + \frac{1}{30} + \frac{1}{60} + \frac{1}{12},\end{aligned}$$

so $a = 5, b = 12, c = 30, d = 60$ is another such solution.

Finally, notice that $\operatorname{Le}(3,1) = \frac{1}{12} = \operatorname{Le}(11,0)$, so that $\frac{1}{12}$ can be rewritten as $\operatorname{Le}(12,0) + \operatorname{Le}(12,1) = \frac{1}{13} + \frac{1}{156}$. Then

$$\begin{aligned}\frac{1}{3} &= \operatorname{Le}(4,0) + \operatorname{Le}(4,1) + \operatorname{Le}(3,1)\\ &= \operatorname{Le}(4,0) + \operatorname{Le}(4,1) + \operatorname{Le}(12,0) + \operatorname{Le}(12,1)\\ &= \frac{1}{5} + \frac{1}{20} + \frac{1}{13} + \frac{1}{156},\end{aligned}$$

yields $a = 5, b = 13, c = 20, d = 156$.

In general, once a triple a, b, c has been found, rewriting (for example) $\frac{1}{a} = \operatorname{Le}(a-1, 0) = \operatorname{Le}(a,0) + \operatorname{Le}(a,1) = \frac{1}{a+1} + \frac{1}{a(a+1)}$ creates an appropriate quadruple, although on occasion these values duplicate another value in the quadruple.

11. The formula is $\operatorname{Le}(n,k) = \frac{1}{(n+1)\cdot \operatorname{Pa}(n,k)}$, or equivalently, $\frac{1}{(k+1)\cdot \operatorname{Pa}(n+1,k+1)}$. Because $\operatorname{Pa}(n,0) = \operatorname{Pa}(n,n) = 1$, when $k = 0$ or $k = n$, the formula is equivalent to the definition of $\operatorname{Le}(n,0) = \operatorname{Le}(n,n) = \frac{1}{n+1}$.

To prove the formula for $1 \le k \le n-1$, use induction on k. The base case $k = 0$ was proved above. If the formula holds for a particular value of $k < n$, then it can be extended to the case $k + 1$ using two identities:

$$\begin{aligned}
\mathrm{Le}(n+1,k+1) &= \mathrm{Le}(n,k) - \mathrm{Le}(n+1,k) \text{ and} \\
(n+1)\mathrm{Pa}(n,k) &= \frac{(n+1)!}{k!(n-k)!}.
\end{aligned}$$

Using the first identity and the inductive hypothesis yields

$$\begin{aligned}
\mathrm{Le}(n+1,k+1) &= \mathrm{Le}(n,k) - \mathrm{Le}(n+1,k) \\
&= \frac{1}{(n+1)\cdot \mathrm{Pa}(n,k)} - \frac{1}{(n+2)\cdot \mathrm{Pa}(n+1,k)}.
\end{aligned}$$

Applying the second identity to the right side of the equation yields:

$$\begin{aligned}
\mathrm{Le}(n+1,k+1) &= \frac{k!\,(n-k)!}{(n+1)!} - \frac{k!(n+1-k)!}{(n+2)!} \\
&= \frac{k!\,(n-k)!}{(n+2)!}\,[(n+2)-(n+1-k)] \\
&= \frac{k!\,(n-k)!}{(n+2)!}\,(k+1) \\
&= \frac{(k+1)!\,((n+1)-(k+1))!}{(n+1)!\,(n+2)} \\
&= \frac{1}{(n+2)\mathrm{Pa}(n+1,k+1)}.
\end{aligned}$$

2011 Relay Problems

R1-1. Compute the number of integers n for which $2^4 < 8^n < 16^{32}$.

R1-2. Let $T = TNYWR$. Compute the number of positive integers b such that the number T has exactly two digits when written in base b.

R1-3. Let $T = TNYWR$. Triangle ABC has a right angle at C, and $AB = 40$. If $AC - BC = T - 1$, compute $[ABC]$, the area of $\triangle ABC$.

R2-1. Let x be a positive real number such that $\log_{\sqrt{2}} x = 20$. Compute $\log_2 \sqrt{x}$.

R2-2. Let $T = TNYWR$. Hannah flips two fair coins, while Otto flips T fair coins. Let p be the probability that the number of heads showing on Hannah's coins is greater than the number of heads showing on Otto's coins. If $p = q/r$, where q and r are relatively prime positive integers, compute $q + r$.

R2-3. Let $T = TNYWR$. In ARMLovia, the unit of currency is the edwah. Janet's wallet contains bills in denominations of 20 and 80 edwahs. If the bills are worth an average of $2T$ edwahs each, compute the smallest possible value of the bills in Janet's wallet.

2011 Relay Answers

R1-1. 41

R1-2. 35

R1-3. 111

R2-1. 5

R2-2. 17

R2-3. 1020

2011 Relay Solutions

R1-1. $8^n = 2^{3n}$ and $16^{32} = 2^{128}$. Therefore $4 < 3n < 128$, and $2 \leq n \leq 42$. Thus there are **41** such integers n.

R1-2. If T has more than one digit when written in base b, then $b \leq T$. If T has fewer than three digits when written in base b, then $b^2 > T$, or $b > \sqrt{T}$. So the desired set of bases b is $\left\{b \mid \sqrt{T} < b \leq T\right\}$. When $T = 41$, $\left\lfloor \sqrt{T} \right\rfloor = 6$ and so $6 < b \leq 41$. There are $41 - 6 = \mathbf{35}$ such integers.

R1-3. Let $AC = b$ and $BC = a$. Then $a^2 + b^2 = 1600$ and $|a - b| = T - 1$. Squaring the second equation yields $a^2 + b^2 - 2ab = (T-1)^2$, so $1600 - 2ab = (T-1)^2$. Hence the area of the triangle is $\frac{1}{2}ab = \frac{1600-(T-1)^2}{4} = 400 - \frac{(T-1)^2}{4}$ or $400 - \left(\frac{T-1}{2}\right)^2$, which for $T = 35$ yields $400 - 289 = \mathbf{111}$.

R2-1. The identity $\log_{b^n} x = \frac{1}{n}\log_b x$ yields $\log_2 x = 10$. Then $\log_2 \sqrt{x} = \log_2 x^{1/2} = \frac{1}{2}\log_2 x = \mathbf{5}$.

Alternate Solution: Use the definition of log to obtain $x = (\sqrt{2}\,)^{20} = (2^{1/2})^{20} = 2^{10}$. Thus $\log_2 \sqrt{x} = \log_2 2^5 = \mathbf{5}$.

Alternate Solution: Use the change of base formula to obtain $\frac{\log x}{\log \sqrt{2}} = 20$, so $\log x = 20 \log \sqrt{2} = 20 \log 2^{1/2} = 10 \log 2$. Thus $x = 2^{10}$, and $\log_2 \sqrt{x} = \log_2 2^5 = \mathbf{5}$.

R2-2. Because Hannah has only two coins, the only ways she can get more heads than Otto are if she gets 1 (and he gets 0), or she gets 2 (and he gets either 1 or 0).

The probability of Hannah getting exactly one head is $\frac{1}{2}$. The probability of Otto getting no heads is $\frac{1}{2^T}$. So the probability of both events occurring is $\frac{1}{2^{T+1}}$.

The probability of Hannah getting exactly two heads is $\frac{1}{4}$. The probability of Otto getting no heads is still $\frac{1}{2^T}$, but the probability of getting exactly one head is $\frac{T}{2^T}$, because there are T possibilities for which coin is heads. So the probability of Otto getting either 0 heads or 1 head is $\frac{1+T}{2^T}$, and combining that with Hannah's result yields an overall probability of $\frac{1+T}{2^{T+2}}$.

Thus the probability that Hannah flips more heads than Otto is $\frac{1}{2^{T+1}} + \frac{1+T}{2^{T+2}} = \frac{3+T}{2^{T+2}}$. For $T = 5$, the value is $\frac{8}{128} = \frac{1}{16}$, giving an answer of $1 + 16 = \mathbf{17}$.

R2-3. Let x be the number of twenty-edwah bills and y be the number of eighty-edwah bills. Then

$$\begin{aligned} \frac{20x + 80y}{x + y} &= 2T \\ 20x + 80y &= 2Tx + 2Ty \\ (80 - 2T)y &= (2T - 20)x. \end{aligned}$$

In the case where $T = 17$ (and hence $2T = 34$), this equation reduces to $46y = 14x$, or $23y = 7x$. Because 23 and 7 are relatively prime, $23 \mid x$ and $7 \mid y$. Therefore the pair that yields the smallest possible value is $(x, y) = (23, 7)$. Then there are $23 + 7 = 30$ bills worth a total of $23 \cdot 20 + 7 \cdot 80 = 460 + 560 = 1020$ edwahs, and $1020/30 = 34$, as required. The answer is **1020**.

Alternate Solution: Consider the equation $\frac{20x+80y}{x+y} = 2T$ derived in the first solution. The identity $\frac{20x+80y}{x+y} = 20 + \frac{60y}{x+y}$ yields the following:

$$\begin{aligned}
\frac{60y}{x+y} &= 2T - 20 \\
\frac{30y}{x+y} &= T - 10 \\
x + y &= \frac{30y}{T-10}.
\end{aligned}$$

For the smallest value of $x + y$, both $x + y$ and y will be relatively prime. Thus the smallest value of $x + y$ is $\frac{30}{\gcd(T-10,30)}$, which occurs when $y = \frac{T-10}{\gcd(T-10,30)}$. Substituting $T = 17$, the numbers $T - 10 = 7$ and 30 are relatively prime, so $y = 7$ and $x = 23$, for a total of **1020** edwahs.

2011 Tiebreaker Problems

TB-1. Spheres centered at points P, Q, R are externally tangent to each other, and are tangent to plane $\mathcal{M}$ at points P', Q', R', respectively. All three spheres are on the same side of the plane. If $P'Q' = Q'R' = 12$ and $P'R' = 6$, compute the area of $\triangle PQR$.

TB-2. Let $f(x) = x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots$. Compute the coefficient of x^{10} in $f(f(x))$.

TB-3. Compute $\lfloor 100000(1.002)^{10} \rfloor$.

2011 Tiebreaker Answers

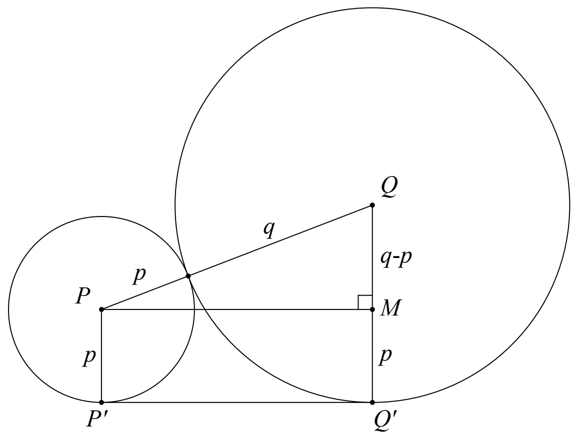
TB-1. $18\sqrt{6}$

TB-2. 40

TB-3. 102,018

2011 Tiebreaker Solutions

TB-1. Let the radii be p, q, r respectively. Looking at a cross-section of the spheres through $\overline{PQ}$ perpendicular to the plane, the points P', P, Q, Q' form a right trapezoid with $\overline{P'P} \perp \overline{P'Q'}$ and $\overline{Q'Q} \perp \overline{P'Q'}$. Draw $\overline{PM}$ perpendicular to $\overline{QQ'}$ as shown.



Then $PP' = MQ' = p$ and $QM = q - p$, while $PQ = p + q$ and $PM = P'Q'$. By the Pythagorean Theorem, $(q-p)^2 + P'Q'^2 = (p+q)^2$, so $q = \frac{(P'Q')^2}{4p}$. Thus $4pq = P'Q'^2 = 12^2$. Similarly, $4pr = P'R'^2 = 6^2$ and $4qr = Q'R'^2 = 12^2$. Dividing the first equation by the third shows that $p = r$ (which can also be inferred from the symmetry of $\triangle P'Q'R'$) and the equation $pr = 9$ yields 3 as their common value; substitute in either of the other two equations to obtain $q = 12$. Therefore the sides of $\triangle PQR$ are $PQ = QR = 12 + 3 = 15$ and $PR = 6$. The altitude to $\overline{PR}$ has length $\sqrt{15^2 - 3^2} = 6\sqrt{6}$, so the triangle's area is $\frac{1}{2}(6)(6\sqrt{6}) = \mathbf{18\sqrt{6}}$.

TB-2. By the definition of f,

$$f(f(x)) = f(x) + (f(x))^2 + (f(x))^4 + (f(x))^8 + \cdots .$$

Consider this series term by term. The first term, $f(x)$, contains no x^{10} terms, so its contribution is 0. The second term, $(f(x))^2$, can produce terms of x^{10} in two ways: as $x^2 \cdot x^8$ or as $x^8 \cdot x^2$. So its contribution is 2.

Now consider the third term:

$$\begin{aligned}
(f(x))^4 &= f(x) \cdot f(x) \cdot f(x) \cdot f(x) \\
&= (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot \\
&\quad (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots).
\end{aligned}$$

Each x^{10} term in the product is the result of multiplying four terms whose exponents sum to 10, one from each factor of $f(x)$. Thus this product contains a term of x^{10} for each quadruple of nonnegative integers (i, j, k, l) such that $2^i + 2^j + 2^k + 2^l = 10$; the order of the quadruple is relevant because rearrangements of the integers correspond to choosing terms from different factors. Note that none of the exponents can exceed 2 because $2^3+2^0+2^0+2^0 > 10$. Therefore $i, j, k, l \leq 2$. Considering cases from largest values to smallest yields two basic cases. First, $10 = 4 + 4 + 1 + 1 = 2^2 + 2^2 + 2^0 + 2^0$, which yields $\frac{4!}{2! \cdot 2!} = 6$ ordered quadruples. Second, $10 = 4+2+2+2 = 2^2+2^1+2^1+2^1$, which yields 4 ordered quadruples. Thus the contribution of the $(f(x))^4$ term is $6 + 4 = 10$.

The last term to consider is $f(x)^8$, because $(f(x))^n$ contains no terms of degree less than n. An analogous analysis to the case of $(f(x))^4$ suggests that the expansion of $(f(x))^8$ has an x^{10} term for every ordered partition of 10 into a sum of eight powers of two. Up to order, there is only one such partition: $2^1 + 2^1 + 2^0 + 2^0 + 2^0 + 2^0 + 2^0 + 2^0$, which yields $\frac{8!}{6! \cdot 2!} = 28$ ordered quadruples.

Therefore the coefficient of x^{10} is $2 + 10 + 28 = \mathbf{40}$.

TB-3. Consider the expansion of $(1.002)^{10}$ as $(1 + 0.002)^{10}$. Using the Binomial Theorem yields the following:

$$(1+0.002)^{10} = 1 + \binom{10}{1}(0.002) + \binom{10}{2}(0.002)^2 + \binom{10}{3}(0.002)^3 + \cdots + (0.002)^{10}.$$

However, when $k > 3$, the terms $\binom{10}{k}(0.002)^k$ do not affect the final answer, because $0.002^4 = 0.000000000016 = \frac{16}{10^{12}}$, and the maximum binomial coefficient is $\binom{10}{5} = 252$, so

$$\binom{10}{4}(0.002)^4 + \binom{10}{5}(0.002)^5 + \cdots + (0.002)^{10} < \frac{252 \cdot 16}{10^{12}} + \frac{252 \cdot 16}{10^{12}} + \cdots + \frac{252 \cdot 16}{10^{12}},$$

where the right side of the inequality contains seven terms, giving an upper bound of $\frac{7 \cdot 252 \cdot 16}{10^{12}}$. The numerator is approximately 28000, but $\frac{28000}{10^{12}} = 2.8 \times 10^{-8}$. So even when multiplied by $100000 = 10^5$, these terms contribute at most 3×10^{-3} to the value of the expression before rounding.

The result of adding the first four terms ($k = 0$ through $k = 3$) and multiplying by 100,000 is given by the following sum:

$$100000 + 10(200) + 45(0.4) + 120(0.0008) = 100000 + 2000 + 18 + 0.096 = 102018.096.$$

Then the desired quantity is $\lfloor 102018.096 \rfloor = \mathbf{102{,}018}$.

2011 Super Relay Problems

1. Suppose that neither of the three-digit numbers $M = \underline{4}\,\underline{A}\,\underline{6}$ and $N = \underline{1}\,\underline{B}\,\underline{7}$ is divisible by 9, but the product $M \cdot N$ **is** divisible by 9. Compute the largest possible value of $A + B$.

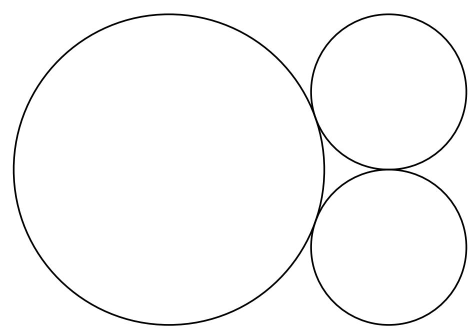
2. Let $T = TNYWR$. Each interior angle of a regular T-gon has measure d°. Compute d.

3. Let $T = TNYWR$, and let k be the sum of the distinct prime factors of T. Suppose that r and s are the two roots of the equation $F_k x^2 + F_{k+1} x + F_{k+2} = 0$, where F_n denotes the n^{th} Fibonacci number. Compute the value of $(r+1)(s+1)$.

4. Let $T = TNYWR$. Compute the product of $-T - i$ and $i - T$, where $i = \sqrt{-1}$.

5. Let $T = TNYWR$. Compute the number of positive divisors of the number $20^4 \cdot 11^T$ that are perfect cubes.

6. Let $T = TNYWR$. As shown, three circles are mutually externally tangent. The large circle has a radius of T, and the smaller two circles each have radius $\frac{T}{2}$. Compute the area of the triangle whose vertices are the centers of the three circles.



7. Let $T = TNYWR$, and let $K = (\frac{T}{12})^2$. In the sequence $0.5, 1, -1.5, 2, 2.5, -3, \ldots$, every third term is negative, and the absolute values of the terms form an arithmetic sequence. Compute the sum of the first K terms of this sequence.

8. Let A be the sum of the digits of the number you will receive from position 7, and let B be the sum of the digits of the number you will receive from position 9. Let (x, y) be a point randomly selected from the interior of the triangle whose consecutive vertices are $(1,1)$, $(B,7)$ and $(17,1)$. Compute the probability that $x > A - 1$.

9. Let $T = TNYWR$. If $\log_2 x^T - \log_4 x = \log_8 x^k$ is an identity for all $x > 0$, compute the value of k.

10. Let $T = TNYWR$. An isosceles trapezoid has an area of $T + 1$, a height of 2, and the shorter base is 3 units shorter than the longer base. Compute the sum of the length of the shorter base and the length of one of the congruent sides.

11. Let $T = TNYWR$. Susan flips a fair coin T times. Leo has an unfair coin such that the probability of flipping heads is $\frac{1}{3}$. Leo gets to flip his coin the least number of times so that Leo's expected number of heads will exceed Susan's expected number of heads. Compute the number of times Leo gets to flip his coin.

12. Let $T = TNYWR$. Dennis and Edward each take 48 minutes to mow a lawn, and Shawn takes 24 minutes to mow a lawn. Working together, how many lawns can Dennis, Edward, and Shawn mow in $2 \cdot T$ hours? (For the purposes of this problem, you may assume that after they complete mowing a lawn, they immediately start mowing the next lawn.)

13. Let $T = TNYWR$. Compute $\sin^2 \dfrac{T\pi}{2} + \sin^2 \dfrac{(5-T)\pi}{2}$.

14. Let $T = TNYWR$. Compute the value of x that satisfies $\sqrt{20 + \sqrt{T + x}} = 5$.

15. The sum of the interior angles of an n-gon equals the sum of the interior angles of a pentagon plus the sum of the interior angles of an octagon. Compute n.

2011 Super Relay Answers

1. 12
2. 150
3. 2
4. 5
5. 12
6. $72\sqrt{2}$
7. 414
8. $\dfrac{79}{128}$
9. 27
10. 9.5
11. 16
12. 10
13. 1
14. 14
15. 11

2011 Super Relay Solutions

1. In order for the conditions of the problem to be satisfied, M and N must both be divisible by 3, but not by 9. Thus the largest possible value of A is 5, and the largest possible value of B is 7, so $A + B = \mathbf{12}$.

2. From the angle sum formula, $d^\circ = \frac{180^\circ \cdot (T-2)}{T}$. With $T = 12$, $d = \mathbf{150}$.

3. Distributing, $(r+1)(s+1) = rs+(r+s)+1 = \frac{F_{k+2}}{F_k} + \left(-\frac{F_{k+1}}{F_k}\right) + 1 = \frac{F_{k+2}-F_{k+1}}{F_k} + 1 = \frac{F_k}{F_k} + 1 = \mathbf{2}$.

4. Multiplying, $(-T-i)(i-T) = -(i+T)(i-T) = -(i^2-T^2) = 1+T^2$. With $T = 2$, $1+T^2 = \mathbf{5}$.

5. Let $N = 20^4 \cdot 11^T = 2^8 \cdot 5^4 \cdot 11^T$. If $m \mid N$, then $m = 2^a \cdot 5^b \cdot 11^c$ where a, b, and c are nonnegative integers such that $a \leq 8, b \leq 4$, and $c \leq T$. If m is a perfect cube, then a, b, and c must be divisible by 3. So $a = 0, 3$, or 6; $b = 0$ or 3, and $c \in \{0, 3, \ldots, 3 \cdot \lfloor T/3 \rfloor\}$. There are a total of $3 \cdot 2 \cdot (\lfloor T/3 \rfloor + 1)$ possible values of m. For $T = 5$, $\lfloor T/3 \rfloor + 1 = 2$, so the number of possible values of m is $\mathbf{12}$.

6. The desired triangle is an isosceles triangle whose base vertices are the centers of the two smaller circles. The congruent sides of the triangle have length $T + \frac{T}{2}$. Thus the altitude to the base has length $\sqrt{\left(\frac{3T}{2}\right)^2 - \left(\frac{T}{2}\right)^2} = T\sqrt{2}$. Thus the area of the triangle is $\frac{1}{2} \cdot \left(\frac{T}{2} + \frac{T}{2}\right) \cdot T\sqrt{2} = \frac{T^2\sqrt{2}}{2}$. With $T = 12$, the area is $\mathbf{72\sqrt{2}}$.

7. The general sequence looks like $x, x+d, -(x+2d), x+3d, x+4d, -(x+5d), \ldots$. The sum of the first three terms is $x - d$; the sum of the second three terms is $x + 2d$; the sum of the third three terms is $x + 5d$, and so on. Thus the sequence of sums of terms $3k-2, 3k-1$, and $3k$ is an arithmetic sequence. Notice that $x = d = 0.5$ and so $x - d = 0$. If there are n triads of terms of the original sequence, then their common difference is 1.5 and their sum is $n \cdot \left(\frac{0+0+(n-1)\cdot 1.5}{2}\right)$. $T = 72\sqrt{2}$, so $K = 72$, and $n = 24$. Thus the desired sum is $\mathbf{414}$.

8. Let $P = (1, 1)$, $Q = (17, 1)$, and $R = (B, 7)$ be the vertices of the triangle, and let $X = (B, 1)$ be the foot of the perpendicular from R to $\overleftrightarrow{PQ}$. Let $M = (A-1, 1)$ and let ℓ be the vertical line through M; then the problem is to determine the fraction of the area of $\triangle PQR$ that lies to the right of ℓ.

   Note that $B \geq 0$ and $A \geq 0$ because they are digit sums of integers. Depending on their values, the line ℓ might intersect any two sides of the triangle or none at all. Each case requires a separate computation. There are two cases where the computation is trivial. First, when ℓ passes to the left of or through the leftmost vertex of $\triangle PQR$, which occurs when $A-1 \leq \min(B,1)$, the probability is 1. Second, when ℓ passes to the right of or through the rightmost vertex of $\triangle PQR$, which occurs when $A-1 \geq \max(B,17)$, the probability is 0. The remaining cases are as follows.

Case 1: The line ℓ intersects $\overline{PQ}$ and $\overline{PR}$ when $1 \leq A-1 \leq 17$ and $A-1 \leq B$.
Case 2: The line ℓ intersects $\overline{PQ}$ and $\overline{QR}$ when $1 \leq A-1 \leq 17$ and $A-1 \geq B$.
Case 3: The line ℓ intersects $\overline{PR}$ and $\overline{QR}$ when $17 \leq A-1 \leq B$.

Now proceed case by case.

Case 1: Let T be the point of intersection of ℓ and $\overline{PR}$. Then the desired probability is $[MQRT]/[PQR] = 1-[PMT]/[PQR]$. Since $\triangle PMT \sim \triangle PXR$ and the areas of similar triangles are proportional to the squares of corresponding sides, $[PMT]/[PXR] = (PM/PX)^2$. Since $\triangle PXR$ and $\triangle PQR$ both have height XR, their areas are proportional to their bases: $[PXR]/[PQR] = PX/PQ$. Taking the product, $[PMT]/[PQR] = (PM/PX)^2(PX/PQ) = \frac{PM^2}{PX \cdot PQ} = \frac{(A-2)^2}{(B-1)(17-1)}$, and the final answer is

$$\frac{[MQRT]}{[PQR]} = 1 - \frac{[PMT]}{[PQR]} = 1 - \frac{(A-2)^2}{16(B-1)}.$$

Case 2: Let U be the point of intersection of ℓ and $\overline{QR}$. A similar analysis to the one in the previous case yields

$$\frac{[MQU]}{[PQR]} = \frac{[MQU]}{[XQR]} \cdot \frac{[XQR]}{[PQR]} = \left(\frac{MQ}{XQ}\right)^2 \frac{XQ}{PQ} = \frac{(18-A)^2}{16(17-B)}.$$

Case 3: Let T be the point of intersection of ℓ and $\overline{PR}$ and let U be the point of intersection of ℓ and $\overline{QR}$ as in the previous cases. Let S be the point on $\overline{PR}$ such that $\overline{QS} \perp \overline{PQ}$. Then $\triangle TUR \sim \triangle SQR$, so the areas of these two triangles are proportional to the squares of the corresponding *altitudes* MX and QX. Thinking of $\overleftrightarrow{PR}$ as the common base, $\triangle SQR$ and $\triangle PQR$ have a common altitude, so the ratio of their areas is SR/PR. Since $\triangle PQS \sim \triangle PXR$, $PS/PR = PQ/PX$ and so $\frac{SR}{PR} = 1 - \frac{PS}{PR} = 1 - \frac{PQ}{PX} = \frac{QX}{PX}$. Therefore the desired probability is

$$\frac{[TUR]}{[PQR]} = \frac{[TUR]}{[SQR]} \cdot \frac{[SQR]}{[PQR]} = \left(\frac{MX}{QX}\right)^2 \frac{QX}{PX} = \frac{(B-A+1)^2}{(B-17)(B-1)}.$$

Using the answers from positions 7 and 9, $A = 4+1+4 = 9$ and $B = 2+7 = 9$. The first case applies, so the probability is

$$1 - \frac{(9-2)^2}{16(9-1)} = 1 - \frac{49}{128} = \mathbf{\frac{79}{128}}.$$

9. Note that in general, $\log_b c = \log_{b^n} c^n$. Using this identity yields $\log_2 x^T = \log_{2^2}(x^T)^2 = \log_4 x^{2T}$. Thus the left hand side of the given equation simplifies to $\log_4 x^{2T-1}$. Express each side in base 64: $\log_4 x^{2T-1} = \log_{64} x^{6T-3} = \log_{64} x^{2k} = \log_8 x^k$. Thus $k = 3T - \frac{3}{2}$. With $T = 9.5$, $k = \mathbf{27}$.

10. Let x be the length of the shorter base of the trapezoid. The area of the trapezoid is $\frac{1}{2} \cdot 2 \cdot (x + x + 3) = T + 1$, so $x = \frac{T}{2} - 1$. Drop perpendiculars from each vertex of the shorter base to the longer base, and note that by symmetry, the feet of these perpendiculars lie $\frac{3}{2} = 1.5$ units away from their nearest vertices of the trapezoid. Hence the congruent sides have length $\sqrt{1.5^2 + 2^2} = 2.5$. With $T = 16$, $x = 7$, and the desired sum of the lengths is $\mathbf{9.5}$.

11. The expected number of heads for Susan is $\frac{T}{2}$. If Leo flips his coin N times, the expected number of heads for Leo is $\frac{N}{3}$. Thus $\frac{N}{3} > \frac{T}{2}$, so $N > \frac{3T}{2}$. With $T = 10$, the smallest possible value of N is $\mathbf{16}$.

12. Working together, Dennis and Edward take $\frac{48}{2} = 24$ minutes to mow a lawn. When the three of them work together, it takes them $\frac{24}{2} = 12$ minutes to mow a lawn. Thus they can mow 5 lawns per hour. With $T = 1$, they can mow $5 \cdot 2 = \mathbf{10}$ lawns in 2 hours.

13. Note that $\sin \dfrac{(5-T)\pi}{2} = \cos\left(\dfrac{\pi}{2} - \dfrac{(5-T)\pi}{2}\right) = \cos\left(\dfrac{T\pi}{2} - 2\pi\right) = \cos \dfrac{T\pi}{2}$. Thus the desired quantity is $\sin^2 \dfrac{T\pi}{2} + \cos^2 \dfrac{T\pi}{2} = \mathbf{1}$ (independent of T).

14. Squaring each side gives $20 + \sqrt{T + x} = 25$, thus $\sqrt{T + x} = 5$, and $x = 25 - T$. With $T = 11$, $x = \mathbf{14}$.

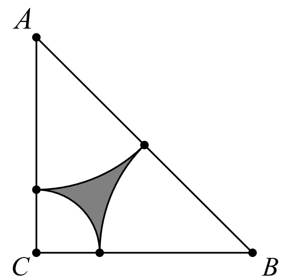
15. Using the angle sum formula, $180° \cdot (n - 2) = 180° \cdot 3 + 180° \cdot 6 = 180° \cdot 9$. Thus $n - 2 = 9$, and $n = \mathbf{11}$.

2012 Contest

Contents

2012 Team Problems

T-1. In $\triangle ABC$, $\mathrm{m}\angle A = \mathrm{m}\angle B = 45^\circ$ and $AB = 16$. Mutually tangent circular arcs are drawn centered at all three vertices; the arcs centered at A and B intersect at the midpoint of $\overline{AB}$. Compute the area of the region inside the triangle and outside of the three arcs.



T-2. Compute the number of ordered pairs of integers (a, b) such that $1 < a \leq 50$, $1 < b \leq 50$, and $\log_b a$ is rational.

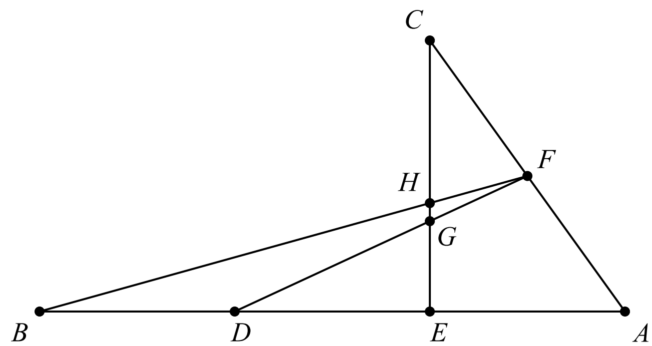
T-3. Suppose that 5-letter "words" are formed using only the letters A, R, M, and L. Each letter need not be used in a word, but each word must contain at least two distinct letters. Compute the number of such words that use the letter A more than any other letter.

T-4. Positive integers $a_1, a_2, a_3, \ldots$ form an arithmetic sequence. If $a_1 = 10$ and $a_{a_2} = 100$, compute $a_{a_{a_3}}$.

T-5. The graphs of $y = x^2 - |x| - 12$ and $y = |x| - k$ intersect at distinct points A, B, C, and D, in order of increasing x-coordinates. If $AB = BC = CD$, compute k.

T-6. The zeros of $f(x) = x^6 + 2x^5 + 3x^4 + 5x^3 + 8x^2 + 13x + 21$ are distinct complex numbers. Compute the average value of $A + BC + DEF$ over all possible permutations (A, B, C, D, E, F) of these six numbers.

T-7. Given noncollinear points A, B, C, segment $\overline{AB}$ is trisected by points D and E, and F is the midpoint of segment $\overline{AC}$. $\overline{DF}$ and $\overline{BF}$ intersect $\overline{CE}$ at G and H, respectively. If $[DEG] = 18$, compute $[FGH]$.



T-8. Let $N = \lfloor (3+\sqrt{5})^{34} \rfloor$. Compute the remainder when N is divided by 100.

T-9. Let ABC be a triangle with $m\angle B = m\angle C = 80^\circ$. Compute the number of points P in the plane such that triangles PAB, PBC, and PCA are all isosceles and non-degenerate. Note: the approximation $\cos 80^\circ \approx 0.17$ may be useful.

T-10. If $\lceil u \rceil$ denotes the least integer greater than or equal to u, and $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u, compute the largest solution x to the equation

$$\left\lfloor \frac{x}{3} \right\rfloor + \lceil 3x \rceil = \sqrt{11} \cdot x.$$

2012 Team Answers

T-1. $64 - 64\pi + 32\pi\sqrt{2}$

T-2. 81

T-3. 165

T-4. 820

T-5. $10 + 2\sqrt{2}$

T-6. $-\dfrac{23}{60}$

T-7. $\dfrac{9}{5}$

T-8. 47

T-9. 6

T-10. $\dfrac{189\sqrt{11}}{11}$

2012 Team Solutions

T-1. Because $AB = 16$, $AC = BC = \frac{16}{\sqrt{2}} = 8\sqrt{2}$. Then each of the large arcs has radius 8, and the small arc has radius $8\sqrt{2} - 8$. Each large arc has measure $45°$ and the small arc has measure $90°$. Therefore the area enclosed by each large arc is $\frac{45}{360} \cdot \pi \cdot 8^2 = 8\pi$, and the area enclosed by the small arc is $\frac{90}{360} \cdot \pi \cdot \left(8\sqrt{2} - 8\right)^2 = 48\pi - 32\pi\sqrt{2}$. Thus the sum of the areas enclosed by the three arcs is $64\pi - 32\pi\sqrt{2}$. On the other hand, the area of the triangle is $\frac{1}{2}\left(8\sqrt{2}\right)^2 = 64$. So the area of the desired region is $\mathbf{64 - 64\pi + 32\pi\sqrt{2}}$.

T-2. Begin by partitioning $\{2, 3, \ldots, 50\}$ into the subsets

$$\begin{aligned} A &= \{2, 4, 8, 16, 32\} \\ B &= \{3, 9, 27\} \\ C &= \{5, 25\} \\ D &= \{6, 36\} \\ E &= \{7, 49\} \\ F &= \text{all other integers between 2 and 50, inclusive.} \end{aligned}$$

If $\log_b a$ is rational, then either a and b are both members of one of the sets A, B, C, D, or E, or $a = b \in F$ (see note below for proof). Then the number of possible ordered pairs is

$$\begin{aligned} |A|^2 + |B|^2 + |C|^2 + |D|^2 + |E|^2 + |F| &= 25 + 9 + 4 + 4 + 4 + 35 \\ &= \mathbf{81}. \end{aligned}$$

Note: The solution essentially relies on one fact: if $\log_b a$ is rational, then b and a are both integral powers of some integer d. This fact can be demonstrated as follows. Suppose that $\log_b a = m/n$; then $b^{m/n} = a \Rightarrow (b^{1/n})^m = a$. Let $d = b^{1/n}$. The proof that d is an integer requires a significant theorem in number theory: if m and n are relatively prime positive integers, there exist integers x and y such that $mx - ny = 1$. Now consider the set $S = \left\{\frac{a^x}{b^y} \mid x, y \in \mathbb{Z}\right\}$. Because a and b are rational and x and y are integers, the elements $\frac{a^x}{b^y}$ are all rational numbers. But because $a = d^m$ and $b = d^n$, the expression $\frac{a^x}{b^y} = \frac{d^{mx}}{d^{ny}} = d^{mx-ny}$. By the theorem mentioned previously, there exist integers x and y such that $mx - ny = 1$, and $d^{mx-ny} \in S$. So d^1 is rational, hence d is rational. Because the n^{th} root of an integer is either an integer or an irrational number, d must be an integer.

T-3. Condition on the number n of A's that appear in the word; n is at least two, because of the requirement that A occur more often than any other letter, and n is at most 4, because of the requirement that there be at least two distinct letters. In the case $n = 4$, there are 3 choices for the other letter, and 5 choices for where to place it, for a total of 15 possibilities. In the case $n = 3$, there are two possibilities to consider: either a second letter occurs twice, or there are two distinct letters besides A. If a second letter occurs twice, there are 3 choices for the other letter, and $\frac{5!}{3!\cdot 2!} = 10$ ways to arrange the three A's and two non-A's, for their locations, for a total of 30 choices. If there are two distinct letters besides A, then there are $\binom{3}{2} = 3$ ways to pick the two letters, and $\frac{5!}{3!\cdot 1!\cdot 1!} = 20$ ways to arrange them, for a total of 60 words. Thus there are a combined total of 90 words when $n = 3$. In the case $n = 2$, no other letter can occur twice, so all the letters R, M, L, must appear in the word; they can be arranged in $\frac{5!}{2!\cdot 1!\cdot 1!\cdot 1!} = 60$ ways. The total number of words satisfying the conditions is therefore $15 + 90 + 60 = \mathbf{165}$.

T-4. Let d be the common difference of the sequence. Then $a_{a_2} = a_1 + (a_2 - 1)d = 100 \Rightarrow (a_2 - 1)d = 90$. But $a_2 = a_1 + d = 10 + d$, so $(9 + d)d = 90$. Solving the quadratic yields $d = -15$ or $d = 6$, but the requirement that a_i be positive for all i rules out the negative value, so $d = 6$ and $a_n = 10 + (n - 1) \cdot 6$. Thus $a_3 = 10 + 2(6) = 22$, and $a_{a_3} = a_{22} = 10 + 21(6) = 136$. Finally, $a_{a_{a_3}} = a_{136} = 10 + 135(6) = \mathbf{820}$.

T-5. First, note that both graphs are symmetric about the y-axis, so C and D must be reflections of B and A, respectively, across the y-axis. Thus $x_C = -x_B$ and $y_C = y_B$, so $BC = 2x_C$. For $x < 0$, the equations become $y = x^2 + x - 12$ and $y = -x - k$; setting the x-expressions equal to each other yields the equation $x^2 + 2x + (k - 12) = 0$, from which $x = -1 \pm \sqrt{13 - k}$. Therefore $x_B = -1 + \sqrt{13 - k}$ and $BC = 2 - 2\sqrt{13 - k}$. (Note that the existence of two distinct negative values of $-1 \pm \sqrt{13 - k}$ forces $12 < k \leq 13$.)

Thus the x-coordinates of the four points are

$$\begin{aligned}
x_A &= -1 - \sqrt{13 - k} \\
x_B &= -1 + \sqrt{13 - k} \\
x_C &= 1 - \sqrt{13 - k} \\
x_D &= 1 + \sqrt{13 - k}.
\end{aligned}$$

To compute y_A, use the second equation $y = |x| - k$ to obtain $y_A = 1 + \sqrt{13 - k} - k = (1 - k) + \sqrt{13 - k}$; similarly, $y_B = (1 - k) - \sqrt{13 - k}$. Therefore

$$\begin{aligned}
AB &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\
&= \sqrt{(2\sqrt{13 - k})^2 + (-2\sqrt{13 - k})^2} \\
&= 2\sqrt{2(13 - k)}.
\end{aligned}$$

Because $AB = BC$, $2\sqrt{2(13 - k)} = 2 - 2\sqrt{13 - k}$. Let $u = \sqrt{13 - k}$; then $2\sqrt{2}\,u = 2 - 2u$, from which $u = \frac{2}{2 + 2\sqrt{2}} = \frac{1}{1 + \sqrt{2}}$, which equals $\sqrt{2} - 1$ by rationalizing the denominator. Thus

$$13 - k = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}, \text{ so } k = \mathbf{10 + 2\sqrt{2}}.$$

Because $10 + 2\sqrt{2} \approx 12.8$, the value of k determined algebraically satisfies the inequality $12 < k \leq 13$ observed above.

Alternate Solution: Let $C = (a, b)$. Because C and D lie on a line with slope 1, $D = (a + h, b + h)$ for some $h > 0$. Because both graphs are symmetric about the y-axis, the other two points of intersection are $A = (-a - h, b + h)$ and $B = (-a, b)$, and $a > 0$.

In terms of these coordinates, the distances are $AB = CD = \sqrt{2}\,h$ and $BC = 2a$. Thus the condition $AB = BC = CD$ holds if and only if $\sqrt{2}\,h = 2a$, or $h = \sqrt{2}\,a$.

The foregoing uses the condition that C and D lie on a line of slope 1, so now use the remaining equation and subtract:

$$\begin{aligned} b &= a^2 - a - 12; \\ b + h &= (a + h)^2 - (a + h) - 12; \\ h &= 2ah + h^2 - h. \end{aligned}$$

Because the points are distinct, $h \neq 0$. Dividing by h yields $2 - 2a = h = \sqrt{2}\,a$. Thus $a = \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}$.

Finally, because C lies on the two graphs, $b = a^2 - a - 12 = -8 - 3\sqrt{2}$ and $k = a - b = \mathbf{10 + 2\sqrt{2}}$.

T-6. There are $6! = 720$ permutations of the zeros, so the average value is the sum, S, divided by 720. Setting any particular zero as A leaves $5! = 120$ ways to permute the other five zeros, so over the 720 permutations, each zero occupies the A position 120 times. Similarly, fixing any ordered pair (B, C) of zeros allows $4! = 24$ permutations of the other four zeros, and $BC = CB$ means that each *value* of BC occurs 48 times. Finally, fixing any ordered triple (D, E, F) allows $3! = 6$ permutations of the other variables, and there are $3! = 6$ equivalent arrangements within each product DEF, so that the product of any three zeros occurs 36 times within the sum. Let $S_1 = A + B + C + D + E + F$ (i.e., the sum of the zeros taken singly), $S_2 = AB + AC + \cdots + AF + BC + \cdots + EF$ (i.e., the sum of the zeros taken two at a time), and $S_3 = ABC + ABD + \cdots + DEF$ be the sum of the zeros three at a time. Then $S = 120S_1 + 48S_2 + 36S_3$. Using the sums and products of roots formulas, $S_1 = -2/1 = -2$, $S_2 = 3/1 = 3$, and $S_3 = -5/1 = -5$. Thus $S = 120(-2) + 48(3) + 36(-5) = -276$. The average value is thus $-\dfrac{276}{720} = \boldsymbol{-\dfrac{23}{60}}$.

T-7. Compute the desired area as $[EGFB] - [EHB]$. To compute the area of concave quadrilateral $EGFB$, draw segment $\overline{BG}$, which divides the quadrilateral into three triangles, $\triangle DEG$, $\triangle BDG$, and $\triangle BGF$. Then $[BDG] = [DEG] = 18$ because the triangles have equal bases and heights. Because D, G, and F are collinear, to compute $[BGF]$ it suffices to find the ratio DG/GF. Use Menelaus's Theorem on $\triangle ADF$ with Menelaus Line $\overline{EC}$ to obtain

$$\frac{AE}{ED} \cdot \frac{DG}{GF} \cdot \frac{FC}{CA} = 1.$$

Because E and F are the midpoints of $\overline{AD}$ and $\overline{CA}$ respectively, $AE/ED = 1$ and $FC/CA = 1/2$. Therefore $DG/GF = 2/1$, and $[BGF] = \frac{1}{2}[BDG] = 9$. Thus $[EGFB] = 18+18+9 = 45$.

To compute $[EHB]$, consider that its base $\overline{EB}$ is twice the base of $\triangle DEG$. The ratio of their heights equals the ratio EH/EG because the altitudes from H and G to $\overleftrightarrow{BE}$ are parallel to each other. Use Menelaus's Theorem twice more on $\triangle AEC$ to find these values:

$$\frac{AD}{DE}\cdot\frac{EG}{GC}\cdot\frac{CF}{FA} = 1 \Rightarrow \frac{EG}{GC} = \frac{1}{2} \Rightarrow EG = \frac{1}{3}EC, \text{ and}$$
$$\frac{AB}{BE}\cdot\frac{EH}{HC}\cdot\frac{CF}{FA} = 1 \Rightarrow \frac{EH}{HC} = \frac{2}{3} \Rightarrow EH = \frac{2}{5}EC.$$

Therefore $\frac{EH}{EG} = \frac{2/5}{1/3} = \frac{6}{5}$. Thus $[EHB] = \frac{6}{5}\cdot 2\cdot[DEG] = \frac{216}{5}$. Thus $[FGH] = 45 - \frac{216}{5} = \mathbf{\frac{9}{5}}$.

Alternate Solution: The method of mass points leads to the same results as Menelaus's Theorem, but corresponds to the physical intuition that masses on opposite sides of a fulcrum balance if and only if the products of the masses and their distances from the fulcrum are equal (in physics-speak, the net torque is zero). If a mass of weight 1 is placed at vertex B and masses of weight 2 are placed at vertices A and C, then $\triangle ABC$ balances on the line $\overleftrightarrow{BF}$ and also on the line $\overleftrightarrow{CE}$. Thus it balances on the point H where these two lines intersect. Replacing the masses at A and C with a single mass of weight 4 at their center of mass F, the triangle still balances at H. Thus $BH/HF = 4$.

Next, consider $\triangle BEF$. Placing masses of weight 1 at the vertices B and E and a mass of weight 4 at F, the triangle balances at G. A similar argument shows that $DG/GF = 2$ and that $EG/GH = 5$. Because $\triangle DEG$ and $\triangle FHG$ have congruent (vertical) angles at G, it follows that $[DEG]/[FHG] = (DG/FG)\cdot(EG/HG) = 2\cdot 5 = 10$. Thus $[FGH] = [DEG]/10 = \frac{18}{10} = \mathbf{\frac{9}{5}}$.

T-8. Let $\alpha = 3+\sqrt{5}$ and $\beta = 3-\sqrt{5}$, so that $N = \lfloor\alpha^{34}\rfloor$, and let $M = \alpha^{34}+\beta^{34}$. When the binomials in M are expanded, terms in which $\sqrt{5}$ is raised to an odd power have opposite signs, and so cancel each other out. Therefore M is an integer. Because $0 < \beta < 1$, $0 < \beta^{34} < 1$, and so $M - 1 < \alpha^{34} < M$. Therefore $M - 1 = N$. Note that α and β are the roots of $x^2 = 6x - 4$. Therefore $\alpha^{n+2} = 6\alpha^{n+1} - 4\alpha^n$ and $\beta^{n+2} = 6\beta^{n+1} - 4\beta^n$. Hence $\alpha^{n+2}+\beta^{n+2} = 6(\alpha^{n+1}+\beta^{n+1}) - 4(\alpha^n+\beta^n)$. Thus the sequence of numbers $\{\alpha^n+\beta^n\}$ satisfies the recurrence relation $c_{n+2} = 6c_{n+1} - 4c_n$. All members of the sequence are determined by the initial values c_0 and c_1, which can be computed by substituting 0 and 1 for n in the expression $\alpha^n+\beta^n$, yielding $c_0 = (3+\sqrt{5})^0 + (3-\sqrt{5})^0 = 2$, and $c_1 = (3+\sqrt{5})^1 + (3-\sqrt{5})^1 = 6$. Then

$$\begin{aligned} c_2 &= (3+\sqrt{5})^2 + (3-\sqrt{5})^2 = 6c_1 - 4c_0 = 36 - 8 = 28, \\ c_3 &= (3+\sqrt{5})^3 + (3-\sqrt{5})^3 = 6c_2 - 4c_1 = 168 - 24 = 144, \end{aligned}$$

and because the final result is only needed modulo 100, proceed using only remainders modulo 100.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_n mod 100	6	28	44	52	36	8	4	92	36	48	44	72	56	48	64	92	96

n	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
c_n mod 100	8	64	52	56	28	44	52	36	8	4	92	36	48	44	72	56	48

Thus N leaves a remainder of $48 - 1 = \mathbf{47}$ when divided by 100.

Alternate Solution: As in the previous solution, let $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$, so that $N = \alpha^{34} + \beta^{34} - 1$ as argued above.

A straightforward way to compute powers of α and β is by successive squaring. Paying attention to just the last two digits of the integer parts yields the following values:

$$\begin{aligned}
\alpha^2 &= 14 + 6\sqrt{5};\\
\alpha^4 &= 196 + 180 + 168\sqrt{5} \equiv 76 + 68\sqrt{5};\\
\alpha^8 &\equiv 96 + 36\sqrt{5};\\
\alpha^{16} &\equiv 96 + 12\sqrt{5};\\
\alpha^{32} &\equiv 36 + 4\sqrt{5};\\
\alpha^{34} = \alpha^2 \cdot \alpha^{32} &\equiv 24 + 72\sqrt{5}.
\end{aligned}$$

Similarly, replacing $\sqrt{5}$ with $-\sqrt{5}$ yields $\beta^{34} \equiv 24 - 72\sqrt{5}$. Thus

$$N \equiv \left(24 + 72\sqrt{5}\right) + \left(24 - 72\sqrt{5}\right) - 1 \equiv \mathbf{47} \pmod{100}.$$

Alternate Solution: As in the previous solutions, let $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$, so that $N = \alpha^{34} + \beta^{34} - 1$ as argued above.

Now consider the binomial expansions more carefully:

$$\begin{aligned}
\alpha^{34} &= 3^{34} + \binom{34}{1}3^{33}\sqrt{5} + \binom{34}{2}3^{32}\cdot 5 + \binom{34}{3}3^{31}\cdot 5\sqrt{5} + \cdots + \binom{34}{33}3\cdot 5^{16}\sqrt{5} + 5^{17};\\
\beta^{34} &= 3^{34} - \binom{34}{1}3^{33}\sqrt{5} + \binom{34}{2}3^{32}\cdot 5 - \binom{34}{3}3^{31}\cdot 5\sqrt{5} + \cdots - \binom{34}{33}3\cdot 5^{16}\sqrt{5} + 5^{17};\\
N &= 2\left(3^{34} + \binom{34}{2}3^{32}\cdot 5 + \cdots + \binom{34}{32}3^2\cdot 5^{16} + 5^{17}\right) - 1.
\end{aligned}$$

The following argument shows that every term that is summarized by the ellipsis $(\cdots)$ in the expression for N is a multiple of 50. First, each such term has the form $\binom{34}{2k}3^{34-2k}5^k$, where $2 \le k \le 15$.

Thus it is enough to show that the binomial coefficient is even. Because $\binom{34}{2k} = \binom{34}{34-2k}$, it is enough to check this for $2 \le k \le 8$. Keep track of powers of 2: $\binom{34}{2}$ is an integer, so

$\binom{34}{4} = \binom{34}{2} \cdot \frac{32\cdot 31}{3\cdot 4}$ is a multiple of 2^3; $\binom{34}{6} = \binom{34}{4} \cdot \frac{30\cdot 29}{5\cdot 6}$ is also a multiple of 2^3; $\binom{34}{8} = \binom{34}{6} \cdot \frac{28\cdot 27}{7\cdot 8}$ is a multiple of 2^2; and so on.

It can also be shown that the sum of the last two terms is a multiple of 50. Again, there are plenty of factors of 5, so it is enough to note that both terms are odd, because $\binom{34}{32} = \frac{34\cdot 33}{1\cdot 2} = 17 \cdot 33$.

Thanks to the initial factor of 2 in the expression for N (outside the parentheses), the previous paragraphs show that $N \equiv 2\big(3^{34} + \binom{34}{2}3^{32} \cdot 5\big) - 1 \pmod{100}$.

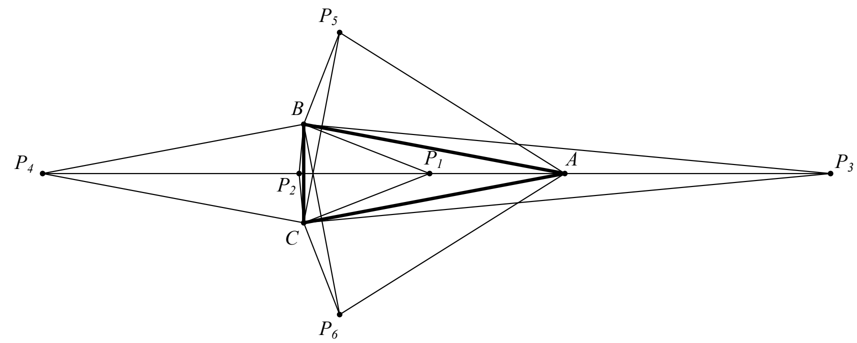
Now consider the powers of 3. Because $3^4 = 81$, we find that $3^8 = 80^2 + 2 \cdot 80 + 1 \equiv 61 \pmod{100}$, $3^{12} \equiv 41 \pmod{100}$, $3^{16} \equiv 21 \pmod{100}$, and $3^{20} \equiv 1 \pmod{100}$. (Note: those familiar with Euler's generalization of Fermat's Little Theorem will recognize this as an example, because $\phi(25) = 25 - 5 = 20$.) Therefore $3^{32} = 3^{20} \cdot 3^{12} \equiv 41 \pmod{100}$ and $3^{34} = 3^2 \cdot 3^{32} \equiv 69 \pmod{100}$.

Finally, $N \equiv 2\big(69 + 17 \cdot 33 \cdot 41 \cdot 5\big) - 1 \equiv 2 \cdot 69 + 10 \cdot \big(17 \cdot 33 \cdot 41\big) - 1 \equiv 38 + 10 - 1 \equiv \mathbf{47} \pmod{100}$.

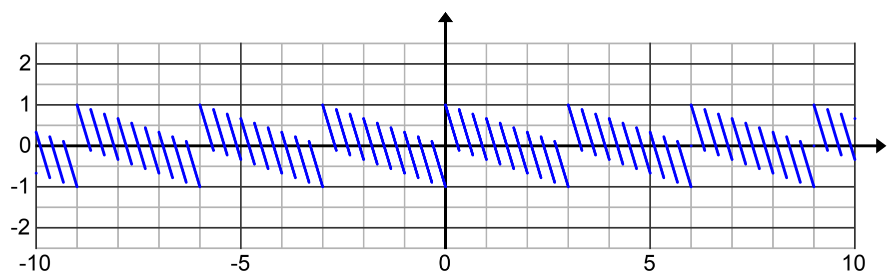
T-9. Focus on $\triangle PBC$. Either $PB = PC$ or $PB = BC$ or $PC = BC$.

If $PB = PC$, then P lies on the perpendicular bisector l of side $\overline{BC}$. Considering now $\triangle PAB$, if $PA = PB$, then $PA = PC$, and P must be the circumcenter of $\triangle ABC$; call this location P_1. If $PA = AB$, then $PA = AC$, and P, B, C all lie on a circle with center A and radius AB. There are two intersection points of that circle with l, one on each arc with endpoints B and C; label the one on the major arc P_2 and on the minor arc P_3. Finally, if $PB = AB$, then $PB = AC$ by the transitive property and $PC = AC$ by the perpendicular bisector theorem, so $PBAC$ is a rhombus; P is the reflection of A across $\overline{BC}$. Call this point P_4.

If $PB = BC$, then P must lie on the circle centered at B with radius BC. Considering $\triangle PAB$, if $PA = AB$, then P lies on the circle centered at A with radius AB. Now $\odot A$ and $\odot B$ intersect at two points, but one of them is C, so the other intersection must be the location of P, which is P_5. The condition $PB = AB$ is impossible, because it implies that $AB = BC$, which is false because in $\triangle ABC$, $m\angle C > m\angle A = 20°$, so $AB > BC$. The third possibility for $\triangle PAB$ is that $PA = PB$, implying that the perpendicular bisector of $\overline{AB}$ intersects $\odot B$, which only occurs if $BC/AB \geq 1/2$ (although if $BC/AB = 1/2$, the triangle is degenerate). But $BC/AB = 2\cos 80°$, and the given approximation $\cos 80° \approx 0.17$ implies that $BC/AB \approx 0.34$. Hence the perpendicular bisector of $\overline{AB}$ does not intersect $\odot B$. Thus the assumption $PB = BC$ yields only one additional location for P, P_5. Similarly, $PC = BC$ yields exactly one more location, P_6, for a total of **6** points. All six points, and their associated triangles, are pictured below.



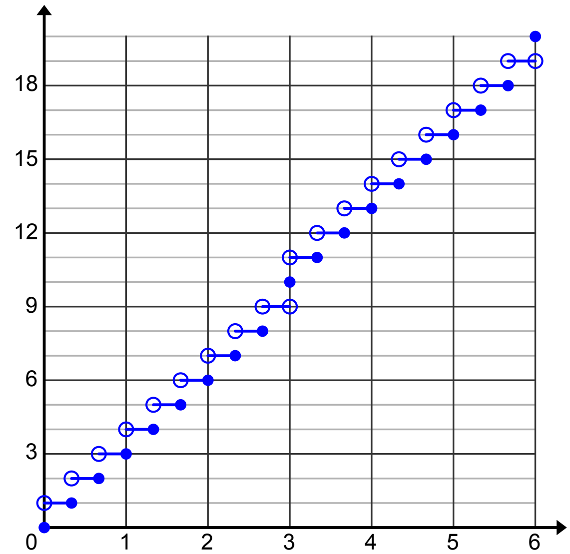
T-10. Let $f(x) = \left\lfloor \frac{x}{3} \right\rfloor + \lceil 3x \rceil$. Observe that $f(x+3) = f(x)+1+9 = f(x)+10$. Let $g(x) = f(x) - \frac{10}{3}x$. Then g is periodic, because $g(x+3) = f(x) + 10 - \frac{10x}{3} - \frac{10 \cdot 3}{3} = g(x)$. The graph of g is shown below:



Because $g(x)$ is the (vertical) distance between the graph of $y = f(x)$ and the line $y = \frac{10}{3}x$, the fact that g is periodic implies that f always stays within some fixed distance D of the line $y = \frac{10}{3}x$. On the other hand, because $\frac{10}{3} > \sqrt{11}$, the graph of $y = \frac{10}{3}x$ gets further and further away from the graph of $y = \sqrt{11}x$ as x increases. Because the graph of $y = f(x)$ remains near $y = \frac{10}{3}x$, the graph of $y = f(x)$ drifts upward from the line $y = \sqrt{11}x$.

For each integer n, define the open interval $I_n = (\frac{n-1}{3}, \frac{n}{3})$. In fact, f is constant on I_n, as the following argument shows. For $x \in I_n$, $\frac{n}{9} - \frac{1}{9} < \frac{x}{3} < \frac{n}{9}$. Because n is an integer, there are no integers between $\frac{n}{9} - \frac{1}{9}$ and $\frac{n}{9}$, so $\left\lfloor \frac{x}{3} \right\rfloor$ is constant; similarly, $\lceil 3x \rceil$ is constant on the same intervals. Let l_n be the value of f on the interval I_n, and let $L_n = f(\frac{n}{3})$, the value at the right end of the interval I_n. If n is not a multiple of 9, then $l_n = L_n$, because as x increases from $n - \varepsilon$ to n, the floor function does not increase. This means that f is actually constant on the half-closed interval $(\frac{n-1}{3}, \frac{n}{3}]$. If neither n nor $n+1$ are multiples of 9, then $l_{n+1} = l_n + 1$. However if n is a multiple of 9, then $L_n = l_n + 1$ and $l_{n+1} = L_n + 1$. (The value of $f(x)$ increases when x increases from $n - \varepsilon$ to n, as well as going from n to $n + \varepsilon$.)

Hence on each interval of the form $(3n-3, 3n)$, the graph of f looks like 9 steps of height 1 and width $\frac{1}{3}$, all open on the left and closed on the right except for the last step, which is open on both ends. Between the intervals $(3n-3, 3n)$ and $(3n, 3n+3)$, $f(x)$ increases by 2, with $f(3n)$ halfway between steps. This graph is shown below:



On each interval $(3n-3, 3n)$, the average rate of change is $3 < \sqrt{11}$, so the steps move down relative $y = \sqrt{11}x$ within each interval. At the end of each interval, the graph of f rises relative to $y = \sqrt{11}x$. Thus the last intersection point between $f(x)$ and $\sqrt{11}x$ will be on the ninth step of one of these intervals. Suppose this intersection point lies in the interval $(3k-3, 3k)$. The ninth step is of height $10k-1$. Set $x = 3k - r$, where $r < \frac{1}{3}$. Then the solution is the largest k for which

$$\begin{aligned} 10k-1 &= \sqrt{11}(3k-r) \qquad \left(0 < r < \frac{1}{3}\right) \\ k(10-3\sqrt{11}) &= 1-\sqrt{11}r < 1 \\ k &< \frac{1}{10-3\sqrt{11}} = 10+3\sqrt{11} < 20. \end{aligned}$$

Because $0 < 19(10-3\sqrt{11}) < 1$, $k = 19$ implies a value of r between 0 and $\frac{1}{\sqrt{11}}$. And because $\frac{1}{\sqrt{11}} < \frac{1}{3}$,

$$x = 3k - r = \frac{10k-1}{\sqrt{11}} = \mathbf{\frac{189\sqrt{11}}{11}}$$

is the largest solution to $f(x) = \sqrt{11}x$.

Alternate Solution: Let x be the largest real number for which $\left\lfloor \frac{x}{3} \right\rfloor + \lceil 3x \rceil = \sqrt{11}x$. Because the left-hand side of this equation is an integer, it is simpler to work with $n = \sqrt{11}x$ instead of x. The equation becomes

$$\left\lfloor \frac{n}{3\sqrt{11}} \right\rfloor + \left\lceil \frac{3n}{\sqrt{11}} \right\rceil = n.$$

A little bit of computation shows that $\frac{1}{3\sqrt{11}} + \frac{3}{\sqrt{11}} > 1$, so the equation cannot hold for large values of n. To make this explicit, write

$$\left\lfloor \frac{n}{3\sqrt{11}} \right\rfloor = \frac{n}{3\sqrt{11}} - r \quad \text{and} \quad \left\lceil \frac{3n}{\sqrt{11}} \right\rceil = \frac{3n}{\sqrt{11}} + s,$$

where r and s are real numbers between 0 and 1. (If $n \neq 0$, then r and s are strictly between 0 and 1.) Then

$$\begin{aligned} 1 > r - s &= \left(\frac{n}{3\sqrt{11}} - \left\lfloor \frac{n}{3\sqrt{11}} \right\rfloor\right) - \left(\left\lceil \frac{3n}{\sqrt{11}} \right\rceil - \frac{3n}{\sqrt{11}}\right) \\ &= \left(\frac{n}{3\sqrt{11}} + \frac{3n}{\sqrt{11}}\right) - \left(\left\lfloor \frac{n}{3\sqrt{11}} \right\rfloor + \left\lceil \frac{3n}{\sqrt{11}} \right\rceil\right) \\ &= n\left(\frac{1}{3\sqrt{11}} + \frac{3}{\sqrt{11}} - 1\right), \end{aligned}$$

so $n < 1/\left(\frac{1}{3\sqrt{11}} + \frac{3}{\sqrt{11}} - 1\right) = 99 + 30\sqrt{11} = 198.45\ldots.$

Use trial and error with $n = 198, 197, 196, \ldots,$ to find the value of n that works. Computing the first row of the following table to three decimal digits, and computing both $\frac{1}{3\sqrt{11}}$ and $\frac{3}{\sqrt{11}}$ to the same degree of accuracy, allows one to calculate the remaining rows with acceptable round-off errors.

n	$n/(3\sqrt{11})$	$3n/\sqrt{11}$
198	19.900	179.098
197	19.799	178.193
196	19.699	177.289
195	19.598	176.384
194	19.498	175.480
193	19.397	174.575
192	19.297	173.671
191	19.196	172.766
190	19.096	171.861
189	18.995	170.957

Because $n = 189 = 18 + 171$, the final answer is $x = \boldsymbol{\dfrac{189\sqrt{11}}{11}}$.

2012 Individual Problems

I-1. Compute the largest prime divisor of $15! - 13!$.

I-2. Three non-overlapping squares of positive integer side lengths each have one vertex at the origin and sides parallel to the coordinate axes. Together, the three squares enclose a region whose area is 41. Compute the largest possible perimeter of the region.

I-3. A circle with center O and radius 1 contains chord $\overline{AB}$ of length 1, and point M is the midpoint of $\overline{AB}$. If the perpendicular to $\overline{AO}$ through M intersects $\overline{AO}$ at P, compute $[MAP]$.

I-4. Suppose that p and q are two-digit prime numbers such that $p^2 - q^2 = 2p + 6q + 8$. Compute the largest possible value of $p + q$.

I-5. The four zeros of the polynomial $x^4 + jx^2 + kx + 225$ are distinct real numbers in arithmetic progression. Compute the value of j.

I-6. Compute the smallest positive integer n such that

$$n, \lfloor \sqrt{n} \rfloor, \lfloor \sqrt[3]{n} \rfloor, \lfloor \sqrt[4]{n} \rfloor, \lfloor \sqrt[5]{n} \rfloor, \lfloor \sqrt[6]{n} \rfloor, \lfloor \sqrt[7]{n} \rfloor, \text{ and } \lfloor \sqrt[8]{n} \rfloor$$

are distinct.

I-7. If n is a positive integer, then $n!!$ is defined to be $n(n-2)(n-4)\cdots 2$ if n is even and $n(n-2)(n-4)\cdots 1$ if n is odd. For example, $8!! = 8\cdot 6\cdot 4\cdot 2 = 384$ and $9!! = 9\cdot 7\cdot 5\cdot 3\cdot 1 = 945$. Compute the number of positive integers n such that $n!!$ divides $2012!!$.

I-8. On the complex plane, the parallelogram formed by the points 0, z, $\frac{1}{z}$, and $z + \frac{1}{z}$ has area $\frac{35}{37}$, and the real part of z is positive. If d is the smallest possible value of $\left|z + \frac{1}{z}\right|$, compute d^2.

I-9. One face of a $2\times 2\times 2$ cube is painted (not the entire cube), and the cube is cut into eight $1\times 1\times 1$ cubes. The small cubes are reassembled randomly into a $2\times 2\times 2$ cube. Compute the probability that no paint is showing.

I-10. In triangle ABC, $AB = BC$. A trisector of $\angle B$ intersects $\overline{AC}$ at D. If AB, AC, and BD are integers and $AB - BD = 7$, compute AC.

2012 Individual Answers

I-1. 19

I-2. 32

I-3. $\dfrac{\sqrt{3}}{32}$

I-4. 162

I-5. -50

I-6. 4096

I-7. 1510

I-8. $\dfrac{50}{37}$

I-9. $\dfrac{1}{16}$

I-10. 146

2012 Individual Solutions

I-1. Factor $15! - 13!$ to obtain $13!(15 \cdot 14 - 1) = 13! \cdot 209$. The largest prime divisor of $13!$ is 13, so continue by factoring $209 = 11 \cdot 19$. Thus the largest prime divisor of $15! - 13!$ is **19**.

I-2. Proceed in two steps: first, determine the possible sets of side lengths for the squares; then determine which arrangement of squares produces the largest perimeter. Let the side lengths of the squares be positive integers $m \geq n \geq p$. Then $m^2 + n^2 + p^2 = 41$, so $m \leq 6$, and because $3^2 + 3^2 + 3^2 < 41$, it follows that $m > 3$. If $m = 6$, then $n^2 + p^2 = 5$, so $n = 2$ and $p = 1$. If $m = 5$, then $n^2 + p^2 = 16$, which has no positive integral solutions. If $m = 4$, then $n^2 + p^2 = 25$, which is possible if $n = 4$ and $p = 3$. So the two possible sets of values are $m = 6$, $n = 2$, $p = 1$ or $m = 4$, $n = 4$, $p = 3$.

First consider $m = 6$, $n = 2$, $p = 1$. Moving counterclockwise around the origin, one square is between the other two; by symmetry, it suffices to consider only the three possibilities for this "middle" square. If the middle square is the 6-square, then each of the other two squares has a side that is a subset of a side of the 6-square. To compute the total perimeter, add the perimeters of the three squares and subtract twice the lengths of the shared segments (because they contribute 0 to the perimeter). Thus the total perimeter is $4\cdot6+4\cdot2+4\cdot1-2\cdot2-2\cdot1 = 30$. If the middle square is the 2-square, then one of its sides is a subset of the 6-square's side, and one of its sides is a superset of the 1-square's side, for a total perimeter of $4\cdot6+4\cdot2+4\cdot1-2\cdot2-2\cdot1 = 30$. But if the middle square is the 1-square, then two of its sides are subsets of the other squares' sides, and the total perimeter is $4 \cdot 6 + 4 \cdot 2 + 4 \cdot 1 - 2 \cdot 1 - 2 \cdot 1 = 32$.

If $m = 4$, $n = 4$, and $p = 3$, similar logic to the foregoing suggests that the maximal perimeter is obtained when the smallest square is between the other two, yielding a total perimeter of $4 \cdot 4 + 4 \cdot 4 + 4 \cdot 3 - 2 \cdot 3 - 2 \cdot 3 = 32$. Either of the other two arrangements yields a total perimeter of $4 \cdot 4 + 4 \cdot 4 + 4 \cdot 3 - 2 \cdot 3 - 2 \cdot 4 = 30$. So the maximum perimeter is **32**.

Alternate Solution: Let the side lengths be a, b, and c, and let P be the perimeter. If the $a \times a$ square is placed in between the other two (going either clockwise or counterclockwise around the origin), then

$$P = 3b + |b - a| + 2a + |c - a| + 3c.$$

To obtain a more symmetric expression, note that for any real numbers x and y,

$$|x - y| = \max\{x, y\} - \min\{x, y\} = x + y - 2\min\{x, y\}.$$

Using this identity,

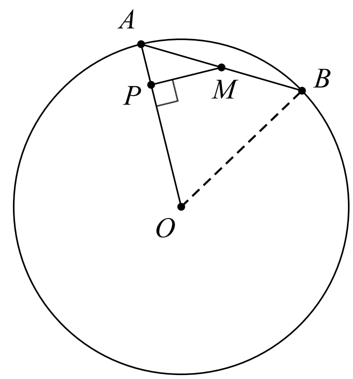
$$P = 4a + 4b + 4c - 2\min\{a, b\} - 2\min\{a, c\}.$$

Thus P is the sum of the perimeters of the three, less twice the overlaps. To maximize P, choose a to be the smallest of the three, which leads to $P = 4b + 4c$.

As in the first solution, the two possible sets of values are $c = 6$, $b = 2$, $a = 1$ and $c = b = 4$, $a = 3$.

In the first case, the maximum length of the boundary is $P = 4 \cdot 2 + 4 \cdot 6 = 32$, and in the second case it is $P = 4 \cdot 4 + 4 \cdot 4 = 32$. So the maximum perimeter is **32**.

I-3. Draw auxiliary segment $\overline{OB}$, as shown in the diagram below.



Triangle OAB is equilateral, so $\mathrm{m}\angle OAB = 60°$. Then $\triangle MAP$ is a $30°-60°-90°$ triangle with hypotenuse $AM = 1/2$. Thus $AP = 1/4$ and $MP = \sqrt{3}/4$, so

$$[MAP] = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{4}\right) = \boldsymbol{\frac{\sqrt{3}}{32}}.$$

I-4. Subtract from both sides and regroup to obtain $p^2 - 2p - (q^2 + 6q) = 8$. Completing both squares yields $(p-1)^2 - (q+3)^2 = 0$. The left side is a difference of two squares; factor to obtain $((p-1)+(q+3))((p-1)-(q+3)) = 0$, whence $(p+q+2)(p-q-4) = 0$. For positive primes p and q, the first factor $p+q+2$ must also be positive. Therefore the second factor $p-q-4$ must be zero, hence $p - 4 = q$. Now look for primes starting with 97 and working downward. If $p = 97$, then $q = 93$, which is not prime; if $p = 89$, then $q = 85$, which is also not prime. But if $p = 83$, then $q = 79$, which is prime. Thus the largest possible value of $p+q$ is $83 + 79 = \mathbf{162}$.

I-5. Let the four zeros be $p \le q \le r \le s$. The coefficient of x^3 is 0, so $p + q + r + s = 0$. The mean of four numbers in arithmetic progression is the mean of the middle two numbers, so $q = -r$. Then the common difference is $r - q = r - (-r) = 2r$, so $s = r + 2r = 3r$ and $p = q - 2r = -3r$. Therefore the four zeros are $-3r$, $-r$, r, $3r$. The product of the zeros is $9r^4$; referring to the original polynomial and using the product of roots formula gives $9r^4 = 225$. Thus $r = \sqrt{5}$, the zeros are $-3\sqrt{5}, -\sqrt{5}, \sqrt{5}, 3\sqrt{5}$, and the polynomial can be factored as $(x-\sqrt{5})(x+\sqrt{5})(x-3\sqrt{5})(x+3\sqrt{5})$. Expanding this product yields $(x^2-5)(x^2-45) = x^4 - 50x^2 + 225$, so $j = \mathbf{-50}$.

Alternate Solution: Proceed as in the original solution, finding the values $-3\sqrt{5}, -\sqrt{5}, \sqrt{5}$, and $3\sqrt{5}$ for the zeros. By the sums and products of roots formulas, the coefficient of x^2 is the sum of all six possible products of pairs of roots:

$$(-3\sqrt{5})(-\sqrt{5}) + (-3\sqrt{5})(\sqrt{5}) + (-3\sqrt{5})(3\sqrt{5}) + (-\sqrt{5})(\sqrt{5}) + (-\sqrt{5})(3\sqrt{5}) + (\sqrt{5})(3\sqrt{5})$$

Observing that some of these terms will cancel yields the simpler expression

$$(-3\sqrt{5})(3\sqrt{5}) + (-\sqrt{5})(\sqrt{5}) = -45 + -5 = \mathbf{-50}.$$

I-6. Inverting the problem, the goal is to find seven positive integers $a < b < c < d < e < f < g$ and a positive integer n such that $a^8, b^7, c^6, \ldots, g^2 \leq n$ and $n < (a+1)^8, (b+1)^7, \ldots, (g+1)^2$. Proceed by cases starting with small values of a.

If $a = 1$, then because $n < (a+1)^8$, $n < 256$. But because $n \geq (a+3)^5$, $n \geq 4^5 = 1024$. So it is impossible for a to be 1.

If $a = 2$, then $a^8 = 256$ and $(a+1)^8 = 6561$, so $256 \leq n < 6561$. Then $b \geq 3 \Rightarrow b^7 \geq 2187$ and $c \geq 4 \Rightarrow c^6 \geq 4096$. So $n \geq 4096$. Because $(3+1)^7 = 16384$ and $(4+1)^6 = 15625$, the condition $n < 6561$ found previously guarantees that $\lfloor \sqrt[7]{n} \rfloor = 3$ and $\lfloor \sqrt[6]{n} \rfloor = 4$. Notice that if $4096 \leq n < 6561$, then $\lfloor \sqrt[5]{n} \rfloor = 5$, $\lfloor \sqrt[4]{n} \rfloor = 8$, and $\lfloor \sqrt[3]{n} \rfloor \geq 16$. In fact, $\left\lfloor \sqrt[3]{4096} \right\rfloor = 2^4 = 16$, and $\left\lfloor \sqrt{4096} \right\rfloor = 2^6 = 64$. So the desired value of n is **4096**.

I-7. If n is even and $n \leq 2012$, then $n!! \mid 2012!!$ trivially, while if $n > 2012$, $2012!! < n!!$, so $n!!$ cannot divide $2012!!$. Thus there are a total of 1006 even values of n such that $n!! \mid 2012!!$. If n is odd and $n < 1006$, then $n!! \mid 2012!!$. To show this, rearrange the terms of $2012!!$ and factor:

$$\begin{aligned} 2012!! &= 2 \cdot 4 \cdot 6 \cdots 2010 \cdot 2012 \\ &= (2 \cdot 6 \cdot 10 \cdots 2010)(4 \cdot 8 \cdot 12 \cdots 2012) \\ &= 2^{503}(1 \cdot 3 \cdot 5 \cdots 1005)(4 \cdot 8 \cdot 12 \cdots 2012). \end{aligned}$$

However, the condition $n < 1006$ is not necessary, only sufficient, because $n!!$ also divides 2012 if $1007 \cdot 1009 \cdots n \mid (4 \cdot 8 \cdot 12 \cdots 2012)$. (The factor of 2^{503} is irrelevant because all the factors on the left side are odd.) The expression $(4 \cdot 8 \cdot 12 \cdots 2012)$ can be factored as $4^{503}(1 \cdot 2 \cdot 3 \cdot \cdots \cdot 503) = 4^{503} \cdot 503!$. Examining the numbers $1007, 1009, \ldots$ in sequence shows that 1007 is satisfactory, because $1007 = 19 \cdot 53$. On the other hand, 1009 is prime, so it cannot be a factor of $4^{503} \cdot 503!$. Thus the largest possible odd value of n is 1007, and there are 504 odd values of n altogether. The total is $1006 + 504 = \mathbf{1510}$.

Note: The primality of 1009 is not difficult to check by hand. First, standard divisibility tests show that 1009 is not divisible by 2, 3, 5, or 11. The well-known fact that $1001 = 7 \cdot 11 \cdot 13$ eliminates both 7 and 13 as factors, leaving only $17, 19, 23, 29, 31$ to check by hand. Each of these can be eliminated using long division or by finding a multiple too close to 1009. For example, $17 \mid 1020 - 17 = 1003$, so 17 cannot also be a factor of $1003 + 6 = 1009$. Similar logic applies to the other potential factors: $19 \mid 950 + 57 = 1007$, $23 \mid 920 + 92 = 1012$, $29 \mid 870 + 145 = 1015$, and $31 \mid 930 + 93 = 1023$.

I-8. As is usual, let $\arg z$ refer to measure of the directed angle whose vertex is the origin, whose initial ray passes through 1 (i.e., the point $(1, 0)$), and whose terminal ray passes through z. Then $\arg 1/z = -\arg z$. Using the formula $ab \sin\gamma$ for the area of the parallelogram with sides a and b and included angle γ yields the equation

$$\frac{35}{37} = |z| \cdot \left|\frac{1}{z}\right| \cdot \sin(2\arg z).$$

However, $|1/z| = 1/|z|$, so the right side simplifies to $\sin(2\arg z)$.

To compute the length c of the diagonal from 0 to $z + 1/z$, use the Law of Cosines and the fact that consecutive angles of a parallelogram are supplementary:

$$\begin{aligned} c^2 &= |z|^2 + \left|\frac{1}{z}\right|^2 - 2|z| \cdot \left|\frac{1}{z}\right| \cos(\pi - 2\arg z) \\ &= |z|^2 + \left|\frac{1}{z}\right|^2 - 2\cos(\pi - 2\arg z) \\ &= |z|^2 + \left|\frac{1}{z}\right|^2 + 2\cos(2\arg z). \end{aligned}$$

This expression separates into two parts: the first, $|z|^2 + |1/z|^2$, is independent of the argument (angle) of z, while the second, $2\cos(2\arg z)$, is determined by the condition that $\sin(2\arg z) = 35/37$. The minimum value of $|z|^2 + |1/z|^2$ is 2, as can be shown by the Arithmetic Mean-Geometric Mean inequality applied to $|z|^2$ and $|1/z|^2$:

$$|z|^2 + |1/z|^2 \geq 2\sqrt{|z|^2 \cdot |1/z|^2} = 2.$$

The value of $\cos(2\arg z)$ is given by the Pythagorean Identity:

$$\cos(2\arg z) = \pm\sqrt{1 - \left(\frac{35}{37}\right)^2} = \pm\sqrt{1 - \frac{1225}{1369}} = \pm\sqrt{\frac{144}{1369}} = \pm\frac{12}{37}.$$

Because the goal is to minimize the diagonal's length, choose the negative value to obtain

$$d^2 = 2 - 2 \cdot \frac{12}{37} = \mathbf{\frac{50}{37}}.$$

Alternate Solution: Using polar coordinates, write

$$z = r\big(\cos\theta + i\sin\theta\big),$$

so that

$$\frac{1}{z} = r^{-1}\big(\cos\theta - i\sin\theta\big).$$

Without loss of generality, assume that z is in the first quadrant, so that $\theta > 0$. Then the angle between the sides $\overline{0z}$ and $\overline{0z^{-1}}$ is 2θ, and the side lengths are r and r^{-1}, so the area of the parallelogram is

$$\frac{35}{37} = r \cdot r^{-1} \cdot \sin(2\theta) = \sin\, 2\theta.$$

Note that $0 < \theta < \pi/2$, so $0 < 2\theta < \pi$, and there are two values of θ that satisfy this equation. Adding the expressions for z and z^{-1} and calculating the absolute value yields

$$\begin{aligned} \left| z + \frac{1}{z} \right|^2 &= (r + r^{-1})^2 \cos^2\theta + (r - r^{-1})^2 \sin^2\theta \\ &= (r^2 + r^{-2})(\cos^2\theta + \sin^2\theta) + 2r \cdot r^{-1}(\cos^2\theta - \sin^2\theta) \\ &= r^2 + r^{-2} + 2\cos 2\theta. \end{aligned}$$

Minimize the terms involving r using the Arithmetic-Geometric Mean inequality:

$$r^2 + r^{-2} \geq 2\sqrt{r^2 \cdot r^{-2}} = 2,$$

with equality when $r^2 = r^{-2}$, that is, when $r = 1$. For the term involving θ, recall that there are two possible values:

$$\cos\, 2\theta = \pm\sqrt{1 - \sin^2\, 2\theta} = \pm\sqrt{\frac{37^2 - 35^2}{37^2}} = \pm\frac{\sqrt{(37+35)(37-35)}}{37} = \pm\frac{12}{37}.$$

To minimize this term, take the negative value, yielding

$$d^2 = 2 - 2 \cdot \frac{12}{37} = \mathbf{\frac{50}{37}}.$$

Alternate Solution: If $z = x + yi$, then compute $1/z$ by rationalizing the denominator:

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2}i.$$

The area of the parallelogram is given by the absolute value of the 2×2 determinant

$$\begin{vmatrix} x & y \\ x/(x^2+y^2) & -y/(x^2+y^2) \end{vmatrix} = \frac{1}{x^2+y^2}\begin{vmatrix} x & y \\ x & -y \end{vmatrix} = \frac{-2xy}{x^2+y^2}.$$

That is,

$$\frac{2xy}{x^2+y^2} = \frac{35}{37}.$$

Calculation shows that

$$\left|z+\frac{1}{z}\right|^2 = \left(x+\frac{x}{x^2+y^2}\right)^2 + \left(y-\frac{y}{x^2+y^2}\right)^2 = (x^2+y^2) + \frac{1}{x^2+y^2} + 2\left(\frac{x^2-y^2}{x^2+y^2}\right).$$

As in the previous solution, the sum of the first two terms is at least 2, when $x^2+y^2=1$. The trick for relating the third term to the area is to express both the third term and the area in terms of the ratio

$$t = \frac{y}{x}.$$

Indeed,

$$\frac{2xy}{x^2+y^2} = \frac{2t}{1+t^2} \qquad \text{and} \qquad \frac{x^2-y^2}{x^2+y^2} = \frac{1-t^2}{1+t^2} = \frac{(1+t)(1-t)}{1+t^2}.$$

As in the previous solution, assume without loss of generality that z is in the first quadrant, so that $t>0$. As found above,

$$\frac{2t}{1+t^2} = \frac{35}{37}.$$

It is not difficult to solve for t using the quadratic formula, but the value of t is not needed to solve the problem. Observe that

$$\frac{(1\pm t)^2}{1+t^2} = 1 \pm \frac{2t}{1+t^2} = 1 \pm \frac{35}{37},$$

so that

$$\left(\frac{1-t^2}{1+t^2}\right)^2 = \frac{(1+t)^2}{1+t^2}\cdot\frac{(1-t)^2}{1+t^2} = \frac{72}{37}\cdot\frac{2}{37} = \left(\frac{12}{37}\right)^2.$$

In order to minimize d, take the negative square root, leading to

$$d^2 = 2 + 2\cdot\frac{1-t^2}{1+t^2} = 2 - \frac{24}{37} = \mathbf{\frac{50}{37}}.$$

Note: The relation between the two alternate solutions is that if

$$z = x+yi = r(\cos\theta + i\sin\theta),$$

then $t = y/x = \tan\theta$. Similarly

$$z^2 = (x^2-y^2) + (2xy)i = r^2(\cos 2\theta + i\sin 2\theta)$$

and $|z^2| = |z|^2 = x^2+y^2 = r^2$, whence

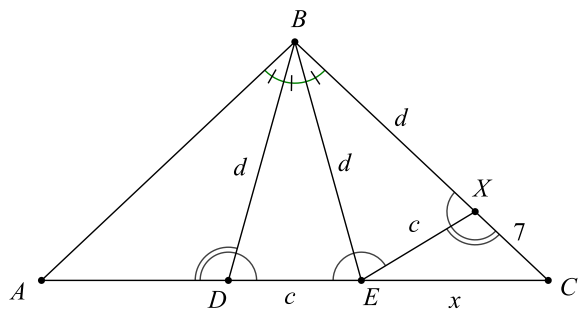
$$\cos 2\theta = \frac{x^2-y^2}{x^2+y^2}, \qquad \sin 2\theta = \frac{2xy}{x^2+y^2}.$$

Thus the trick used in the alternate solution corresponds precisely to the trigonometric identities

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}, \qquad \sin 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

I-9. Call each $1\times1\times1$ cube a *cubelet*. Then four cubelets are each painted on one face, and the other four cubelets are completely unpainted and can be ignored. For each painted cubelet, the painted face can occur in six positions, of which three are hidden from the outside, so the probability that a particular painted cubelet has no paint showing is $3/6 = 1/2$. Thus the probability that all four painted cubelets have no paint showing is $(1/2)^4 = \mathbf{\frac{1}{16}}$.

I-10. Let E be the point where the other trisector of $\angle B$ intersects side $\overline{AC}$. Let $AB = BC = a$, and let $BD = BE = d$. Draw X on $\overline{BC}$ so that $BX = d$. Then $CX = 7$.



The placement of point X guarantees that $\triangle BEX \cong \triangle BDE$ by Side-Angle-Side. Therefore $\angle BXE \cong \angle BEX \cong \angle BDE$, and so $\angle CXE \cong \angle ADB \cong \angle CEB$. By Angle-Angle, $\triangle CEX \sim \triangle CBE$. Let $EX = c$ and $EC = x$. Then comparing ratios of corresponding sides yields

$$\frac{c}{d} = \frac{7}{x} = \frac{x}{d+7}.$$

Using the right proportion, $x^2 = 7(d+7)$. Because d is an integer, x^2 is an integer, so either x is an integer or irrational. The following argument shows that x cannot be irrational. Applying the Angle Bisector Theorem to $\triangle BCD$ yields $DE = c = \frac{d}{d+7}\cdot x$. Then $AC = 2x + c = x(2+\frac{d}{d+7})$. Because the expression $(2+\frac{d}{d+7})$ is rational, AC will not be an integer if x is irrational.

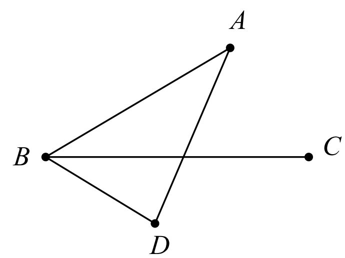
Hence x is an integer, and because x^2 is divisible by 7, x must also be divisible by 7. Let $x = 7k$ so that $d = ck$. Rewrite the original proportion using $7k$ for x and ck for d:

$$\begin{aligned}
\frac{c}{d} &= \frac{x}{d+7}\\
\frac{c}{ck} &= \frac{7k}{ck+7}\\
7k^2 &= ck+7\\
7k &= c+\frac{7}{k}.
\end{aligned}$$

Because the left side of this last equation represents an integer, $7/k$ must be an integer, so either $k = 1$ or $k = 7$. The value $k = 1$ gives the extraneous solution $c = 0$. So $k = 7$, from which $c = 48$. Then $d = 336$ and $AC = 2x + c = 2 \cdot 49 + 48 = \mathbf{146}$.

Power Question 2012: Cops and Robbers

This Power Question involves one Robber and one or more Cops. After robbing a bank, the Robber retreats to a network of hideouts, represented by dots in the diagram below. Every day, the Robber stays holed up in a single hideout, and every night, the Robber moves to an adjacent hideout. Two hideouts are *adjacent* if and only if they are connected by an edge in the diagram, also called a *hideout map* (or *map*). For the purposes of this Power Question, the map must be connected; that is, given any two hideouts, there must be a path from one to the other. To clarify, the Robber may not stay in the same hideout for two consecutive days, although he may return to a hideout he has previously visited. For example, in the map below, if the Robber holes up in hideout C for day 1, then he would have to move to B for day 2, and would then have to move to either A, C, or D on day 3.

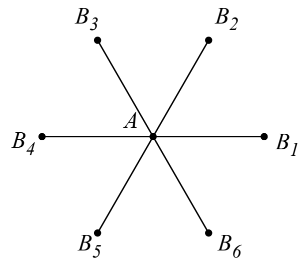


Every day, each Cop searches one hideout: the Cops know the location of all hideouts and which hideouts are adjacent to which. Cops are thorough searchers, so if the Robber is present in the hideout searched, he is found and arrested. If the Robber is not present in the hideout searched, his location is *not* revealed. That is, the Cops only know that the Robber was not caught at any of the hideouts searched; they get no specific information (other than what they can derive by logic) about what hideout he was in. Cops are not constrained by edges on the map: a Cop may search any hideout on any day, regardless of whether it is adjacent to the hideout searched the previous day. A Cop may search the same hideout on consecutive days, and multiple Cops may search different hideouts on the same day. In the map above, a Cop could search A on day 1 and day 2, and then search C on day 3.

The focus of this Power Question is to determine, given a hideout map and a fixed number of Cops, whether the Cops can be sure of catching the Robber within some time limit.

Map Notation: The following notation may be useful when writing your solutions. For a map M, let $h(M)$ be the number of hideouts and $e(M)$ be the number of edges in M. The *safety* of a hideout H is the number of hideouts adjacent to H, and is denoted by $s(H)$.

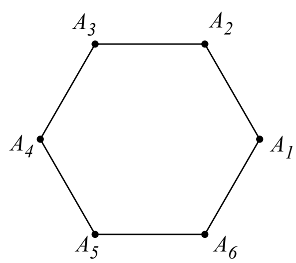
1a. Consider the hideout map M below.



Show that one Cop can always catch the Robber. [1]

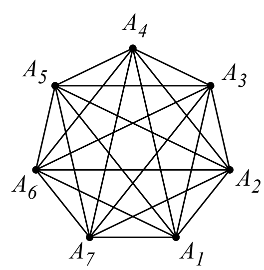
The *Cop number* of a map M, denoted $C(M)$, is the minimum number of Cops required to guarantee that the Robber is caught. In 1a, you have shown that $C(M) = 1$.

1b. The map shown below is $\mathcal{C}_6$, the cyclic graph with six hideouts. Show that three Cops are sufficient to catch the Robber on $\mathcal{C}_6$, so that $C(\mathcal{C}_6) \leq 3$. [2]

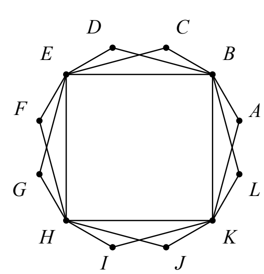


1c. Show that for all maps M, $C(M) < h(M)$. [2]

2a. Find $C(M)$ for the map below. [2]



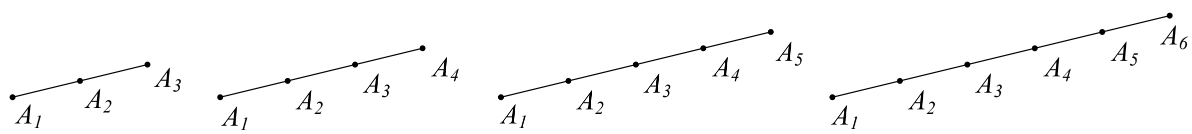
2b. Show that $C(M) \leq 3$ for the map below. [2]



2c. Generalize the result of 2a: if $\mathcal{K}_n$ is a map with n hideouts in which every hideout is adjacent to every other hideout (also called the *complete map* on n hideouts), determine $C(\mathcal{K}_n)$. [3]

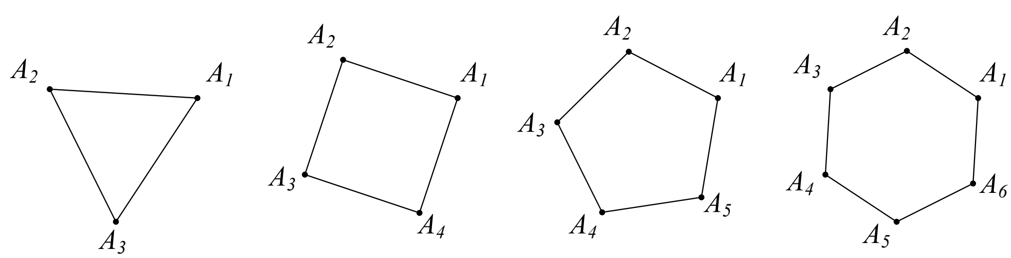
The police want to catch the Robber with a minimum number of Cops, but time is of the essence. For a map M and a fixed number of Cops $c \geq C(M)$, define the *capture time*, denoted $D(M, c)$, to be the minimum number of days required to *guarantee* a capture using c Cops. For example, in the graph from 1b, if three Cops are deployed, they might catch the Robber in the first day, but if they don't, there is a strategy that will guarantee they will capture the Robber within two days. Therefore the capture time is $D(\mathcal{C}_6, 3) = 2$.

3. A *path on n hideouts* is a map with n hideouts, connected in one long string. (More formally, a map is a path if and only if two hideouts are adjacent to exactly one hideout each and all other hideouts are adjacent to exactly two hideouts each.) It is denoted by $\mathcal{P}_n$. The maps $\mathcal{P}_3$ through $\mathcal{P}_6$ are shown below.



Show that $D(\mathcal{P}_n, 1) \leq 2n$ for $n \geq 3$. [4]

4. A *cycle on n hideouts* is a map with n hideouts, connected in one loop. (More formally, a map is a cycle if every hideout is adjacent to exactly two hideouts.) It is denoted by $\mathcal{C}_n$. The maps $\mathcal{C}_3$ through $\mathcal{C}_6$ are shown below.



a. Determine $C(\mathcal{C}_n)$ for $n \geq 3$. (Note that this is *determine* and not *compute*.) [2]

b. Show that $D(\mathcal{C}_n, C(\mathcal{C}_n)) < \frac{3n}{2}$ for $n \geq 3$. [2]

Definition: The *workday number* of M, denoted $W(M)$, is the minimum number of *Cop workdays* needed to guarantee the Robber's capture. For example, a strategy that guarantees capture within three days using 17 Cops on the first day, 11 Cops on the second day, and only 6 Cops on the third day would require a total of $17 + 11 + 6 = 34$ Cop workdays.

5. Determine $W(M)$ for each of the maps in problem 2a and 2b. [4]

6. Let M be a map with $n \geq 3$ hideouts. Prove that $2 \leq W(M) \leq n$, and that these bounds cannot be improved. In other words, prove that for each $n \geq 3$, there exist maps M_1 and M_2 such that $W(M_1) = 2$ and $W(M_2) = n$. [6]

Definition: A map is *bipartite* if it can be partitioned into two sets of hideouts, $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A} \cap \mathcal{B} = \emptyset$, and each hideout in $\mathcal{A}$ is adjacent only to hideouts in $\mathcal{B}$, and each hideout in $\mathcal{B}$ is adjacent only to hideouts in $\mathcal{A}$.

7a. Prove that if M is bipartite, then $C(M) \leq n/2$. [3]

7b. Prove that $C(M) \leq n/2$ for *any* map M with the property that, for all hideouts H_1 and H_2, either all paths from H_1 to H_2 contain an odd number of edges, or all paths from H_1 to H_2 contain an even number of edges. [3]

8. A map M is called *k-perfect* if its hideout set H can be partitioned into equal-sized subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ such that for any j, the hideouts of $\mathcal{A}_j$ are only adjacent to hideouts in $\mathcal{A}_{j+1}$ or $\mathcal{A}_{j-1}$. (The indices are taken modulo k: the hideouts of $\mathcal{A}_1$ may be adjacent to the hideouts of $\mathcal{A}_k$.) Show that if M is k-perfect, then $C(M) \leq \frac{2n}{k}$. [4]

9. Find an example of a map M with 2012 hideouts such that $C(M) = 17$ and $W(M) = 34$, or prove that no such map exists. [5]

10. Find an example of a map M with 4 or more hideouts such that $W(M) = 3$, or prove that no such map exists. [5]

Solutions to 2012 Power Question

1. a. Have the Cop stay at A for 2 days. If the Robber is not at A the first day, he must be at one of B_1–B_6, and because the Robber must move along an edge every night, he will be forced to go to A on day 2.

   b. The Cops should stay at $\{A_1, A_3, A_5\}$ for 2 days. If the Robber evades capture the first day, he must have been at an even-numbered hideout. Because he must move, he will be at an odd-numbered hideout the second day. Equivalently, the Cops could stay at $\{A_2, A_4, A_6\}$ for 2 days.

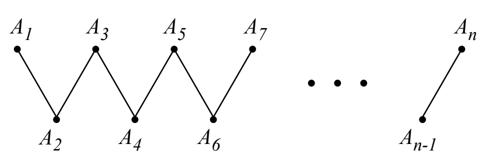
   c. Let $n = h(M)$. The following strategy will always catch a Robber within two days using $n-1$ Cops, which proves that $C(M) \leq n-1$. Choose any subset $\mathcal{S}$ of $n-1$ hideouts and position $n-1$ Cops at the hideouts of $\mathcal{S}$ for 2 days. If the Robber is not caught on the first day, he must have been at the hideout not in $\mathcal{S}$, and therefore must move to a hideout in $\mathcal{S}$ on the following day.

2. a. $C(M) = 6$. The result of 1c applies, so $C(M) \leq 6$. To see that 6 is minimal, note that it is possible for the Robber to go from any hideout to any other hideout in a single night. Suppose that on a given day, the Robber successfully evades capture. Without loss of generality, assume that the Robber was at hideout A_1. Then the only valid conclusion the Cops can draw is that on the next day, the Robber will *not* be at A_1. Thus the only hideout the Cops can afford to leave open is A_1 itself. If they leave another hideout open, the Robber might have chosen to hide there, and the process will repeat.

   b. The following strategy guarantees capture using three Cops for four consecutive days, so $C(M) \leq 3$. Position three Cops at $\{B, E, H\}$ for two days, which will catch any Robber who starts out at B, C, D, E, F, G, or H, because a Robber at C or D would have to move to either B or E, and a Robber at F or G would have to move to either E or H. So if the Robber is not yet caught, he must have been at I, J, K, L, or A for those two days. In this case, an analogous argument shows that placing Cops at B, H, K for two consecutive days will guarantee a capture.

   c. The result in 1c applies, so $C(\mathcal{K}_n) \leq n-1$. It remains to show that $n-1$ is minimal. As was argued in 2a, the Robber always has $n-1$ choices of where to move the next day, so a strategy using less than $n-1$ Cops can never guarantee capture. Thus $C(\mathcal{K}_n) > n-2$.

3. The following argument shows that $C(\mathcal{P}_n) = 1$, and that capture occurs in at most $2n-4$ days. It helps to draw the hideout map as in the following diagram, so that odd-numbered hideouts are all on one level and even-numbered hideouts are all on another level; the case where n is odd is shown below. (A similar argument applies where n is even.)



Each night, the Robber has the choice of moving diagonally left or diagonally right on this map, but he is required to move from top to bottom or vice-versa.

For the first $n-2$ days, the Cop should search hideouts A_2, A_3, ..., A_{n-1}, in that order. For the next $n-2$ days, the Cop should search hideouts A_{n-1}, A_{n-2}, ..., A_2, in that order.

The Cop always moves from top to bottom, or vice-versa, except that he searches A_{n-1} two days in a row. If the Cop is lucky, the Robber chose an even-numbered hideout on the first day. In this case, the Cop and the Robber are always on the same level (top or bottom) for the first half ($n-2$ days) of the search. The Cop is searching from left to right, and the Robber started out to the right of the Cop (or at A_2, the first hideout searched), so eventually the Robber is caught (at A_{n-1}, if not earlier).

If the Robber chose an odd-numbered hideout on the first day, then the Cop and Robber will be on different levels for the first half of the search, but on the same level for the second half. For the second half of the search, the Robber is to the left of the Cop (or possibly at A_{n-1}, which would happen if the Robber moved unwisely or if he spent day $n-2$ at hideout A_n). But now the Cop is searching from right to left, so again, the Cop will eventually catch the Robber.

The same strategy can be justified without using the zig-zag diagram above. Suppose that the Robber is at hideout A_R on a given day, and the Cop searches hideout A_C. Whenever the Cop moves to the next hideout or the preceding hideout, C changes by ± 1, and the Robber's constraint forces R to change by ± 1. Thus if the Cop uses the strategy above, on each of the first $n-2$ days, either the difference $R-C$ stays the same or it decreases by 2. If on the first day $R-C$ is even, either $R-C$ is 0 (the Robber was at A_2 and is caught immediately) or is positive (because $C=2$ and $R \geq 4$). Because the difference $R-C$ is even and decreases (if at all) by 2 each day, it cannot go from positive to negative without being zero. If on the first day $R-C$ is odd, then the Robber avoids capture through day $n-2$ (because $R-C$ is still odd), but then on day $n-1$, R changes (by ± 1) while C does not. So $R-C$ is now even (and is either 0 or negative), and henceforth either remains the same or increases by 2 each day, so again, $R-C$ must be zero at some point between day $n-1$ and day $2n-4$, inclusive.

4. a. $C(\mathcal{C}_n) = 2$. Position one Cop at A_1 and have the Cop search that hideout each day. This strategy reduces $\mathcal{C}_n$ to $\mathcal{P}_{n-1}$, because the Robber cannot get past A_1. Then the second Cop can pursue the strategy from problem 3 on hideouts $\{A_2, \ldots, A_n\}$ to guarantee capture. Thus $C(\mathcal{C}_n) \leq 2$. On the other hand, one Cop is obviously insufficient because the Robber has two choices of where to go every day, so it is impossible for one Cop to cover both possible choices.

   b. If the second Cop makes it all the way to hideout A_n without capturing the Robber, then the second part of the strategy from part (a) can be improved by having the Cops walk back towards each other in a "pincers" move. The details are as follows. If n is odd, then when the second Cop reaches A_n, his and the first Cop's hideouts are both odd. By hypothesis, the Robber's hideout has the opposite (even) parity. (Otherwise the second Cop would have caught the Robber the first time around.) So the two Cops should each "wait" (that is, search A_1 and A_n again) for one day, after which they will have the same parity as the Robber. Then the first Cop should search hideouts in increasing order while the second Cop searches in decreasing order. This part of the search takes at most $\frac{n-1}{2}$ days, for a total of $(n-1)+1+\frac{n-1}{2}$ days, or $\frac{3n-1}{2}$ days. If n is even, then when the second Cop reaches hideout A_n, the Robber is on an odd hideout (but not A_1), and so the first Cop's hideout is *already* the same parity as the Robber's hideout. So on the next day, the first Cop should search A_2 while the second Cop should re-search A_n. Thereafter, the first Cop searches consecutively increasing hideouts while the second Cop searches consecutively decreasing ones as in the case of n odd. In this case, the search takes $(n-1)+1+\frac{n-2}{2}=\frac{3n-2}{2}$ days.

5. For the map M from 2a, $W(M)=7$. The most efficient strategy is to use 7 Cops to blanket all the hideouts on the first day. Any strategy using fewer than 7 Cops would require 6 Cops on each of two consecutive days: given that any hideout can be reached from any other hideout, leaving more than one hideout unsearched on one day makes it is impossible to eliminate any hideouts the following day. So any other strategy would require a minimum of 12 Cop workdays.

   For the map M from 2b, $W(M)=8$. The strategy outlined in 2b used three Cops for a maximum of four workdays, yielding 12 Cop workdays. The most efficient strategy is to use 4 Cops, positioned at $\{B,E,H,K\}$ for 2 days each. The following argument demonstrates that 8 Cop workdays is in fact minimal. First, notice that there is no advantage to searching one of the hideouts between vertices of the square (for example, C) without searching the other hideout between the same vertices (for example, D). The Robber can reach C on day n if and only if he is at either B or E on day $n-1$, and in either case he could just as well go to D instead of C. So there is no situation in which the Robber is certain to be caught at C rather than at D. Additionally, the Robber's possible locations on day $n+1$ are the same whether he is at C or D on day n, so searching one rather than the other fails to rule out any locations for future days. So any successful strategy that involves searching C should also involve searching D on the same day, and similarly for F and G, I and J, and L and A. On the other hand, if the Robber must be at one of C and D on day n, then he must be at either B or E on day $n+1$, because those are the only adjacent hideouts. So any strategy that involves searching both hideouts of one of the off-the-square pairs on day n is equivalent to a strategy that searches the adjacent on-the-square hideouts on day $n+1$; the two strategies use the same number of Cops for the same number of workdays. Thus the optimal number of Cop workdays can be achieved using strategies that only search the "corner" hideouts B,E,H,K.

   Restricting the search to only those strategies searching corner hideouts B,E,H,K, a total of 8 workdays can be achieved by searching all four hideouts on two consecutive days: if the Robber is at one of the other eight hideouts the first day, he must move to one of the two adjacent corner hideouts the second day. But each of these corner hideouts is adjacent to two other corner hideouts. So if only one hideout is searched, for no matter how many consecutive days, the following day, the Robber could either be back at the previously-searched hideout or be at any other hideout: no possibilities are ruled out. If two adjacent corners are searched, the Cops do no better, as the following argument shows. Suppose that B and E are both searched for two consecutive days. Then the Cops can rule out B, E, C, and D as possible locations, but if the Cops then switch to searching either H or K instead of B or E, the Robber can go back to C or D within two days. So searching two adjacent corner hideouts for two days is fruitless and costs four Cop workdays. Searching diagonally opposite corner hideouts is even less fruitful, because doing so rules out none of the other hideouts as possible Robber locations. Using three Cops each day, it is easy to imagine scenarios in which the Robber evades capture for three days before being caught: for example, if B, E, H are searched for two consecutive days, the Robber goes from I or J to K to L. Therefore if three Cops are used, four days are required for a total of 12 Cop workdays.

If there are more than four Cops, the preceding arguments show that the number of Cops must be even to produce optimal results (because there is no advantage to searching one hideout between vertices of the square without searching the other). Using six Cops with four at corner hideouts yields no improvement, because the following day, the Robber could get to any of the four corner hideouts, requiring at least four Cops the second day, for ten Cop workdays. If two or fewer Cops are at corner hideouts, the situation is even worse, because if the Robber is not caught that day, he has at least nine possible hideouts the following day (depending on whether the unsearched corners are adjacent or diagonally opposite to each other). Using eight Cops (with four at corner vertices) could eliminate one corner vertex as a possible location for the second day (if the non-corner hideouts searched are on adjacent sides of the square), but eight Cop workdays have already been used on the first day. So 8 Cop workdays is minimal.

6. A single Cop can only search one hideout in a day, so as long as M has two or more hideouts, there is no strategy that guarantees that a lone Cop captures the Robber the first day. Then either more than one Cop will have to search on the first day, or a lone Cop will have to search for at least 2 days; in either of these cases, $W(M) \geq 2$. On the other hand, n Cops can always guarantee a capture simply by searching all n hideouts on the first day, so $W(M) \leq n$.

   To show that the lower bound cannot be improved, consider the *star on n hideouts*; that is, the map $\mathcal{S}_n$ with one central hideout connected to $n-1$ outer hideouts, none of which is connected to any other hideout. (The map in 1a is $\mathcal{S}_7$.) Then $W(\mathcal{S}_n) = 2$, because a single Cop can search the central hideout for 2 days; if the Robber is at one of the outer hideouts on the first day, he must go to the central hideout on the second day. For the upper bound, the complete map $\mathcal{K}_n$ is an example of a map with $W(M) = n$. As argued above, the minimum number of Cops needed to guarantee a catch is $n-1$, and using $n-1$ Cops requires two days, for a total of $2n-2$ Cop workdays. So the optimal strategy for $\mathcal{K}_n$ (see 2c) is to use n Cops for a single day.

7. a. Suppose M is bipartite, and let $\mathcal{A}$ and $\mathcal{B}$ be the sets of hideouts referenced in the definition. Because $\mathcal{A}$ and $\mathcal{B}$ are disjoint, either $|\mathcal{A}| \leq n/2$ or $|\mathcal{B}| \leq n/2$ or both. Without loss of generality, suppose that $|\mathcal{A}| \leq n/2$. Then position Cops at each hideout in $\mathcal{A}$ for two days. If the Robber was initially on a hideout in $\mathcal{B}$, he must move the following day, and because no hideout in $\mathcal{B}$ is connected to any other hideout in $\mathcal{B}$, his new hideout must be a hideout in $\mathcal{A}$.

b. The given condition actually implies that the graph is bipartite. Let A_1 be a hideout in M, and let $\mathcal{A}$ be the set of all hideouts V such that all paths from A_1 to V have an even number of edges, as well as A_1 itself; let $\mathcal{B}$ be the set of all other hideouts in M. Notice that there are no edges from A_1 to any other element A_i in $\mathcal{A}$, because if there were, that edge would create a path from A_1 to A_i with an odd number of edges (namely 1). So A_1 has edges only to hideouts in $\mathcal{B}$. In fact, there can be no edge from any hideout A_i in $\mathcal{A}$ to any other hideout A_j in $\mathcal{A}$, because if there were, there would be a path with an odd number of edges from A_1 to A_j via A_i. Similarly, there can be no edge from any hideout B_i in $\mathcal{B}$ to any other hideout B_j in $\mathcal{B}$, because if there were such an edge, there would be a path from A_1 to B_j with an even number of edges via B_i. Thus hideouts in $\mathcal{B}$ are connected only to hideouts in $\mathcal{A}$ and vice versa; hence M is bipartite.

8. Because a Robber in $\mathcal{A}_i$ can only move to a hideout in $\mathcal{A}_{i-1}$ or $\mathcal{A}_{i+1}$, this map is essentially the same as the cyclic map $\mathcal{C}_k$. So the Cops should apply a similar strategy. First, position n/k Cops at the hideouts of set $\mathcal{A}_1$ and n/k Cops at the hideouts of set $\mathcal{A}_2$. On day 2, leave the first n/k Cops at $\mathcal{A}_1$, but move the second group of Cops to $\mathcal{A}_3$. Continue until the second set of Cops is at $\mathcal{A}_k$; then "wait" one turn (search $\mathcal{A}_k$ again), and then search backward $\mathcal{A}_{k-1}$, $\mathcal{A}_{k-2}$, etc.

9. There are many examples. Perhaps the simplest to describe is the *complete bipartite map* on the hideouts $\mathcal{A} = \{A_1, A_2, \ldots, A_{17}\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_{1995}\}$, which is often denoted $M = \mathcal{K}_{17,1995}$. That is, let M be the map with hideouts $\mathcal{A} \cup \mathcal{B}$ such that A_i is adjacent to B_j for all i and j, and that A_i is not adjacent to A_j, nor is B_i adjacent to B_j, for any i and j.

   If 17 Cops search the 17 hideouts in $\mathcal{A}$ for two consecutive days, then they are guaranteed to catch the Robber. This shows that $C(M) \leq 17$ and that $W(M) \leq 34$. If fewer than 17 Cops search on a given day, then the Robber has at least one safe hideout: he has exactly one choice if the Cops search 16 of the hideouts A_i and the Robber was hiding at some B_j the previous day, 1995 choices if the Robber was hiding at some A_i the previous day. Thus $C(M) = 17$.

   On the other hand, fewer than 34 Cop workdays cannot guarantee catching the Robber. Unless the Cops search every A_i (or every B_j) on a given day, they gain no information about where the Robber is (unless he is unlucky enough to be caught).

   The complete bipartite map is not the simplest in terms of the number of edges. Try to find a map with 2012 hideouts and as few edges as possible that has a Cop number of 17 and a workday number of 34.

10. No such map exists. Let M be a map with at least four hideouts. The proof below shows that either $W(M) > 3$ or else M is a star (as in the solution to problem 6), in which case $W(M) = 2$.

    First, suppose that there are four distinct hideouts A_1, A_2, B_1, and B_2 such that A_1 and A_2 are adjacent, as are B_1 and B_2. If the Cops make fewer than four searches, then they cannot search $\{A_1, A_2\}$ twice and also search $\{B_1, B_2\}$ twice. Without loss of generality, assume the Cops search $\{A_1, A_2\}$ at most once. Then the Robber can evade capture by moving from A_1 to A_2 and back again, provided that he is lucky enough to start off at the right one. Thus

$W(M) > 3$ in this case. For the second case, assume that whenever A_1 is adjacent to A_2 and B_1 is adjacent to B_2, the four hideouts are *not* distinct. Start with two adjacent hideouts, and call them A_1 and A_2. Consider any hideout, say B, that is not one of these two. Then B must be adjacent to some hideout, say C. By assumption, A_1, A_2, B, and C are not distinct, so $C = A_1$ or $C = A_2$. That is, every hideout B is adjacent to A_1 or to A_2.

If there are hideouts B_1 and B_2, distinct from A_1 and A_2 and from each other, such that B_1 is adjacent to A_1 and B_2 is adjacent to A_2, then it clearly violates the assumption of this case. That is, either every hideout B, distinct from A_1 and A_2, is adjacent to A_1, or every such hideout is adjacent to A_2. Without loss of generality, assume the former.

Because every hideout is adjacent to A_1 (except for A_1 itself), the foregoing proves that the map M is a star. It remains to show that there are no "extra" edges in the map. For the sake of contradiction, suppose that B_1 and B_2 are adjacent hideouts, both distinct from A_1. Because the map has at least four hideouts, choose one distinct from these three and call it A_2. Then these four hideouts violate the assumption of this case.

2012 Relay Problems

R1-1. The rational number r is the largest number less than 1 whose base-7 expansion consists of two distinct repeating digits, $r = 0.\underline{A}\,\underline{B}\,\underline{A}\,\underline{B}\,\underline{A}\,\underline{B}\ldots$. Written as a reduced fraction, $r = \dfrac{p}{q}$. Compute $p + q$ (in base 10).

R1-2. Let $T = TNYWR$. Triangle ABC has $AB = AC$. Points M and N lie on $\overline{BC}$ such that $\overline{AM}$ and $\overline{AN}$ trisect $\angle BAC$, with M closer to C. If $\mathrm{m}\angle AMC = T^\circ$, then $\mathrm{m}\angle ACB = U^\circ$. Compute U.

R1-3. Let $T = TNYWR$. At Wash College of Higher Education (Wash Ed.), the entering class has n students. Each day, two of these students are selected to oil the slide rules. If the entering class had two more students, there would be T more ways of selecting the two slide rule oilers. Compute n.

R2-1. Compute the least positive integer n such that the set of angles

$$\{123^\circ, 246^\circ, \ldots, n \cdot 123^\circ\}$$

contains at least one angle in each of the four quadrants.

R2-2. Let $T = TNYWR$. In ARMLvania, license plates use only the digits 1–9, and each license plate contains exactly $T - 3$ digits. On each plate, all digits are distinct, and for all $k \leq T - 3$, the k^{th} digit is at least k. Compute the number of valid ARMLvanian license plates.

R2-3. Let $T = TNYWR$. Let $\mathcal{R}$ be the region in the plane defined by the inequalities $x^2 + y^2 \geq T$ and $|x| + |y| \leq \sqrt{2T}$. Compute the area of region $\mathcal{R}$.

2012 Relay Answers

R1-1. 95

R1-2. 75

R1-3. 37

R2-1. 11

R2-2. 256

R2-3. $1024 - 256\pi$

2012 Relay Solutions

R1-1. In base 7, the value of r must be $0.656565\ldots = 0.\overline{65}_7$. Then $100_7 \cdot r = 65.\overline{65}_7$, and $(100_7 - 1)r = 65_7$. In base 10, $65_7 = 6 \cdot 7 + 5 = 47_{10}$ and $100_7 - 1 = 7^2 - 1 = 48_{10}$. Thus $r = 47/48$, and $p + q = \mathbf{95}$.

R1-2. Because $\triangle ABC$ is isosceles with $AB = AC$, $\mathrm{m}\angle ABC = U^\circ$ and $\mathrm{m}\angle BAC = (180 - 2U)^\circ$. Therefore $\mathrm{m}\angle MAC = \left(\frac{180-2U}{3}\right)^\circ = \left(60 - \frac{2}{3}U\right)^\circ$. Then $\left(60 - \frac{2}{3}U\right) + U + T = 180$, so $\frac{1}{3}U = 120 - T$ and $U = 3(120 - T)$. Substituting $T = 95$ yields $U = \mathbf{75}$.

R1-3. With n students, Wash Ed. can choose slide-rule oilers in $\binom{n}{2} = \frac{n(n-1)}{2}$ ways. With $n + 2$ students, there would be $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$ ways of choosing the oilers. The difference is $\frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} = T$. Simplifying yields $\frac{(n^2+3n+2)-(n^2-n)}{2} = 2n + 1 = T$, so $n = \frac{T-1}{2}$. Because $T = 75$, $n = \mathbf{37}$.

R2-1. The first angle is 123°, which is in Quadrant II, the second (246°) is in Quadrant III, and the third is in Quadrant I, because $3 \cdot 123^\circ = 369^\circ \equiv 9^\circ \bmod 360^\circ$. The missing quadrant is IV, which is $270^\circ - 246^\circ = 24^\circ$ away from the second angle in the sequence. Because $3 \cdot 123^\circ \equiv 9^\circ \bmod 360^\circ$, the terminal ray of the $(n+3)^{\text{rd}}$ angle is rotated 9° counterclockwise from the n^{th} angle. Thus three full cycles are needed to reach Quadrant IV starting from the second angle: the fifth angle is 255°, the eighth angle is 264°, and the eleventh angle is 273°. So $n = \mathbf{11}$.

R2-2. There are 9 valid one-digit plates. For a two-digit plate to be valid, it has to be of the form $\underline{A}\,\underline{B}$, where $B \in \{2, \ldots, 9\}$, and either $A \in \{2, \ldots, 9\}$ with $A \neq B$ or $A = 1$. So there are 8 ways to choose B and $8 - 1 + 1 = 8$ ways to choose A, for a total of $8 \cdot 8 = 64$ plates. In general, moving from the last digit to the first, if there are k ways to choose digit n, then there are $k - 1$ ways to choose digit $n - 1$ from the same set of possibilities as digit n had, plus one additional way, for a total of $k - 1 + 1 = k$ choices for digit $n - 1$. So if a license plate has d digits, there are $10 - d$ choices for the last digit and for each digit before it, yielding $(10 - d)^d$ possible d-digit plates. Using $d = T - 3 = 8$, there are $2^8 = \mathbf{256}$ plates.

R2-3. The first inequality states that the point (x, y) is outside the circle centered at the origin with radius $\sqrt{T}$, while the second inequality states that (x, y) is inside the tilted square centered at the origin with diagonal $2\sqrt{2T}$. The area of the square is $4 \cdot \frac{1}{2}\left(\sqrt{2T}\right)^2 = 4T$, while the area of the circle is simply πT, so the area of $\mathcal{R}$ is $4T - \pi T = \mathbf{1024 - 256\boldsymbol{\pi}}$.

2012 Tiebreaker Problems

TB-1. Triangle ABC has $m\angle A > m\angle B > m\angle C$. The angle between the altitude and the angle bisector at vertex A is 6°. The angle between the altitude and the angle bisector at vertex B is 18°. Compute the degree measure of angle C.

TB-2. Compute the number of ordered pairs of integers (b, c), with $-20 \le b \le 20$, $-20 \le c \le 20$, such that the equations $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ share at least one root.

TB-3. A seventeen-sided die has faces numbered 1 through 17, but it is not fair: 17 comes up with probability $1/2$, and each of the numbers 1 through 16 comes up with probability $1/32$. Compute the probability that the sum of two rolls is either 20 or 12.

2012 Tiebreaker Answers

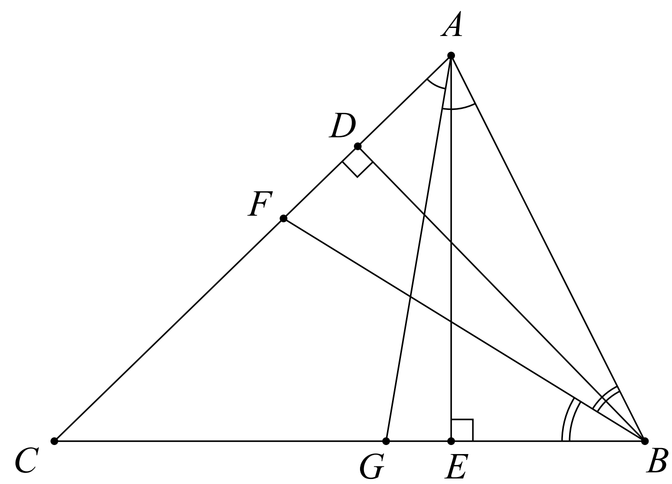
TB-1. $44°$

TB-2. 81

TB-3. $\dfrac{7}{128}$

2012 Tiebreaker Solutions

TB-1. Let the feet of the altitudes from A and B be E and D, respectively, and let F and G be the intersection points of the angle bisectors with $\overline{AC}$ and $\overline{BC}$, respectively, as shown below.



Then $m\angle GAE = 6°$ and $m\angle DBF = 18°$. Suppose $m\angle FBC = x°$ and $m\angle CAG = y°$. So $m\angle CAE = (y+6)°$ and $m\angle CBD = (x+18)°$. Considering right triangle BDC, $m\angle C = 90° - (x+18)° = (72-x)°$, while considering right triangle AEC, $m\angle C = 90° - (y+6)° = (84-y)°$. Thus $84 - y = 72 - x$ and $y - x = 12$. Considering $\triangle ABE$, $m\angle EAB = (y-6)°$ and $m\angle EBA = 2x°$, so $(y-6)+2x = 90$, or $2x + y = 96$. Solving the system yields $x = 28, y = 40$. Therefore $m\angle A = 80°$ and $m\angle B = 56°$, so $m\angle C = \mathbf{44}°$.

Alternate Solution: From right triangle ABE, $90° = \left(\frac{1}{2}A - 6°\right) + B$, and from right triangle ABD, $90° = \left(\frac{1}{2}B - 18°\right) + A$. Adding the two equations gives $180° = \frac{3}{2}(A+B) - 24°$, so $A + B = \frac{2}{3} \cdot 204° = 136°$ and $C = 180° - (A+B) = \mathbf{44°}$.

TB-2. Let r be the common root. Then $r^2 + br + c = r^2 + cr + b \Rightarrow br - cr = b - c$. So either $b = c$ or $r = 1$. In the latter case, $1 + b + c = 0$, so $c = -1 - b$.

There are 41 ordered pairs where $b = c$. If $c = -1 - b$ and $-20 \le b \le 20$, then $-21 \le c \le 19$. Therefore there are 40 ordered pairs $(b, -1-b)$ where both terms are in the required intervals. Thus there are $41 + 40 = \mathbf{81}$ solutions.

TB-3. The rolls that add up to 20 are $17+3$, $16+4$, $15+5$, $14+6$, $13+7$, $12+8$, $11+9$, and $10+10$. Accounting for order, the probability of $17+3$ is $\frac{1}{2} \cdot \frac{1}{32} + \frac{1}{32} \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} \cdot \frac{1}{32} = \frac{32}{1024}$. The combination $10+10$ has probability $\frac{1}{32} \cdot \frac{1}{32} = \frac{1}{1024}$; the other six combinations have probability $2 \cdot \frac{1}{32} \cdot \frac{1}{32} = \frac{2}{1024}$, for a total of $\frac{32+1+6\cdot 2}{1024} = \frac{45}{1024}$ (again, accounting for two possible orders per combination). The rolls that add up to 12 are $1+11$, $2+10$, $3+9$, $4+8$, $5+7$, $6+6$, all of which have probability $2 \cdot \frac{1}{32} \cdot \frac{1}{32} = \frac{2}{1024}$ except the last, which has probability $\left(\frac{1}{32}\right)^2$, for a total of $\frac{11}{1024}$. Thus the probability of either sum appearing is $\frac{45}{1024} + \frac{11}{1024} = \frac{56}{1024} = \mathbf{\frac{7}{128}}$.

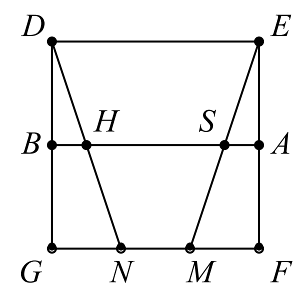
2012 Super Relay Problems

1. If x, y, and z are positive integers such that $xy = 20$ and $yz = 12$, compute the smallest possible value of $x + z$.

2. Let $T = TNYWR$. Let $A = (1, 5)$ and $B = (T - 1, 17)$. Compute the value of x such that $(x, 3)$ lies on the perpendicular bisector of $\overline{AB}$.

3. Let $T = TNYWR$. Let N be the smallest positive T-digit number that is divisible by 33. Compute the product of the last two digits of N.

4. Let $T = TNYWR$. In the square $DEFG$ diagrammed at right, points M and N trisect $\overline{FG}$, points A and B are the midpoints of $\overline{EF}$ and $\overline{DG}$, respectively, and $\overline{EM} \cap \overline{AB} = S$ and $\overline{DN} \cap \overline{AB} = H$. If the side length of square $DEFG$ is T, compute $[DESH]$.

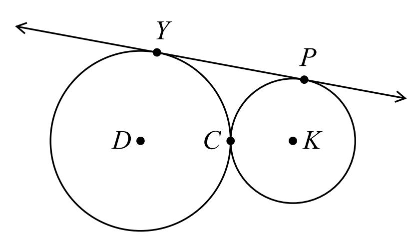


5. Let $T = TNYWR$. For complex z, define the function $f_1(z) = z$, and for $n > 1$, $f_n(z) = f_{n-1}(\overline{z})$. If $f_1(z) + 2f_2(z) + 3f_3(z) + 4f_4(z) + 5f_5(z) = T + Ti$, compute $|z|$.

6. Let $T = TNYWR$. Compute the number of ordered pairs of positive integers (a, b) with the property that $ab = T^{20} \cdot 210^{12}$, and the greatest common divisor of a and b is 1.

7. Let $T = TNYWR$. Given that $\sin\theta = \dfrac{\sqrt{T^2 - 64}}{T}$, compute the largest possible value of the infinite series $\cos\theta + \cos^2\theta + \cos^3\theta + \dots$.

8. Let R be the larger number you will receive, and let r be the smaller number you will receive. In the diagram at right (not drawn to scale), circle D has radius R, circle K has radius r, and circles D and K are tangent at C. Line $\overleftrightarrow{YP}$ is tangent to circles D and K. Compute YP.



9. Let $T = TNYWR$. When T is expressed as a reduced fraction, let m and n be the numerator and denominator, respectively. A square pyramid has base $ABCD$, the distance from vertex P to the base is $n - m$, and $PA = PB = PC = PD = n$. Compute the area of square $ABCD$.

10. Let $T = TNYWR$, and let $d = |T|$. A person whose birthday falls between July 23 and August 22 inclusive is called a *Leo*. A person born in July is randomly selected, and it is given that her birthday is before the d^{th} day of July. Another person who was also born in July is randomly selected, and it is given that his birthday is *after* the d^{th} day of July. Compute the probability that exactly one of these people is a Leo.

11. Let $T = TNYWR$. Given that $\log_2 4^{8!} + \log_4 2^{8!} = 6! \cdot T \cdot x$, compute x.

12. Let $T = TNYWR$. For some real constants a and b, the solution sets of the equations $x^2 + (5b - T - a)x = T + 1$ and $2x^2 + (T + 8a - 2)x = -10b$ are the same. Compute a.

13. Let $T = TNYWR$, and let $K = T - 2$. If K workers can produce 9 widgets in 1 hour, compute the number of workers needed to produce $\dfrac{720}{K}$ widgets in 4 hours.

14. Let $T = TNYWR$, and append the digits of T to $\underline{A}\,\underline{A}\,\underline{B}$ (for example, if $T = 17$, then the result would be $\underline{1}\,\underline{7}\,\underline{A}\,\underline{A}\,\underline{B}$). If the resulting number is divisible by 11, compute the largest possible value of $A + B$.

15. Given that April 1^{st}, 2012 fell on a Sunday, what is the next year in which April 1^{st} will fall on a Sunday?

2012 Super Relay Answers

1. 8
2. 20
3. 6
4. 15
5. $\sqrt{26}$
6. 32
7. $\frac{1}{3}$
8. $10\sqrt{6}$
9. 450
10. $\frac{9}{17}$
11. -14
12. -10
13. 20
14. 14
15. 2018

2012 Super Relay Solutions

1. Note that x and z can each be minimized by making y as large as possible, so set $y = \text{lcm}(12, 20) = 4$. Then $x = 5, z = 3$, and $x + z = \mathbf{8}$.

2. The midpoint of $\overline{AB}$ is $(\frac{T}{2}, 11)$, and the slope of $\overleftrightarrow{AB}$ is $\frac{12}{T-2}$. Thus the perpendicular bisector of $\overline{AB}$ has slope $\frac{2-T}{12}$ and passes through the point $(\frac{T}{2}, 11)$. Thus the equation of the perpendicular bisector of $\overline{AB}$ is $y = (\frac{2-T}{12})x + (11 - \frac{2T-T^2}{24})$. Plugging $y = 3$ into this equation and solving for x yields $x = \frac{96}{T-2} + \frac{T}{2}$. With $T = 8$, it follows that $x = \frac{96}{6} + \frac{8}{2} = 16 + 4 = \mathbf{20}$.

3. The sum of the digits of N must be a multiple of 3, and the alternating sum of the digits must be a multiple of 11. Because the number of digits of N is fixed, the minimum N will have the alternating sum of its digits equal to 0, and therefore the sum of the digits of N will be even, so it must be 6. Thus if T is even, then $N = 1\underbrace{0\ldots0}_{T-3\text{ 0's}}23$, and if T is odd, then $N = 1\underbrace{0\ldots0}_{T-3\text{ 0's}}32$. Either way, the product of the last two digits of N is $\mathbf{6}$ (independent of T).

4. Note that $DESH$ is a trapezoid with height $\frac{T}{2}$. Because $\overline{AS}$ and $\overline{BH}$ are midlines of triangles EFM and DGN respectively, it follows that $AS = BH = \frac{T}{6}$. Thus $SH = T - 2 \cdot \frac{T}{6} = \frac{2T}{3}$. Thus $[DESH] = \frac{1}{2}(T + \frac{2T}{3}) \cdot \frac{T}{2} = \frac{5T^2}{12}$. With $T = 6$, the desired area is $\mathbf{15}$.

5. Because $\overline{\overline{z}} = z$, it follows that $f_n(z) = z$ when n is odd, and $f_n(z) = \overline{z}$ when n is even. Taking $z = a+bi$, where a and b are real, it follows that $\sum_{k=1}^{5} kf_k(z) = 15a+3bi$. Thus $a = \frac{T}{15}, b = \frac{T}{3}$, and $|z| = \sqrt{a^2+b^2} = \frac{|T|\sqrt{26}}{15}$. With $T = 15$, the answer is $\sqrt{\mathbf{26}}$.

6. If the prime factorization of ab is $p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$, where the p_i's are distinct primes and the e_i's are positive integers, then in order for $\gcd(a, b)$ to equal 1, each p_i must be a divisor of exactly one of a or b. Thus the desired number of ordered pairs is 2^k because there are 2 choices for each prime divisor (i.e., $p_i \mid a$ or $p_i \mid b$). With $T = \sqrt{26}$, it follows that $(\sqrt{26})^{20} \cdot 210^{12} = (2^{10} \cdot 13^{10}) \cdot 210^{12} = 2^{22} \cdot 3^{12} \cdot 5^{12} \cdot 7^{12} \cdot 13^{10}$. Thus there are five distinct prime divisors, and the answer is $2^5 = \mathbf{32}$.

7. Using $\sin^2\theta + \cos^2\theta = 1$ gives $\cos^2\theta = \frac{64}{T^2}$, so to maximize the sum, take $\cos\theta = \frac{8}{|T|}$. Using the formula for the sum of an infinite geometric series gives $\frac{8/|T|}{1-8/|T|} = \frac{8}{|T|-8}$. With $T = 32$, the answer is $\frac{8}{24} = \mathbf{\frac{1}{3}}$.

8. Note that $\overline{DY}$ and $\overline{KP}$ are both perpendicular to line $\overleftrightarrow{YP}$. Let J be the foot of the perpendicular from K to $\overline{DY}$. Then $PKJY$ is a rectangle and $YP = JK = \sqrt{DK^2 - DJ^2} = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$. With $R = 450$ and $r = \frac{1}{3}$, the answer is $2\sqrt{150} = \mathbf{10\sqrt{6}}$.

9. By the Pythagorean Theorem, half the diagonal of the square is $\sqrt{n^2-(n-m)^2}=\sqrt{2mn-m^2}$. Thus the diagonal of the square is $2\sqrt{2mn-m^2}$, and the square's area is $4mn-2m^2$. With $T=\frac{9}{17}$, $m=9, n=17$, and the answer is **450**.

10. Note that there are 9 days in July in which a person could be a Leo (July 23–31). Let the woman (born before the d^{th} day of July) be called Carol, and let the man (born after the d^{th} day of July) be called John, and consider the possible values of d. If $d \leq 21$, then Carol will not be a Leo, and the probability that John is a Leo is $\frac{9}{31-d}$. If $d=22$ or 23, then the probability is 1. If $d \geq 24$, then John will be a Leo, and Carol will *not* be a Leo with probability $1-\frac{d-23}{d-1}$. With $T=-14$, the first case applies, and the desired probability is $\mathbf{\frac{9}{17}}$.

11. Note that $4^{8!}=2^{2\cdot 8!}$, thus $\log_2 4^{8!}=2\cdot 8!$. Similarly, $\log_4 2^{8!}=\frac{8!}{2}$. Thus $2\cdot 8!+\frac{8!}{2}=6!(2\cdot 7\cdot 8+7\cdot\frac{8}{2})=6!\cdot 140$. Thus $140=Tx$, and with $T=-10$, $x=\mathbf{-14}$.

12. Divide each side of the second equation by 2 and equate coefficients to obtain $5b-T-a=\frac{T}{2}+4a-1$ and $T+1=-5b$. Thus $b=\frac{T+1}{-5}$, and plugging this value into the first equation yields $a=-\frac{T}{2}$. With $T=20$, the answer is $\mathbf{-10}$.

13. Because T workers produce 9 widgets in 1 hour, 1 worker will produce $\frac{9}{T}$ widgets in 1 hour. Thus 1 worker will produce $\frac{36}{T}$ widgets in 4 hours. In order to produce $\frac{720}{T}$ widgets in 4 hours, it will require $\frac{720/T}{36/T}=\mathbf{20}$ workers (independent of T).

14. Let R be the remainder when T is divided by 11. Note that the alternating sum of the digits of the number must be divisible by 11. This sum will be congruent mod 11 to $B-A+A-R=B-R$, thus $B=R$. Because A's value is irrelevant, to maximize $A+B$, set $A=9$ to yield $A+B=9+R$. For $T=2018$, $R=5$, and the answer is $9+5=\mathbf{14}$.

15. Note that $365=7\cdot 52+1$. Thus over the next few years after 2012, the day of the week for April 1$^{\text{st}}$ will advance by one day in a non-leap year, and it will advance by two days in a leap year. Thus in six years, the day of the week will have rotated a complete cycle, and the answer is **2018**.

2013 Contest

Contents

2013 Team Problems

T-1. Let x be the smallest positive integer such that $1584 \cdot x$ is a perfect cube, and let y be the smallest positive integer such that xy is a multiple of 1584. Compute y.

T-2. Emma goes to the store to buy apples and peaches. She buys five of each, hands the shopkeeper one \$5 bill, but then has to give the shopkeeper another; she gets back some change. Jonah goes to the same store, buys 2 apples and 12 peaches, and tries to pay with a single \$10 bill. But that's not enough, so Jonah has to give the shopkeeper another \$10 bill, and also gets some change. Finally, Helen goes to the same store to buy 25 peaches. Assuming that the price in cents of each fruit is an integer, compute the least amount of money, in cents, that Helen can expect to pay.

T-3. Circle O has radius 6. Point P lies outside circle O, and the shortest distance from P to circle O is 4. Chord $\overline{AB}$ is parallel to $\overleftrightarrow{OP}$, and the distance between $\overline{AB}$ and $\overleftrightarrow{OP}$ is 2. Compute $PA^2 + PB^2$.

T-4. A *palindrome* is a positive integer, not ending in 0, that reads the same forwards and backwards. For example, $35253, 171, 44$, and 2 are all palindromes, but 17 and 1210 are not. Compute the least positive integer greater than 2013 that *cannot* be written as the sum of two palindromes.

T-5. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$.

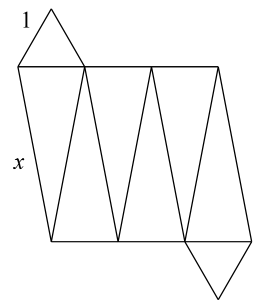
T-6. Compute $\cos^3 \frac{2\pi}{7} + \cos^3 \frac{4\pi}{7} + \cos^3 \frac{8\pi}{7}$.

T-7. In right triangle ABC with right angle C, line ℓ is drawn through C and is parallel to $\overline{AB}$. Points P and Q lie on $\overline{AB}$ with P between A and Q, and points R and S lie on ℓ with C between R and S such that $PQRS$ is a square. Let $\overline{PS}$ intersect $\overline{AC}$ in X, and let $\overline{QR}$ intersect $\overline{BC}$ in Y. The inradius of triangle ABC is 10, and the area of square $PQRS$ is 576. Compute the sum of the inradii of triangles AXP, CXS, CYR, and BYQ.

T-8. Compute the sum of all real numbers x such that

$$\left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor = \frac{x}{7}.$$

T-9. Two equilateral triangles of side length 1 and six isosceles triangles with legs of length x and base of length 1 are joined as shown below; the net is folded to make a solid. If the volume of the solid is 6, compute x.



T-10. Let $S = \{1, 2, \ldots, 20\}$, and let f be a function from S to S; that is, for all $s \in S, f(s) \in S$. Define the sequence $s_1, s_2, s_3, \ldots$ by setting $s_n = \sum_{k=1}^{20} \underbrace{(f \circ \cdots \circ f)}_{n}(k)$. That is, $s_1 = f(1) + \cdots + f(20)$, $s_2 = f(f(1)) + \cdots + f(f(20))$, $s_3 = f(f(f(1))) + f(f(f(2))) + \cdots + f(f(f(20)))$, etc. Compute the smallest integer p such that the following statement is true: The sequence $s_1, s_2, s_3, \ldots$ must be periodic after a certain point, and its period is at most p. (If the sequence is never periodic, then write ∞ as your answer.)

2013 Team Answers

T-1. 12

T-2. 1525

T-3. 272

T-4. 2019

T-5. 50

T-6. $-\dfrac{1}{2}$

T-7. 14

T-8. -21

T-9. $\dfrac{5\sqrt{39}}{3}$

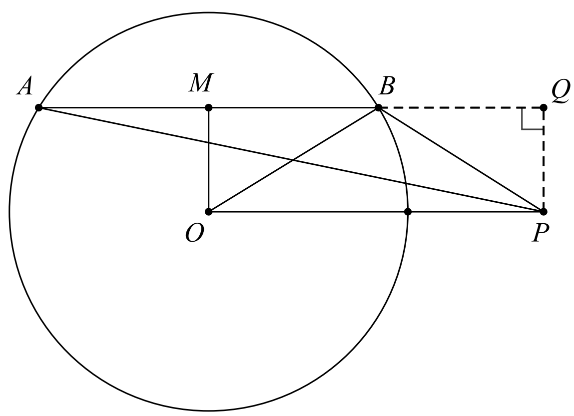
T-10. 140

2013 Team Solutions

T-1. In order for $1584 \cdot x$ to be a perfect cube, all of its prime factors must be raised to powers divisible by 3. Because $1584 = 2^4 \cdot 3^2 \cdot 11$, x must be of the form $2^{3k+2} \cdot 3^{3m+1} \cdot 11^{3n+2} \cdot r^3$, for nonnegative integers k, m, n, r, $r > 0$. Thus the least positive value of x is $2^2 \cdot 3 \cdot 11^2 = 1452$. But in order for xy to be a positive multiple of 1584, xy must be of the form $2^a \cdot 3^b \cdot 11^c \cdot d$, where $a \geq 4$, $b \geq 2$, $c \geq 1$, and $d \geq 1$. Thus y must equal $2^2 \cdot 3^1 = \mathbf{12}$.

T-2. Let a be the price of one apple and p be the price of one peach, in cents. The first transaction shows that $500 < 5a + 5p < 1000$, hence $100 < a + p < 200$. The second transaction shows that $1000 < 2a + 12p < 2000$, so $500 < a + 6p < 1000$. Subtracting the inequalities yields $300 < 5p < 900$, so $60 < p < 180$. Therefore the price of 25 peaches is at least $25 \cdot 61 = \mathbf{1525}$ cents.

T-3. Extend $\overline{AB}$ to point Q such that $\overline{PQ} \perp \overline{AQ}$ as shown, and let M be the midpoint of $\overline{AB}$. (The problem does not specify whether A or B is nearer P, but B can be assumed to be nearer P without loss of generality.)



Then $OP = 10$, $PQ = OM = 2$, and $OB = 6$. Thus $MB = \sqrt{6^2 - 2^2} = 4\sqrt{2}$. Because $QM = OP = 10$, it follows that $QB = 10 - 4\sqrt{2}$ and $QA = 10 + 4\sqrt{2}$. So

$$\begin{aligned} PA^2 + PB^2 &= (QA^2 + QP^2) + (QB^2 + QP^2) \\ &= (10 + 4\sqrt{2})^2 + 2^2 + (10 - 4\sqrt{2})^2 + 2^2 \\ &= \mathbf{272}. \end{aligned}$$

T-4. If $a + b \geq 2014$, then at least one of a, b must be greater than 1006. The palindromes greater than 1006 but less than 2014 are, in descending order, 2002, 1991, 1881, ..., 1111. Let a

represent the larger of the two palindromes. Then for $n = 2014$, $a = 2002$ is impossible, because $2014 - 2002 = 12$. Any value of a between 1111 and 2000 ends in 1, so if $a + b = 2014$, b ends in 3, and because $b < 1000$, it follows that $303 \le b \le 393$. Subtracting 303 from 2014 yields 1711, and so $a \le 1711$. Thus $a = 1661$ and $b = 353$. A similar analysis shows the following results:

$$\begin{aligned} 2015 &= 1551 + 464; \\ 2016 &= 1441 + 575; \\ 2017 &= 1331 + 686; \text{ and} \\ 2018 &= 1221 + 797. \end{aligned}$$

But **2019** cannot be expressed as the sum of two palindromes: b would have to end in 8, so $b = 808 + 10d$ for some digit d. Then $2019 - 898 \le a \le 2019 - 808$, hence $1121 \le a \le 1211$, and there is no palindrome in that interval.

T-5. First consider the problem with x, y, z positive real numbers. If $xy + z = 160$ and z is constant, then $y = \frac{160-z}{x}$, yielding $x + yz = x + \frac{z(160-z)}{x}$. For $a, x > 0$, the quantity $x + \frac{a}{x}$ is minimized when $x = \sqrt{a}$ (proof: use the Arithmetic-Geometric Mean Inequality $\frac{A+B}{2} \ge \sqrt{AB}$ with $A = x$ and $B = \frac{a}{x}$); in this case, $x + \frac{a}{x} = 2\sqrt{a}$. Thus $x + yz \ge 2\sqrt{z(160 - z)}$. Considered as a function of z, this lower bound is increasing for $z < 80$.

These results suggest the following strategy: begin with small values of z, and find a factorization of $160 - z$ such that x is close to $\sqrt{z(160 - z)}$. (Equivalently, such that $\frac{x}{y}$ is close to z.) The chart below contains the triples (x, y, z) with the smallest values of $x + yz$, conditional upon z.

z	(x, y, z)	$x + yz$
1	$(53, 3, 1)$	56
2	$(79, 2, 2)$	83
3	$(157, 1, 3)$	160
4	$(26, 6, 4)$	50
5	$(31, 5, 5)$	56
6	$(22, 7, 6)$	64

Because $x + yz \ge 2\sqrt{z(160 - z)}$, it follows that $x + yz \ge 64$ for $6 \le z \le 80$. And because $x + yz > 80$ for $z \ge 80$, the minimal value of $x + yz$ is **50**.

Note: In fact, 160 is the smallest positive integer such that $xy + z = N$ requires $z \ge 4$ in the minimal solution for $x + yz$. The smallest values of N for which $z = 2, \ldots, 12$, are 8, 48, 160, 720, 790, 1690, 4572, 13815, 22031, 22032, 79965.

T-6. The identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ can be rewritten into the power-reducing identity

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta.$$

Thus if D is the desired sum,

$$\begin{aligned} D &= \cos^3\frac{2\pi}{7}+\cos^3\frac{4\pi}{7}+\cos^3\frac{8\pi}{7} \\ &= \frac{1}{4}\left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}+\cos\frac{24\pi}{7}\right)+\frac{3}{4}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right). \end{aligned}$$

Observe that $\cos\frac{24\pi}{7}=\cos\frac{10\pi}{7}$, so

$$D=\frac{1}{4}\left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}+\cos\frac{10\pi}{7}\right)+\frac{3}{4}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right).$$

Notice also that $\cos\theta=\cos(2\pi-\theta)$ implies $\cos\frac{12\pi}{7}=\cos\frac{2\pi}{7}$, $\cos\frac{10\pi}{7}=\cos\frac{4\pi}{7}$, and $\cos\frac{8\pi}{7}=\cos\frac{6\pi}{7}$. Rewriting D using the least positive equivalent angles yields

$$\begin{aligned} D &= \frac{1}{4}\left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}\right)+\frac{3}{4}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right) \\ &= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}. \end{aligned}$$

To evaluate this sum, use the identity $\cos\theta=\cos(2\pi-\theta)$ again to write

$$2D=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7}.$$

If $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, notice that the right side of the equation above is simply the real part of the sum $\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6$. Because α^n is a solution to the equation $z^7=1$ for $n=0,1,\ldots,6$, the sum $1+\alpha+\alpha^2+\cdots+\alpha^6$ equals 0. Hence $\alpha+\alpha^2+\cdots+\alpha^6=-1$ and $D=\mathbf{-1/2}$.

Alternate Solution: Construct a cubic polynomial in x for which $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{8\pi}{7}$ are zeros; then the sum of their cubes can be found using techniques from the theory of equations. In particular, suppose the three cosines are zeros of x^3+bx^2+cx+d. Then

$$\begin{aligned} b &= -\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right), \\ c &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}, \text{ and} \\ d &= -\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}. \end{aligned}$$

Use complex seventh roots of unity (as in the previous solution) to find $b=1/2$. To find c, use the product-to-sum formula $2\cos A\cos B=\cos(A+B)+\cos(A-B)$ three times:

$$\begin{aligned} 2c &= \left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}\right)+\left(\cos\frac{10\pi}{7}+\cos\frac{6\pi}{7}\right)+\left(\cos\frac{4\pi}{7}+\cos\frac{12\pi}{7}\right) \\ &= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7}\ [\text{because } \cos\theta=\cos(2\pi-\theta)] \\ &= -1. \end{aligned}$$

Thus $c = -1/2$.

To compute d, multiply both sides by $\sin\frac{2\pi}{7}$ and use the identity $2\sin\theta\cos\theta = \sin 2\theta$:

$$\begin{aligned} d\sin\frac{2\pi}{7} &= -\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{8\pi}{7} \\ &= -\frac{1}{2}\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}\cos\frac{8\pi}{7} \\ &= -\frac{1}{4}\sin\frac{8\pi}{7}\cos\frac{8\pi}{7} \\ &= -\frac{1}{8}\sin\frac{16\pi}{7}. \end{aligned}$$

Because $\sin\frac{16\pi}{7} = \sin\frac{2\pi}{7}$, the factors on both sides cancel, leaving

$$d = -1/8.$$

Thus $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{8\pi}{7}$ are roots of $x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8}$; so each value also satisfies the equation $x^3 = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{8}$. Hence the desired sum can be rewritten as

$$\begin{aligned} \cos^3\frac{2\pi}{7} + \cos^3\frac{4\pi}{7} + \cos^3\frac{8\pi}{7} &= -\frac{1}{2}\left(\cos^2\frac{2\pi}{7} + \cos^2\frac{4\pi}{7} + \cos^2\frac{8\pi}{7}\right) \\ &+ \frac{1}{2}\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right) + \frac{3}{8}. \end{aligned}$$

Previous work has already established that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -1/2$, so it remains to compute $\cos^2\frac{2\pi}{7} + \cos^2\frac{4\pi}{7} + \cos^2\frac{8\pi}{7}$. The identity $A^2+B^2+C^2 = (A+B+C)^2 - 2(AB+BC+AC)$ allows the use of previous results: $\cos^2\frac{2\pi}{7} + \cos^2\frac{4\pi}{7} + \cos^2\frac{8\pi}{7} = (-1/2)^2 - 2(-1/2) = 5/4$. Thus

$$\cos^3\frac{2\pi}{7} + \cos^3\frac{4\pi}{7} + \cos^3\frac{8\pi}{7} = -\frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{3}{8} = \boldsymbol{-\frac{1}{2}}.$$

T-7. Note that in right triangle ABC with right angle C, the inradius r is equal to $\frac{a+b-c}{2}$, where $a = BC$, $b = AC$, and $c = AB$, because the inradius equals the distance from the vertex of the right angle C to (either) point of tangency along $\overline{AC}$ or $\overline{BC}$. Thus the sum of the inradii of triangles AXP, CXS, CYR, and BYQ is equal to one-half the difference between the sum of the lengths of the legs of these triangles and the sum of the lengths of the hypotenuses of these triangles. Let t be the side length of square $PQRS$. Then the sum of the lengths of the legs of triangles AXP, CXS, CYR, and BYQ is

$$\begin{aligned} & AP + PX + XS + SC + CR + RY + YQ + QB \\ =\ & AP + PS + SR + RQ + QB \\ =\ & AP + t + t + t + QB \\ =\ & AB - PQ + 3t \\ =\ & c - t + 3t \\ =\ & c + 2t. \end{aligned}$$

The sum of the lengths of the hypotenuses of triangles AXP, CXS, CYR, and BYQ is $AX + XC + CY + YB = AC + CB = b + a$. Hence the sum of the inradii of triangles AXP, CXS, CYR, and BYQ is $\frac{c+2t-(a+b)}{2} = t - r$. Thus the desired sum equals $\sqrt{576} - 10 = 24 - 10 = \mathbf{14}$.

T-8. Because the quantity on the left side is the difference of two integers, $x/7$ must be an integer, hence x is an integer (in fact a multiple of 7). Because the denominators on the left side are 2 and 3, it is convenient to write $x = 6q + r$, where $0 \le r \le 5$, so that $\lfloor x/2 \rfloor = 3q + \lfloor r/2 \rfloor$ and $\lfloor x/3 \rfloor = 2q + \lfloor r/3 \rfloor$. Then for $r = 0, 1, \ldots, 5$ these expressions can be simplified as shown in the table below.

r	0	1	2	3	4	5
$\lfloor \frac{x}{2} \rfloor$	$3q$	$3q$	$3q+1$	$3q+1$	$3q+2$	$3q+2$
$\lfloor \frac{x}{3} \rfloor$	$2q$	$2q$	$2q$	$2q+1$	$2q+1$	$2q+1$
$\lfloor \frac{x}{2} \rfloor - \lfloor \frac{x}{3} \rfloor$	q	q	$q+1$	q	$q+1$	$q+1$

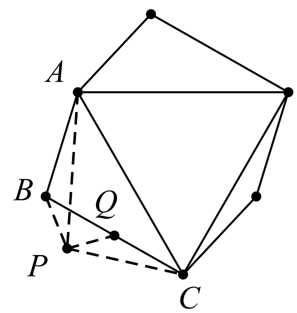
Now proceed by cases:

$r = 0$: Then $q = x/6$. But from the statement of the problem, $q = x/7$, so $x = 0$.
$r = 1$: Then $q = (x-1)/6 = x/7 \Rightarrow x = 7$.
$r = 2$: Then $q = (x-2)/6$ and $q + 1 = x/7$, so $(x+4)/6 = x/7$, and $x = -28$.
$r = 3$: Then $q = (x-3)/6$ and $q = x/7$, so $x = 21$.
$r = 4$: Then $q = (x-4)/6$ and $q + 1 = x/7$, so $(x+2)/6 = x/7$, and $x = -14$.
$r = 5$: Then $q = (x-5)/6$ and $q + 1 = x/7$, so $(x+1)/6 = x/7$, and $x = -7$.

The sum of these values is $0 + 7 + -28 + 21 + -14 + -7 = \mathbf{-21}$.

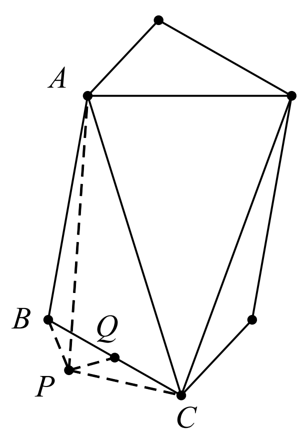
T-9. First consider a regular octahedron of side length 1. To compute its volume, divide it into two square-based pyramids with edges of length 1. Such a pyramid has slant height $\frac{\sqrt{3}}{2}$ and height $\sqrt{(\frac{\sqrt{3}}{2})^2 - (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$, so its volume is $\frac{1}{3} \cdot 1^2 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$. Thus the octahedron has volume twice that, or $\frac{\sqrt{2}}{3}$. The result of folding the net shown is actually the image of a regular octahedron after being stretched along an axis perpendicular to one face by a factor of r. Because the octahedron is only being stretched in one dimension, the volume changes by the same factor r. So the problem reduces to computing the factor r and the edge length of the resulting octahedron.

For convenience, imagine that one face of the octahedron rests on a plane. Seen from above the plane, the octahedron appears as shown below.



Let P be the projection of A onto the plane on which the octahedron rests, and let Q be the foot of the perpendicular from P to $\overline{BC}$. Then $PQ = R_C - R_I$, where R_C is the circumradius and R_I the inradius of the equilateral triangle. Thus $PQ = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6}$. Then $BP^2 = PQ^2 + BQ^2 = \frac{3}{36} + \frac{1}{4} = \frac{1}{3}$, so $AP^2 = AB^2 - BP^2 = \frac{2}{3}$, and $AP = \frac{\sqrt{6}}{3}$.

Now let a vertical stretch take place along an axis parallel to $\overleftrightarrow{AP}$. If the scale factor is r, then $AP = \frac{r\sqrt{6}}{3}$, and because the stretch occurs on an axis perpendicular to $\overline{BP}$, the length BP is unchanged, as can be seen below.



Thus $AB^2 = \frac{6r^2}{9} + \frac{1}{3} = \frac{6r^2+3}{9}$. It remains to compute r. But r is simply the ratio of the new volume to the old volume:

$$r = \frac{6}{\frac{\sqrt{2}}{3}} = \frac{18}{\sqrt{2}} = 9\sqrt{2}.$$

Thus $AB^2 = \frac{6(9\sqrt{2})^2+3}{9} = \frac{975}{9} = \frac{325}{3}$, and $AB = \boldsymbol{\frac{5\sqrt{39}}{3}}$.

T-10. If f is simply a permutation of S, then $\{s_n\}$ is periodic. To understand why, consider a smaller set $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If $f : [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] \to [2, 3, 4, 5, 1, 7, 8, 6, 9, 10]$, then f has one cycle of period 5 and one cycle of period 3, so the period of f is 15. However,

$$f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) + f(9) + f(10) =$$

$$2+3+4+5+1+7+8+6+9+10=55,$$

because f just rearranges the order of the summands. So $s_1 = s_0$, and for all n, $s_n = s_{n+1}$; in short, the period of $\{s_n\}$ is just 1.

In order for $\{s_n\}$ to have a period greater than 1, f must be many-to-one, so that some values occur more than once (and some values do not occur at all) in the sum $f(1)+f(2)+\cdots+f(10)$ (or, in the original problem, $f(1)+f(2)+\cdots+f(20)$). For example, consider the function f_2 below:

$$f_2 : [1,2,3,4,5,6,7,8,9,10] \to [2,3,4,5,1,10,9,10,7,3].$$

Note that $s_1 = 2+3+4+5+1+10+9+10+7+3 \neq 55$, so $\{s_n\}$ is not immediately periodic. But $\{s_n\}$ is eventually periodic, as the following argument shows. The function f_2 has two cycles: $1 \to 2 \to 3 \to 4 \to 5 \to 1$, and $7 \to 9 \to 7$. There are also two paths that meet up with the first cycle: $6 \to 10 \to 3 \to \cdots$ and $8 \to 10 \to 3 \to \cdots$. Thus for all k in T, $f_2(f_2(k))$ is an element of one of these two extended cycles. Thus $\{s_n\}$ eventually becomes periodic.

The criterion that the function be many-to-one is necessary, but not sufficient, for $\{s_n\}$ to have period greater than 1. To see why, consider the function $g : [1,2,3,4,5,6,7,8,9,10] \to [2,3,4,5,6,1,8,7,8,7]$. This function is many-to-one, and contains two cycles, $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1$ and $7 \to 8 \to 7$. But because $g(9) = 8$ and $g(10) = 7$, the sum $s_1 = 2+3+4+5+6+1+8+7+8+7$, while $s_2 = 3+4+5+6+1+2+7+8+7+8$. In fact, for $n > 1$, $s_{n+1} = s_n$, because applying f only permutes the 6-cycle and switches the two 7's and two 8's. That is, in the list $\underbrace{(g \circ \cdots \circ g)}_{n}(1), \ldots, \underbrace{(g \circ \cdots \circ g)}_{n}(10)$, the values 7 and 8 both show up exactly twice. This cycle is *balanced*: each of its elements shows up the same number of times for all n in the list $\underbrace{(g \circ \cdots \circ g)}_{n}(1), \ldots, \underbrace{(g \circ \cdots \circ g)}_{n}(10)$, for all n after a certain point. The conclusion is that not all many-to-one functions produce unbalanced cycles.

There are two ways a function g can produce balanced cycles. First, the cycles can be self-contained, so no element outside of the cycle is ever absorbed into the cycle, as happens with the 6-cycle in the example above. Alternatively, the outside elements that are absorbed into a cycle can all arrive at different points of the cycle, so that each element of the cycle occurs equally often in each iteration of g. In the example above, the values $g(9) = 7$ and $g(10) = 8$ balance the $7 \to 8 \to 7$ cycle. On the other hand, in the function f_2 above, $f(f(6)) = f(f(8)) = f(f(1)) = 3$, making the large cycle unbalanced: in s_2, the value 3 appears three times in s_2, but the value 2 only appears once in s_2.

The foregoing shows that only *unbalanced* cycles can affect the periodicity of $\{s_n\}$. Because each element of a balanced cycle occurs equally often in each iteration, the period of that component of the sum s_n attributed to the cycle is simply 1. (The case where f is a permutation of S is simply a special case of this result.) In the above example, the large cycle is unbalanced. Note the following results under f_2.

n	$\overbrace{(f_2 \circ \cdots \circ f_2)}^{n}(T)$	s_n
1	$[2, 3, 4, 5, 1, 10, 9, 10, 7, 3]$	54
2	$[3, 4, 5, 1, 2, 3, 7, 3, 9, 4]$	41
3	$[4, 5, 1, 2, 3, 4, 9, 4, 7, 5]$	40
4	$[5, 1, 2, 3, 4, 5, 7, 5, 9, 1]$	42
5	$[1, 2, 3, 4, 5, 1, 9, 1, 7, 2]$	35
6	$[2, 3, 4, 5, 1, 2, 7, 2, 9, 3]$	38
7	$[3, 4, 5, 1, 2, 3, 9, 3, 7, 4]$	41
8	$[4, 5, 1, 2, 3, 4, 7, 4, 9, 5]$	40
9	$[5, 1, 2, 3, 4, 5, 9, 5, 7, 1]$	42

The period of $\{s_n\}$ for f_2 is 5, the period of the unbalanced cycle.

The interested reader may inquire whether *all* unbalanced cycles affect the periodicity of $\{s_n\}$; we encourage those readers to explore the matter independently. For the purposes of solving this problem, it is sufficient to note that unbalanced cycles *can* affect $\{s_n\}$'s periodicity.

Finally, note that an unbalanced k-cycle actually requires at least $k+1$ elements: k to form the cycle, plus at least 1 to be absorbed into the cycle and cause the imbalance. For the original set S, one way to create such an imbalance would be to have $f(20) = f(1) = 2, f(2) = 3, f(3) = 4, \ldots, f(19) = 1$. This arrangement creates an unbalanced cycle of length 19. But breaking up into smaller unbalanced cycles makes it possible to increase the period of $\{s_n\}$ even more, because then in most cases the period is the least common multiple of the periods of the unbalanced cycles. For example, $f : [1, 2, 3, \ldots, 20] = [2, 3, 4, 5, 6, 7, 8, 9, 1, 1, 12, 13, 14, 15, 16, 17, 18, 11, 11, 11]$ has an unbalanced cycle of length 9 and an unbalanced cycle of length 8, giving $\{s_n\}$ a period of 72.

So the goal is to maximize $\operatorname{lcm}\{k_1, k_2, \ldots, k_m\}$ such that $k_1 + k_2 + \cdots + k_m + m \leq 20$. With $m = 2$, the maximal period is 72, achieved with $k_1 = 9$ and $k_2 = 8$. With $m = 3$, $k_1 + k_2 + k_3 \leq 17$, but $\operatorname{lcm}\{7, 6, 4\} = 84 < \operatorname{lcm}\{7, 5, 4\} = 140$. This last result can be obtained with unbalanced cycles of length 4, 5, and 7, with the remaining four points entering the three cycles (or with one point forming a balanced cycle of length 1, i.e., a fixed point). Choosing larger values of m decreases the values of k so far that they no longer form long cycles: when $m = 4$, $k_1 + k_2 + k_3 + k_4 \leq 16$, and even if $k_4 = 2$, $k_3 = 3$, and $k_2 = 5$, for a period of 30, the largest possible value of $k_1 = 6$, which does not alter the period. (Even $k_1 = 7$, $k_2 = 5$, and $k_3 = k_4 = 2$ only yields a period of 70.) Thus the maximum period of s_n is **140**. One such function f is given below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f(n)$	2	3	4	1	1	7	8	9	10	6	6	13	14	15	16	17	18	12	12	20

2013 Individual Problems

I-1. Call a positive integer *fibbish* if each digit, after the leftmost two, is at least the sum of the previous two digits. Compute the greatest fibbish number.

I-2. An *ARMLbar* is a 7×7 grid of unit squares with the center unit square removed. A *portion* of an ARMLbar is a square section of the bar, cut along the gridlines of the original bar. Compute the number of different ways there are to cut a single portion from an ARMLbar.

I-3. Regular hexagon $ABCDEF$ and regular hexagon $GHIJKL$ both have side length 24. The hexagons overlap, so that G is on $\overline{AB}$, B is on $\overline{GH}$, K is on $\overline{DE}$, and D is on $\overline{JK}$. If $[GBCDKL] = \frac{1}{2}[ABCDEF]$, compute LF.

I-4. Compute the largest base-10 integer $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$, with $A > 0$, such that $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D} = B! + C! + D!$.

I-5. Let X be the number of digits in the decimal expansion of $100^{1000^{10,000}}$, and let Y be the number of digits in the decimal expansion of $1000^{10,000^{100,000}}$. Compute $\lfloor \log_X Y \rfloor$.

I-6. Compute the smallest possible value of n such that two diagonals of a regular n-gon intersect at an angle of 159 degrees.

I-7. Compute the number of quadratic functions $f(x) = ax^2 + bx + c$ with integer roots and integer coefficients whose graphs pass through the points $(0, 0)$ and $(15, 225)$.

I-8. A bubble in the shape of a hemisphere of radius 1 is on a tabletop. Inside the bubble are five congruent spherical marbles, four of which are sitting on the table and one which rests atop the others. All marbles are tangent to the bubble, and their centers can be connected to form a pyramid with volume V and with a square base. Compute V.

I-9. Compute the smallest positive integer base b for which 16_b is prime and 97_b is a perfect square.

I-10. For a positive integer n, let $C(n)$ equal the number of pairs of consecutive 1's in the binary representation of n. For example, $C(183) = C(10110111_2) = 3$. Compute $C(1) + C(2) + C(3) + \cdots + C(256)$.

2013 Individual Answers

I-1. 10112369

I-2. 96

I-3. 18

I-4. 5762

I-5. 13

I-6. 60

I-7. 8

I-8. $\dfrac{1}{54}$

I-9. 53

I-10. 448

2013 Individual Solutions

I-1. The largest fibbish number is **10112369**. First, if $\underline{A_1}\,\underline{A_2}\cdots\underline{A_n}$ is an n-digit fibbish number with A_1 and $A_2 \neq 0$, the number created by *prepending* the digits A_1 and 0 to the number is larger and still fibbish: $\underline{A_1}\,\underline{0}\,\underline{A_1}\,\underline{A_2}\cdots\underline{A_n} > \underline{A_1}\,\underline{A_2}\cdots\underline{A_n}$. Suppose that $A_2 = 0$ and $A_3 = A_1$, so that the number begins $\underline{A_1}\,\underline{0}\,\underline{A_1}\,\underline{A_4}$. If the number is to be fibbish, $A_4 \geq A_1 > 0$. Then if $A_1 \geq 2$ and $A_4 \geq 2$, because the number is fibbish, $A_5 \geq 4$, and $A_6 \geq 6$. In this case there can be no more digits, because $A_5 + A_6 \geq 10$. So the largest possible fibbish number beginning with 20 is 202246. If $A_1 = 2$ and $A_2 = 1$, then A_3 must be at least 3, and the largest possible number is 21459; changing A_3 to 3 does not increase the length. Now consider $A_1 = 1$. If $A_2 = 1$, then $A_3 \geq 2$, $A_4 \geq 3$, $A_5 \geq 5$, and $A_6 \geq 8$. There can be no seventh digit because that digit would have to be at least 13. Increasing A_3 to 3 yields only two additional digits, because $A_4 \geq 4$, $A_5 \geq 7$. So $A_3 = 2$ yields a longer (and thus larger) number. Increasing A_4 to 4 yields only one additional digit, $A_5 \geq 6$, because $A_4 + A_5 \geq 10$. But if $A_4 = 3$, increasing A_5 to 6 still allows $A_6 = 9$, yielding the largest possible number of digits (8) and the largest fibbish number with that many digits.

I-2. Note that any portion of side length $m \geq 4$ will overlap the center square, so consider only portions of side length 3 or less. If there were no hole in the candy bar, the number of portions could be counted by conditioning on the possible location of the upper-left corner of the portion. If the portion is of size 1×1, then the corner can occupy any of the 7^2 squares of the bar. If the portion is of size 2×2, then the corner can occupy any of the top 6 rows and any of the left 6 columns, for $6^2 = 36$ possible 2×2 portions. In general, the upper-left corner of an $m \times m$ portion can occupy any of the top $8 - m$ rows and any of the left $8 - m$ columns. So the total number of portions from an intact bar would be $7^2 + 6^2 + 5^2$. Now when $m \leq 3$, the number of $m \times m$ portions that include the missing square is simply m^2, because the missing square could be any square of the portion. So the net number of portions is

$$\begin{aligned} 7^2 + 6^2 + 5^2 - 3^2 - 2^2 - 1^2 &= (49 + 36 + 25) - (9 + 4 + 1) \\ &= 110 - 14 \\ &= \mathbf{96}. \end{aligned}$$

Alternate Solution: First ignore the missing square. As in the previous solution, the number of $m \times m$ portions that can fit in the bar is $(8 - m)^2$. So the total number of portions of all sizes is simply

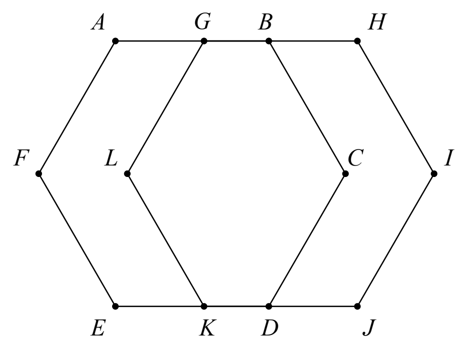
$$7^2 + 6^2 + \cdots + 1^2 = \frac{7(7+1)(2 \cdot 7 + 1)}{6} = 140.$$

To exclude portions that overlap the missing center square, it is useful to consider the location of the missing square within the portion. If an $m \times m$ portion includes the missing center square, and $m \le 4$, then the missing square could be any one of the m^2 squares in the portion. If $m = 5$, then the missing square cannot be in the leftmost or rightmost columns of the portion, because then the entire bar would have to extend at least four squares past the hole, and it only extends three. By similar logic, the square cannot be in the top or bottom rows of the portion. So for $m = 5$, there are $3 \cdot 3 = 9$ possible positions. For $m = 6$, the two left and two right columns are excluded, as are the two top and the two bottom rows, for $2 \cdot 2 = 4$ possible positions for the portion. And in a 7×7 square, the only possible location of the hole is in the center. So the total number of portions overlapping the missing square is

$$1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2 = 44.$$

The difference is thus $140 - 44 = \mathbf{96}$.[1]

I-3. The diagram below shows the hexagons.



The area of hexagon $GBCDKL$ can be computed as $[GBCDKL] = [ABCDEF] - [AGLKEF]$, and $[AGLKEF]$ can be computed by dividing concave hexagon $AGLKEF$ into two parallelograms sharing $\overline{FL}$. If $AB = s$, then the height AE is $s\sqrt{3}$, so the height of parallelogram $AGLF$ is $\frac{s\sqrt{3}}{2}$. Thus $[AGLF] = LF \cdot \frac{s\sqrt{3}}{2}$ and $[AGLKEF] = LF \cdot s\sqrt{3}$. On the other hand, the area of a regular hexagon of side length s is $\frac{3s^2\sqrt{3}}{2}$. Because $[GBCDKL] = \frac{1}{2}[ABCDEF]$, it follows that $[AGLKEF] = \frac{1}{2}[ABCDEF]$, and

$$LF \cdot s\sqrt{3} = \frac{1}{2}\left(\frac{3s^2\sqrt{3}}{2}\right) = \frac{3s^2\sqrt{3}}{4},$$

whence $LF = \frac{3}{4}s$. With $s = 24$, the answer is $\mathbf{18}$.

Alternate Solution: Compute $[BCDKLG]$ as twice the area of trapezoid $BCLG$. If $AB = s$, then $BG = s - LF$ and $CL = 2s - LF$, while the height of the trapezoid is $\frac{s\sqrt{3}}{2}$.

[1] The answer 115 was also accepted for this problem because of an alternate (and unintended) reasonable interpretation of the problem statement. Some students also counted portions that contained the "hole", with the hole being strictly inside the portion, and not along its edges.

Thus the area of the trapezoid is:

$$\frac{1}{2}\left(\frac{s\sqrt{3}}{2}\right)((s-LF)+(2s-LF))=\frac{s\sqrt{3}(3s-2LF)}{4}.$$

Double that area to obtain

$$[BCDKLG]=\frac{s\sqrt{3}(3s-2LF)}{2}.$$

On the other hand, $[ABCDEF]=\frac{3s^2\sqrt{3}}{2}$, so

$$\begin{aligned}\frac{s\sqrt{3}(3s-2LF)}{2} &= \frac{3s^2\sqrt{3}}{4}\\ 3s-2LF &= \frac{3s}{2}\\ LF &= \frac{3}{4}s.\end{aligned}$$

Substituting $s=24$ yields $LF=\mathbf{18}$.

I-4. Let $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}=N$. Because $7!=5040$ and $8!=40{,}320$, N must be no greater than $7!+6!+6!=6480$. This value of N does not work, so work through the list of possible sums in decreasing order: $7!+6!+5!$, $7!+6!+4!$, etc. The first value that works is $N=\mathbf{5762}=7!+6!+2!$.

Alternate Solution: Let $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}=N$. Because $7!=5040$ and $8!=40{,}320$, to find the maximal value, first consider values of N that include 7 as a digit. Suppose then that $N=5040+X!+Y!$. To force a 7 to appear in this sum with maximal N, let $X=6$, which yields $N=5040+720+Y!=5760+Y!$. This value of N has a 7 (and a 6), so search for values of Y to find ones that satisfy the conditions of the problem. Only $Y=1$ and $Y=2$ will do, giving 5761 and 5762. Hence **5762** is the maximum possible value of N.

I-5. The number of digits of n is $\lfloor\log n\rfloor+1$. Because $100^{1000^{10{,}000}}=(10^2)^{1000^{10{,}000}}$, $X=2\cdot 1000^{10{,}000}+1$. Similarly, $Y=3\cdot 10{,}000^{100{,}000}+1$. Using the change-of-base formula,

$$\begin{aligned}\log_X Y=\frac{\log Y}{\log X} &\approx \frac{\log 3+\log 10{,}000^{100{,}000}}{\log 2+\log 1000^{10{,}000}}\\ &= \frac{\log 3+100{,}000\log 10{,}000}{\log 2+10{,}000\log 1000}\\ &= \frac{\log 3+100{,}000\cdot 4}{\log 2+10{,}000\cdot 3}\\ &= \frac{400{,}000+\log 3}{30{,}000+\log 2}.\end{aligned}$$

Both $\log 3$ and $\log 2$ are tiny compared to the integers to which they are being added. If the quotient 400,000/30,000 were an integer (or extremely close to an integer), the values of these logarithms might matter, but $400{,}000/30{,}000 = 40/3 = 13.\overline{3}$, so in this case, they are irrelevant. Hence

$$\lfloor \log_X Y \rfloor = \left\lfloor \frac{400{,}000}{30{,}000} \right\rfloor = \left\lfloor \frac{40}{3} \right\rfloor = \mathbf{13}.$$

I-6. Let the vertices of the polygon be $A_0, A_1, \ldots, A_{n-1}$. Considering the polygon as inscribed in a circle, the angle between diagonals $\overline{A_0A_i}$ and $\overline{A_0A_j}$ is $\frac{1}{2} \cdot \left(\frac{360^\circ}{n}\right) \cdot |j-i| = \left(\frac{180|j-i|}{n}\right)^\circ$. The diagonal $\overline{A_kA_{k+j}}$ can be considered as the rotation of $\overline{A_0A_j}$ through k/n of a circle, or $\left(\frac{360k}{n}\right)^\circ$. So the diagonals A_0A_i and A_kA_{k+j} intersect at a combined angle of $\left(\frac{180|j-i|}{n}\right)^\circ + \left(\frac{360k}{n}\right)^\circ$. Without loss of generality, assume $i < j$ (otherwise relabel vertices in the opposite order, with A_k becoming A_0). Then the desired number is the least n such that

$$\left(\frac{180(j-i)}{n}\right) + \frac{360k}{n} = 159.$$

Multiply both sides of the equation by n and factor to obtain $180(j - i + 2k) = 159n$, thus $60(j - i + 2k) = 53n$. Because 53 and 60 are relatively prime and $(j - i + 2k)$ is an integer, it follows that $60 \mid n$. So the smallest possible value is $n = \mathbf{60}$; one set of values that satisfies the equation is $i = 1$, $j = 54$, $k = 0$.

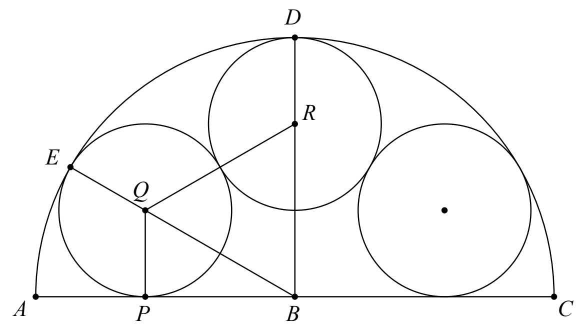
I-7. Because the graph passes through $(0, 0)$, conclude that $c = 0$. Then

$$f(15) = 225 \Rightarrow a(15)^2 + b(15) = 225a + 15b = 225,$$

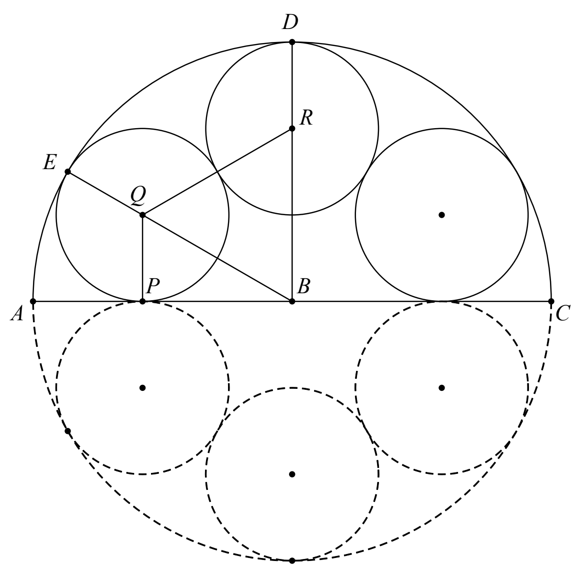
from which $b = 15 - 15a$. On the other hand, f can be factored as $f(x) = ax(x+b/a)$, so if the roots are integers, b/a must be an integer. Divide both sides of the equation $b = 15 - 15a$ by a to obtain $b/a = 15/a - 15$. Thus $15/a$ must be an integer, and $a \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$. Because $b = 15 - 15a$ is linear, each of these values for a yields a unique integer value for b, so there are **8** such ordered pairs. The values of a, b, and the nonnegative root are given in the table below.

a	b	Root
1	0	0
3	−30	10
5	−60	12
15	−210	14
−1	30	30
−3	60	20
−5	90	18
−15	240	16

I-8. The first step is to compute the radius r of one of the marbles. The diagram below shows a cross-section through the centers of two diagonally opposite marbles.



Triangle BQR appears to be equilateral, and in fact, it is. Reflect the diagram in the tabletop $\overline{AC}$ to obtain six mutually tangent congruent circles inside a larger circle:



Because the circles are congruent, their centers are equidistant from B, and the distances between adjacent centers are equal. So Q can be obtained as the image of R under a rotation of $360°/6 = 60°$ counterclockwise around B. Then $PQ = r \Rightarrow BQ = BR = 2r \Rightarrow BD = 3r$, hence $r = 1/3$. Notice too that the height of the pyramid is simply the radius r and the diagonal of the square base is twice the altitude of the equilateral triangle BQR, that is, $2 \cdot \frac{r\sqrt{3}}{2} = r\sqrt{3}$. So the area of the base is $3r^2/2$. Thus the volume of the pyramid is $(1/3)(3r^2/2)(r) = r^3/2$. Because $r = 1/3$, the volume is $\mathbf{1/54}$.

I-9. Because 9 is used as a digit, $b \geq 10$. The conditions require that $b+6$ be prime and $9b+7$ be a perfect square. The numbers modulo 9 whose squares are congruent to 7 modulo 9 are 4 and 5. So $9b+7=(9k+4)^2$ or $(9k+5)^2$ for some integer k. Finally, b must be odd (otherwise $b+6$ is even), so $9b+7$ must be even, which means that for any particular value of k, only one of $9k+4$ and $9k+5$ is possible. Taking these considerations together, $k=0$ is too small. Using $k=1$ makes $9k+4$ odd, and while $(9\cdot 1+5)^2=196=9\cdot 21+7$ is even, because $21+6=27$ is composite, $b \neq 21$. Using $k=2$ makes $9k+4$ even, yielding $22^2=484=9\cdot 53+7$, and $53+6=59$ is prime. Thus $b=\mathbf{53}$, and $53+6=59$ is prime while $9\cdot 53+7=484=22^2$.

I-10. Group values of n according to the number of bits (digits) in their binary representations:

Bits	$C(n)$ values	Total
1	$C(1_2)=0$	0
2	$C(10_2)=0$ $C(11_2)=1$	1
3	$C(100_2)=0$ $C(101_2)=0$ $C(110_2)=1$ $C(111_2)=2$	3
4	$C(1000_2)=0$ $C(1001_2)=0$ $C(1100_2)=1$ $C(1101_2)=1$ $C(1010_2)=0$ $C(1011_2)=1$ $C(1110_2)=2$ $C(1111_2)=3$	8

Let B_n be the set of n-bit integers, and let $c_n=\sum_{k\in B_n} C(k)$ be the sum of the C-values for all n-bit integers. Observe that the integers in B_{n+1} can be obtained by appending a 1 or a 0 to the integers in B_n. Appending a bit does not change the number of consecutive 1's in the previous (left) bits, but each number in B_n generates two different numbers in B_{n+1}. Thus c_{n+1} equals twice $2c_n$ plus the number of new 11 pairs. Appending a 1 will create a new pair of consecutive 1's in (and only in) numbers that previously terminated in 1. The number of such numbers is half the number of elements in B_n. Because there are 2^{n-1} numbers in B_n, there are 2^{n-2} additional pairs of consecutive 1's among the elements in B_{n+1}. Thus for $n\geq 2$, the sequence $\{c_n\}$ satisfies the recurrence relation

$$c_{n+1}=2c_n+2^{n-2}.$$

(Check: the table shows $c_3=3$ and $c_4=8$, and $8=2\cdot 3+2^{3-1}$.) Thus

$$\begin{aligned}
c_5 &= 2\cdot c_4+2^{4-2}=2\cdot 8+4=20,\\
c_6 &= 2\cdot c_5+2^{5-2}=2\cdot 20+8=48,\\
c_7 &= 2\cdot c_6+2^{6-2}=2\cdot 48+16=112, \text{ and}\\
c_8 &= 2\cdot c_7+2^{7-2}=2\cdot 112+32=256.
\end{aligned}$$

Because $C(256)=0$, the desired sum is $c_1+c_2+c_3+c_4+c_5+c_6+c_7+c_8$, which equals **448**.

Power Question 2013: Power of (Urban) Planning

In ARMLopolis, every house number is a positive integer, and City Hall's address is 0. However, due to the curved nature of the cowpaths that eventually became the streets of ARMLopolis, the distance $d(n)$ between house n and City Hall is not simply the value of n. Instead, if $n = 3^k n'$, where $k \geq 0$ is an integer and n' is an integer not divisible by 3, then $d(n) = 3^{-k}$. For example, $d(18) = 1/9$ and $d(17) = 1$. Notice that even though no houses have negative numbers, $d(n)$ is well-defined for negative values of n. For example, $d(-33) = 1/3$ because $-33 = 3^1 \cdot -11$. By definition, $d(0) = 0$. Following the dictum "location, location, location," this Power Question will refer to "houses" and "house numbers" interchangeably.

Curiously, the arrangement of the houses is such that the distance from house n to house m, written $d(m,n)$, is simply $d(m-n)$. For example, $d(3,4) = d(-1) = 1$ because $-1 = 3^0 \cdot -1$. In particular, if $m = n$, then $d(m,n) = 0$.

1a. Compute $d(6)$, $d(16)$, and $d(72)$. [1 pt]

1b. Of the houses with positive numbers less than 100, find, with proof, the house or houses which is (are) closest to City Hall. [2 pts]

1c. Find four houses at a distance of 1/9 from house number 17. [1 pt]

1d. Find an infinite sequence of houses $\{h_n\}$ such that

$$d(17,h_1) > d(17,h_2) > \cdots > d(17,h_n) > d(17,h_{n+1}) > \cdots.$$ [2 pts]

The *neighborhood* of a house n, written $\mathcal{N}(n)$, is the set of all houses that are the same distance from City Hall as n. In symbols, $\mathcal{N}(n) = \{m \mid d(m) = d(n)\}$. Geometrically, it may be helpful to think of $\mathcal{N}(n)$ as a circle centered at City Hall with radius $d(n)$.

2a. Find four houses in $\mathcal{N}(6)$. [1 pt]

2b. Suppose that n is a house with $d(n) = 1$. Find four houses in $\mathcal{N}(n)$. [1 pt]

2c. Suppose that n is a house with $d(n) = 1/27$. Determine the ten smallest positive integers m (in the standard ordering of the integers) such that $m \in \mathcal{N}(n)$. [2 pts]

3a. Notice that $d(16) = d(17) = 1$ and that $d(16,17) = 1$. Is it true that, for all m,n such that $d(m) = d(n) = 1$ and $m \neq n$, $d(m,n) = 1$? Either prove your answer or find a counterexample. [1 pt]

3b. Suppose that $d(m) = d(n) = 1/3^k$ and that $m \neq n$. Determine the possible values of $d(m,n)$. [2 pts]

3c. Suppose that $d(17,m) = 1/81$. Determine the possible values of $d(16,m)$. [2 pts]

A visitor to ARMLopolis is surprised by the ARMLopolitan distance formula, and starts to wonder about ARMLopolitan geometry. An *ARMLopolitan triangle* is a triangle, all of whose vertices are ARMLopolitan houses.

4a. Show that $d(17,51) \leq d(17,34) + d(34,51)$ and that $d(17,95) \leq d(17,68) + d(68,95)$. [1 pt]

4b. If $a = 17$ and $b = 68$, determine all values of c such that $d(a,c) = d(a,b)$. [2 pts]

5a. Prove that, for all a and b, $d(a,b) \leq \max\{d(a), d(b)\}$. [2 pts]

5b. Prove that, for all a, b, and c, $d(a,c) \leq \max\{d(a,b), d(b,c)\}$. [2 pts]

5c. Prove that, for all a, b, and c, $d(a,c) \leq d(a,b) + d(b,c)$. [1 pt]

5d. After thinking about it some more, the visitor announces that all ARMLopolitan triangles have a special property. What is it? Justify your answer. [2 pts]

Unfortunately for new development, ARMLopolis is full: every nonnegative integer corresponds to (exactly one) house (or City Hall, in the case of 0). However, eighteen families arrive and are looking to move in. After much debate, the connotations of using negative house numbers are deemed unacceptable, and the city decides on an alternative plan. On July 17, Shewad Movers arrive and relocate every family from house n to house $n + 18$, for all positive n (so that City Hall does not move). For example, the family in house number 17 moves to house number 35.

6a. Find at least one house whose distance from City Hall changes as a result of the move. [1 pt]

6b. Prove that the distance between houses with consecutive numbers does not change after the move. [2 pts]

7a. The residents of $\mathcal{N}(1)$ value their suburban location and protest the move, claiming it will make them closer to City Hall. Will it? Justify your answer. [2 pts]

7b. Some residents of $\mathcal{N}(9)$ claim that their tightly-knit community will be disrupted by the move: they will be scattered between different neighborhoods after the change. Will they? Justify your answer. [2 pts]

7c. Other residents of $\mathcal{N}(9)$ claim that their community will be disrupted by newcomers: they say that after the move, their new neighborhood will also contain residents previously from several different old neighborhoods (not just the new arrivals to ARMLopolis). Will it? Justify your answer. [2 pts]

7d. Determine all values of n such that $\mathcal{N}(n)$ will either entirely relocate (i.e., all residents r of $\mathcal{N}(n)$ are at a different distance from City Hall than they were before), lose residents (as in 7b), or gain residents besides the new ARMLopolitans (as in 7c). [2 pts]

8a. One day, Paul (house 23) and Sally (house 32), longtime residents of $\mathcal{N}(1)$, are discussing the "2 side" of the neighborhood, that is, the set of houses $\{n \mid n = 3k + 2,\ k \in \mathbb{Z}\}$. Paul says "I feel like I'm at the center of the "2 side": when I looked out my window, I realized that the "2 side" consists of exactly those houses whose distance from me is at most $1/3$." Prove that Paul is correct. [2 pts]

8b. Sally replies "It's not all about you: I have the same experience!" Justify Sally's claim; in other words, prove that $\{n \mid n = 3k + 2,\ k \in \mathbb{Z}\}$ consists exactly of those houses whose distance from Sally's house is at most 1/3. [2 pts]

8c. Paul's and Sally's observations can be generalized. For any x, let $\mathcal{D}_r(x) = \{y \mid d(x, y) \leq r\}$, that is, $\mathcal{D}_r(x)$ is the *disk* of radius r. Prove that if $d(x, z) = r$, then $\mathcal{D}_r(x) = \mathcal{D}_r(z)$. [2 pts]

Paul's and Sally's experiences may seem incredible to a newcomer to ARMLopolis. Given a circle, any point on the circle is the center of the circle (so really *a* center of the circle). But ARMLopolitan geometry is even stranger than that!

9a. Ross takes a walk starting at his house, which is number 34. He first visits house n_1, such that $d(n_1, 34) = 1/3$. He then goes to another house, n_2, such that $d(n_1, n_2) = 1/3$. Continuing in that way, he visits houses $n_3, n_4, \ldots$, and each time, $d(n_i, n_{i+1}) = 1/3$. At the end of the day, what is his maximum possible distance from his original house? Justify your answer. [2 pts]

9b. Generalize your answer to 9a: given a value of n_0, determine the rational values of r for which the sequence $n_1, n_2, \ldots$ with $d(n_i, n_{i+1}) = r$ is entirely contained in a circle of finite radius centered at n_0. [2 pts]

It turns out that the eighteen newcomers are merely the first wave of a veritable deluge of enthusiastic arrivals: infinitely many, in fact. ARMLopolis finally decides on a drastic expansion plan: now house numbers will be *rational* numbers. To define $d(p/q)$, with p and q integers such that $pq \neq 0$, write $p/q = 3^k p'/q'$, where neither p' nor q' is divisible by 3 and k is an integer (not necessarily positive); then $d(p/q) = 3^{-k}$.

10a. Compute $d(3/5), d(5/8)$, and $d(7/18)$. [1 pt]

10b. Determine all pairs of relatively prime integers p and q such that $p/q \in \mathcal{N}(4/3)$. [1 pt]

10c. A longtime resident of IMOpia moves to ARMLopolis and hopes to keep his same address. When asked, he says "My old address was e, that is, the sum $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots$. I'd really like to keep that address, because the addresses of my friends here are all partial sums of this series: Alice's house is $\frac{1}{0!}$, Bob's house is $\frac{1}{0!} + \frac{1}{1!}$, Carol's house is $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}$, and so on. Just let me know what I have to do in order to be near my friends!" After some head-scratching, the ARMLopolitan planning council announces that this request cannot be satisfied: there is no number, rational or otherwise, that corresponds to the (infinite) sum, or that is arbitrarily close to the houses in this sequence. Prove that the council is correct (and not just bureaucratic). [2 pts]

Solutions to 2013 Power Question

1. a. Factoring, $6 = 2 \cdot 3^1$, $16 = 16 \cdot 3^0$, and $72 = 8 \cdot 3^2$, so $d(6) = 1/3$, $d(16) = 1$, and $d(72) = 1/9$.

   b. If $n = 3^k m$ where $3 \nmid m$, then $d(n) = 1/3^k$. So the smallest values of $d(n)$ occur when k is largest. The largest power of 3 less than 100 is $3^4 = 81$, so $d(81) = 1/3^4 = 1/81$ is minimal.

   c. If $d(n, 17) = 1/9$, then $9 \mid (n - 17)$ and $27 \nmid (n - 17)$. So $n = 17 + 9k$ where $k \in \mathbb{Z}$ is not divisible by 3. For $k = -1$, 1, 2, 4, this formula yields $n = 8$, 26, 35, and 53, respectively.

   d. There are many possible answers; the simplest is $h_n = 17 + 3^n$, yielding the sequence $\{20, 26, 44, 98, \ldots\}$.

2. a. Because $d(6) = 1/3$, $\mathcal{N}(6)$ is the set of all houses n such that $d(n) = 1/3$, equivalently, where $n = 3k$ and k is not divisible by 3. So $\mathcal{N}(6) = \{3, 6, 12, 15, 21, 24, \ldots\}$.

   b. Using similar logic to 2a, $\mathcal{N}(n) = \{m \in \mathbb{Z}^+ \mid 3 \nmid m\}$. So $\mathcal{N}(n) = \{1, 2, 4, 5, 7, 8, \ldots\}$.

   c. Here, $\mathcal{N}(n) = \{m \mid m = 27k, \text{ where } 3 \nmid k\}$. The ten smallest elements of $\mathcal{N}(n)$ are 27, 54, 108, 135, 189, 216, 270, 297, 351, and 378.

3. a. The statement is false. $d(n, m) < 1$ if $3 \mid (n - m)$, so for example, if $n = 2$ and $m = 5$.

   b. If $d(m) = d(n) = 1/3^k$, rewrite $m = 3^k m'$ and $n = 3^k n'$ where m' and n' are not divisible by 3. Then $m - n = 3^k(m' - n')$. If $3 \mid (m' - n')$, then $d(m, n) < 1/3^k$. If $3 \nmid (m' - n')$, then $d(m, n) = 1/3^k$. So the set of possible values of $d(m, n)$ is $\{3^{-q} \mid q \geq k, q \in \mathbb{Z}^+\}$.

   c. Because $d(17, m) = 1/81$, $17 - m = 81l$, where $l \in \mathbb{Z}$ and $3 \nmid l$. So $m = 17 - 81l$ and $m - 16 = 1 - 81l$. Hence $3 \nmid m - 16$, and $d(m, 16) = d(m - 16) = 1$.

4. a. Because $d(17, 34) = d(17) = 1$, $d(34, 51) = d(17) = 1$, and $d(17, 51) = d(34) = 1$, it follows that $d(17, 51) \leq d(17, 34) + d(34, 51)$. Because $d(17, 68) = d(51) = 1/3$, $d(68, 95) = d(27) = 1/27$, and $d(17, 95) = d(78) = 1/3$, it follows that $d(17, 95) \leq d(17, 68) + d(68, 95)$.

   b. Because $d(17, 68) = 1/3$, the condition implies that $3 \mid (c - 17)$ but $9 \nmid (c - 17)$. Thus either $c - 17 = 9k + 3$ or $c - 17 = 9k + 6$ for some $k \in \mathbb{Z}$. Solving for c yields $c = 9k + 20$ or $c = 9k + 23$, equivalently $c = 9k + 2$ or $c = 9k + 5$, $k \in \mathbb{Z}$, $k \geq 0$.

5. a. Write $a = 3^\alpha a_0$ and $b = 3^\beta b_0$, where $3 \nmid a_0$ and $3 \nmid b_0$. First consider the case $\alpha = \beta$. Then $a - b = 3^\alpha(a_0 - b_0)$. In this case, if $3 \mid (a_0 - b_0)$, then $3^\alpha(a_0 - b_0) = 3^\gamma c$, where $3 \nmid c$ and $\alpha < \gamma$; so $d(a - b) = 3^{-\gamma} < 3^{-\alpha} = d(a) = d(b)$. If $3 \nmid (a_0 - b_0)$, then $d(a, b) = 3^{-\alpha} = d(a) = d(b)$. If $\alpha \neq \beta$, suppose, without loss of generality, that $\alpha < \beta$ (so that $d(a) > d(b)$). Then $a - b = 3^\alpha(a_0 - 3^{\beta - \alpha} b_0)$. In this second factor, notice that the second term, $3^{\beta - \alpha} b_0$ is divisible by 3 but the first term a_0 is not, so their difference is not divisible by 3. Thus $d(a, b) = 3^{-\alpha} = d(a)$. Therefore $d(a, b) = d(a)$ when $d(a) > d(b)$, and similarly $d(a, b) = d(b)$ when $d(b) > d(a)$. Hence $d(a, b) \leq \max\{d(a), d(b)\}$.

b. Note that $d(a,c) = d(a-c) = d((a-b)-(c-b)) = d(a-b, c-b)$. Then $d(a,c) \le \max\{d(a,b), d(c,b)\}$ by part 5a, and the fact that $d(c,b) = d(b,c)$.

c. Because $d(x,y) > 0$ for all $x \ne y$, $\max\{d(a,b), d(b,c)\} < d(a,b) + d(b,c)$ whenever $a \ne b \ne c$. Thus $d(a,c) < d(a,b) + d(b,c)$ when a, b, c are all different. If $a = b \ne c$, then $d(a,b) = 0$, $d(a,c) = d(b,c)$, so $d(a,c) \le d(a,b) + d(b,c)$. And if $a = c \ne b$, then $d(a,c) = 0$ while $d(a,b) = d(b,c) > 0$, so $d(a,c) < d(a,b) + d(b,c)$.

d. The foregoing shows that all ARMLopolitan triangles are isosceles! Examining the proof in 5a, note that if $d(a) \ne d(b)$, then $d(a,b) = \max\{d(a), d(b)\}$. Applying that observation to the proof in 5b, if $d(a,b) \ne d(b,c)$, then $d(a,c) = \max\{d(a,b), d(b,c)\}$. So either $d(a,b) = d(b,c)$ or, if not, then either $d(a,c) = d(a,b)$ or $d(a,c) = d(b,c)$. Thus all ARMLopolitan triangles are isosceles.

6. a. For example, $d(9) = 1/9$ and $d(9+18) = 1/27$. In general, if $n = 9+27k$, then $d(n) = 1/9$, but $d(n+18) = d(27+27k) \le 1/27$.

b. Note that $d(m+18, n+18) = d((m+18)-(n+18)) = d(m-n) = d(m,n)$. In particular, if $m = n+1$, then $d(m,n) = 1$, thus $d(m+18, n+18)$ is also 1.

7. a. No, they won't move. If $a \in \mathcal{N}(1)$, then $3 \nmid a$. Thus $3 \nmid (a+18)$, and so $d(a+18) = 1$.

b. Yes, they will be split apart. If $a \in \mathcal{N}(9)$, then $9 \mid a$ but $27 \nmid a$, so $a = 27k+9$ or $a = 27k+18$. Adding 18 has different effects on these different sets of houses: $27k+18 \mapsto 27k+36$, which is still in $\mathcal{N}(9)$, while $27k+9 \mapsto 27k+27$, which is in $\mathcal{N}(27r)$ for some $r \in \mathbb{Z}^+$, depending on the exact value of a. For example, $a = 9 \mapsto 27$, while $a = 63 \mapsto 81$, and $d(81) = 3^{-4}$. That is, $d(a+18) \le 3^{-3}$ whenever $a = 27k+9$, with equality unless $a \equiv 63 \bmod 81$.

c. The question is whether it is possible that $x \notin \mathcal{N}(9)$ but $x+18 \in \mathcal{N}(9)$. Because $d(9) = 1/9$, there are two cases to consider: $d(x) > 1/9$ and $d(x) < 1/9$. In the first case, $9 \nmid x$ (either x is not divisible by 3, or x is divisible by 3 and not 9). These houses do not move into $\mathcal{N}(9)$: $x \not\equiv 0 \bmod 9 \Rightarrow x+18 \not\equiv 0 \bmod 9$. On the other hand, if $d(x) < 1/9$, then $27 \mid x$, i.e., $x = 27k$. Then $x+18 = 27k+18$ which is divisible by 9 but not by 27, so $d(x+18) = 9$. So in fact **every** house x of ARMLopolis with $d(x) < 1/9$ will move into $\mathcal{N}(9)$!

d. As noted in 7c, all houses in $\mathcal{N}(3^k)$, where $k \ge 3$, will move into $\mathcal{N}(9)$. On the other hand, houses of the form $27k+9$ will find that their distance from City Hall *decreases*, so each of those now-vacated neighborhoods will be filled by residents of the old $\mathcal{N}(9)$. None of $\mathcal{N}(1)$, $\mathcal{N}(2)$, or $\mathcal{N}(3)$ will be affected, however.

8. a. If $n = 3k+2$, then $d(23, n) = d(3k-21)$, and because $3k-21 = 3(k-7)$, $d(3k-21) \le 1/3$. On the other hand, if $d(23,n) \le 1/3$, then $3 \mid (n-23)$, so $n-23 = 3k$, and $n = 3k+23 = 3(k+7)+2$.

b. If $n = 3k+2$, then $d(32,n) = d(3k-30) \le 1/3$. Similarly, if $d(32,n) \le 1/3$, then $3 \mid n-32 \Rightarrow n = 3k+32 = 3(k+10)+2$.

c. Show that $\mathcal{D}_r(x) \subseteq \mathcal{D}_r(z)$ and conversely. Suppose that $y \in \mathcal{D}_r(x)$. Then $d(x,y) \leq r$. Because $d(x,z) = r$, by 5b, $d(y,z) \leq \max\{d(x,y), d(x,z)\} = r$. So $y \in \mathcal{D}_r(z)$. Thus $\mathcal{D}_r(x) \subseteq \mathcal{D}_r(z)$. Similarly, $\mathcal{D}_r(z) \subseteq \mathcal{D}_r(x)$, so the two sets are equal.

9. a. The maximum possible distance $d(34, n_k)$ is $1/3$. This can be proved by induction on k: $d(n_1, 34) \leq 1/3$, and if both $d(n_{k-1}, 34) \leq 1/3$ and $d(n_{k-1}, n_k) \leq 1/3$, then $\max\{d(n_{k-1}, 34), d(n_{k-1}, n_k)\} \leq 1/3$ so by 5b, $d(34, n_k) \leq 1/3$.

b. The sequence is contained in the disk for all rational values of r, by the same logic as in 9a.

10. a. $d(3/5) = 1/3$, $d(5/8) = 1$, and $d(7/18) = 9$.

b. Because $d(4/3) = 1/3$, $\mathcal{N}(4/3) = \{p/q$ in lowest terms such that $3 \mid q$ and $9 \nmid q\}$. Thus the set of all possible (p,q) consists precisely of those ordered pairs $(p'r, q'r)$, where r is any non-zero integer, p' and q' are relatively prime integers, $3 \mid q'$, and $9 \nmid q'$.

c. The houses in this sequence actually get arbitrarily far apart from each other, and so there's no single house which they approach. If $H_n = \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}$, then $d(H_n, H_{n-1}) = d(1/n!) = 3^k$, where $k = \lfloor n/3 \rfloor + \lfloor n/9 \rfloor + \lfloor n/27 \rfloor \ldots$. In particular, if $n! = 3^k \cdot n_0$, where $k > 0$ and $3 \nmid n_0$, then $d(H_n, H_{n-1}) = 3^k$, which can be made arbitrarily large as n increases.

2013 Relay Problems

R1-1. A set S contains thirteen distinct positive integers whose sum is 120. Compute the largest possible value for the median of S.

R1-2. Let $T = TNYWR$. Compute the least positive integer b such that, when expressed in base b, the number $T!$ ends in exactly two zeroes.

R1-3. Let $T = TNYWR$. Suppose that $a_1 = 1$, and that for all positive integers n, $a_{n+1} = \left\lceil \sqrt{a_n^2 + 34} \right\rceil$. Compute the least value of n such that $a_n > 100T$.

R2-1. Compute the smallest n such that in the regular n-gon $A_1A_2A_3 \cdots A_n$, $\mathrm{m}\angle A_1A_{20}A_{13} < 60^\circ$.

R2-2. Let $T = TNYWR$. A cube has edges of length T. Square holes of side length 1 are drilled from the center of each face of the cube through the cube's center and across to the opposite face; the edges of each hole are parallel to the edges of the cube. Compute the surface area of the resulting solid.

R2-3. Let $T = TNYWR$. Compute $\left\lfloor \log_4(1 + 2 + 4 + \cdots + 2^T) \right\rfloor$.

2013 Relay Answers

R1-1. 11

R1-2. 5

R1-3. 491

R2-1. 37

R2-2. 8640

R2-3. 4320

2013 Relay Solutions

R1-1. Let S_L be the set of the least six integers in S, let m be the median of S, and let S_G be the set of the greatest six integers in S. In order to maximize the median, the elements of S_L should be as small as possible, so start with $S_L = \{1, 2, 3, 4, 5, 6\}$. Then the sum of S_L's elements is 21, leaving 99 as the sum of m and the six elements of S_G. If $m = 11$ and $S_G = \{12, 13, 14, 15, 16, 17\}$, then the sum of all thirteen elements of S is 119. It is impossible to increase m any further, because then the smallest set of numbers for S_G would be $\{13, 14, 15, 16, 17, 18\}$, and the sum would be at least 126. To get the sum to be exactly 120, simply increase either 6 to 7 or 17 to 18. The answer is **11**.

R1-2. For any integers n and b, define $d(n, b)$ to be the unique nonnegative integer k such that $b^k \mid n$ and $b^{k+1} \nmid n$; for example, $d(9, 3) = 2$, $d(9, 4) = 0$, and $d(18, 6) = 1$. So the problem asks for the smallest value of b such that $d(T!, b) = 2$. If p is a prime and $p \mid b$, then $d(T!, b) \leq d(T!, p)$, so the least value of b such that $d(T!, b) = 2$ must be prime. Also, if b is prime, then $d(T!, b) = \lfloor T/b \rfloor + \lfloor T/b^2 \rfloor + \lfloor T/b^3 \rfloor + \cdots$. The only way that $d(T, b)$ can equal 2 is if the first term $\lfloor T/b \rfloor$ equals 2 and all other terms equal zero. (If $T \geq b^2$, then $b \geq 2$ implies $T/b \geq b \geq 2$, which would mean the first two terms by themselves would have a sum of at least 3.) Thus $2b \leq T < 3b$, hence $b \leq T/2$ and $T/3 < b$. For $T = 11$, the only such b is **5**.

R1-3. Start by computing the first few terms of the sequence: $a_1 = 1, a_2 = \left\lceil \sqrt{35} \right\rceil = 6, a_3 = \left\lceil \sqrt{70} \right\rceil = 9$, and $a_4 = \left\lceil \sqrt{115} \right\rceil = 11$. Note that when $m \geq 17$, $(m+1)^2 = m^2 + 2m + 1 > m^2 + 34$, so if $a_n \geq 17$, $a_{n+1} = \left\lceil \sqrt{a_n^2 + 34} \right\rceil = a_n + 1$. So it remains to continue the sequence until $a_n \geq 17$: $a_5 = 13$, $a_6 = 15$, $a_7 = 17$. Then for $n > 7$, $a_n = 17 + (n - 7) = n + 10$, and $a_n > 100T \Rightarrow n > 100T - 10$. With $T = 5$, $n > 490$, and the least value of n is **491**.

R2-1. If the polygon is inscribed in a circle, then the arc $\widehat{A_1A_{13}}$ intercepted by $\angle A_1A_{20}A_{13}$ has measure $12(360°/n)$, and thus $\mathrm{m}\angle A_1A_{20}A_{13} = 6(360°/n)$. If $6(360/n) < 60$, then $n > 6(360)/60 = 36$. Thus the smallest value of n is **37**.

R2-2. After the holes have been drilled, each face of the cube has area $T^2 - 1$. The three holes meet in a $1 \times 1 \times 1$ cube in the center, forming six holes in the shape of rectangular prisms whose bases are 1×1 squares and whose heights are $(T-1)/2$. Each of these holes thus contributes $4(T-1)/2 = 2(T-1)$ to the surface area, for a total of $12(T-1)$. Thus the total area is $6(T^2 - 1) + 12(T - 1)$, which can be factored as $6(T-1)(T+1+2) = 6(T-1)(T+3)$. With $T = 37$, the total surface area is $6(36)(40) = \mathbf{8640}$.

R2-3. Let $S = \log_4(1 + 2 + 4 + \cdots + 2^T)$. Because $1 + 2 + 4 + \cdots + 2^T = 2^{T+1} - 1$, the change-of-base formula yields

$$S = \frac{\log_2(2^{T+1} - 1)}{\log_2 4}.$$

Let $k = \log_2(2^{T+1} - 1)$. Then $T < k < T + 1$, so $T/2 < S < (T + 1)/2$. If T is even, then $\lfloor S \rfloor = T/2$; if T is odd, then $\lfloor S \rfloor = (T - 1)/2$. With $T = 8640$, the answer is **4320**.

2013 Tiebreaker Problems

TB-1. Let $ARML$ be a trapezoid with bases $\overline{AR}$ and $\overline{ML}$, such that $MR = RA = AL$ and $LR = AM = ML$. Point P lies inside the trapezoid such that $\angle RMP = 12^\circ$ and $\angle RAP = 6^\circ$. Diagonals AM and RL intersect at D. Compute the measure, in degrees, of angle APD.

TB-2. A regular hexagon has side length 1. Compute the average of the areas of the 20 triangles whose vertices are vertices of the hexagon.

TB-3. Paul was planning to buy 20 items from the ARML shop. He wanted some mugs, which cost \$10 each, and some shirts, which cost \$6 each. After checking his wallet he decided to put 40% of the mugs back. Compute the number of dollars he spent on the remaining items.

2013 Tiebreaker Answers

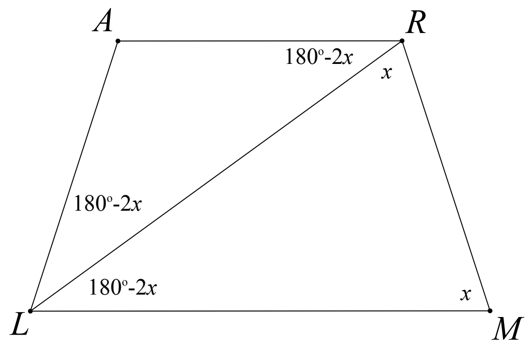
TB-1. 48

TB-2. $\dfrac{9\sqrt{3}}{20}$

TB-3. 120

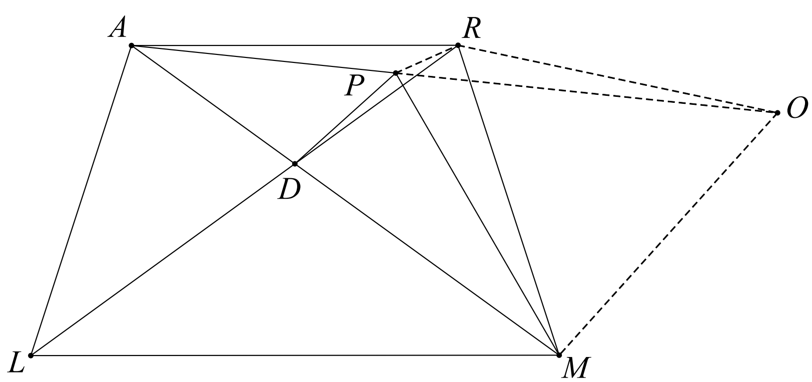
2013 Tiebreaker Solutions

TB-1. First, determine the angles of $ARML$. Let $\mathrm{m}\angle M = x$. Then $\mathrm{m}\angle LRM = x$ because $\triangle LRM$ is isosceles, and $\mathrm{m}\angle RLM = 180^\circ - 2x$. Because $\overline{AR} \parallel \overline{LM}$, $\mathrm{m}\angle ARM = 180^\circ - x$ and $\mathrm{m}\angle ARL = 180^\circ - 2x$, as shown in the diagram below.



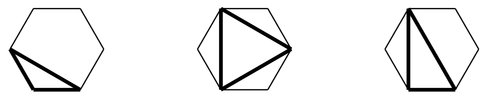
However, $\triangle ARL$ is also isosceles (because $AR = AL$), so $\mathrm{m}\angle ALR = 180^\circ - 2x$, yielding $\mathrm{m}\angle ALM = 360^\circ - 4x$. Because $\mathrm{m}\angle RML = \mathrm{m}\angle ALM$, conclude that $360^\circ - 4x = x$, so $x = 72^\circ$. Therefore the base angles L and M have measure 72° while the other base angles A and R have measure 108°. Finally, the angle formed by diagonals $\overline{AM}$ and $\overline{LR}$ is as follows: $\mathrm{m}\angle RDM = 180^\circ - \mathrm{m}\angle LRM - \mathrm{m}\angle AMR = 180^\circ - 72^\circ - 36^\circ = 72^\circ$.

Now construct equilateral $\triangle ROM$ with O on the exterior of the trapezoid, as shown below.



Because $AR = RM = RO$, triangle OAR is isosceles with base $\overline{AO}$. The measure of $\angle ARO$ is $108^\circ + 60^\circ = 168^\circ$, so $\mathrm{m}\angle RAO = (180 - 168)^\circ/2 = 6^\circ$. Thus P lies on $\overline{AO}$. Additionally, $\mathrm{m}\angle POM = \mathrm{m}\angle AOM = 60^\circ - 6^\circ = 54^\circ$, and $\mathrm{m}\angle PMO = 60^\circ + 12^\circ = 72^\circ$ by construction. Thus $\mathrm{m}\angle MPO = 180^\circ - 72^\circ - 54^\circ = 54^\circ$, hence $\triangle PMO$ is isosceles with $PM = OM$. But because $OM = RM$, $\triangle RMP$ is isosceles with $RM = MP$, and $RM = DM$ implies that $\triangle PDM$ is also isosceles. But $\mathrm{m}\angle RMP = 12^\circ$ implies that $\mathrm{m}\angle PMD = 36^\circ - 12^\circ = 24^\circ$, so $\mathrm{m}\angle DPM = 78^\circ$. Thus $\mathrm{m}\angle APD = 180^\circ - \mathrm{m}\angle OPM - \mathrm{m}\angle DPM = 180^\circ - 54^\circ - 78^\circ = \mathbf{48}^\circ$.

TB-2. There are 6 triangles of side lengths 1, 1, $\sqrt{3}$; 2 equilateral triangles of side length $\sqrt{3}$; and 12 triangles of side lengths 1, $\sqrt{3}$, 2. One triangle of each type is shown in the diagram below.



Each triangle in the first set has area $\sqrt{3}/4$; each triangle in the second set has area $3\sqrt{3}/4$; and each triangle in the third set has area $\sqrt{3}/2$. The average is

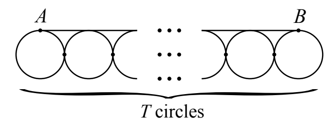
$$\frac{6\left(\frac{\sqrt{3}}{4}\right)+2\left(\frac{3\sqrt{3}}{4}\right)+12\left(\frac{\sqrt{3}}{2}\right)}{20}=\frac{\frac{6\sqrt{3}}{4}+\frac{6\sqrt{3}}{4}+\frac{24\sqrt{3}}{4}}{20}=\boldsymbol{\frac{9\sqrt{3}}{20}}.$$

TB-3. The problem does not state the number of mugs Paul intended to buy, but the actual number is irrelevant. Suppose Paul plans to buy M mugs and $20-M$ shirts. The total cost is $10M+6(20-M)$ However, he puts back 40% of the mugs, so he ends up spending $10(0.6M)+6(20-M)=6M+120-6M=\mathbf{120}$ dollars.

2013 Super Relay Problems

1. Compute the smallest positive integer n such that $n^2 + n^0 + n^1 + n^3$ is a multiple of 13.

2. Let $T = TNYWR$. The diagram at right consists of T congruent circles, each of radius 1, whose centers are collinear, and each pair of adjacent circles are externally tangent to each other. Compute the length of the tangent segment $\overline{AB}$.



3. Let $T = TNYWR$. Compute $2^{\log_T 8} - 8^{\log_T 2}$.

4. Let $T = TNYWR$. At some point during a given week, a law enforcement officer had issued $T + 2$ traffic warnings, 20 tickets, and had made $T + 5$ arrests. How many more tickets must the officer issue in order for the combined number of tickets and arrests to be 20 times the number of warnings issued that week?

5. Let $T = TNYWR$. In parallelogram $ARML$, points P and Q trisect $\overline{AR}$ and points W, X, Y, Z divide $\overline{ML}$ into fifths (where W is closest to M, and points X and Y are both between W and Z). If $[ARML] = T$, compute $[PQWZ]$.

6. Let $T = TNYWR$. Compute the number of positive perfect cubes that are divisors of $(T + 10)!$.

7. Let $T = TNYWR$. The graph of $y = x^2 + 2x - T$ intersects the x-axis at points A and M, which are diagonally opposite vertices of square $ARML$. Compute $[ARML]$.

8. Let S be the set of prime factors of the numbers you receive from positions 7 and 9, and let p and q be the two least distinct elements of S, with $p < q$. Hexagon $HEXAGO$ is inscribed in circle ω, and every angle of $HEXAGO$ is $120°$. If $HE = XA = GO = p$ and $EX = AG = OH = q$, compute the area of circle ω.

9. Let $T = TNYWR$. A group of n friends goes camping; two of them are selected to set up the campsite when they arrive and two others are selected to take down the campsite the next day. Compute the smallest possible value of n such that there are at least T ways of selecting the four helpers.

10. Let $T = TNYWR$. The parabola $y = x^2 + Tx$ is tangent to the parabola $y = -(x - 2T)^2 + b$. Compute b.

11. Let $T = TNYWR$. The first two terms of a sequence are $a_1 = 3/5$ and $a_2 = 4/5$. For $n > 2$, if n is odd, then $a_n = a_{n-1}^2 - a_{n-2}^2$, while if n is even, then $a_n = 2a_{n-2}a_{n-3}$. Compute the sum of the squares of the first $T - 3$ terms of the sequence.

12. Let $T = TNYWR$. A regular n-gon has exactly T more diagonals than a regular $(n-1)$-gon. Compute the value of n.

13. Let $T = TNYWR$. The sequence $a_1, a_2, a_3, \ldots,$ is arithmetic with $a_{16} = 13$ and $a_{30} = 20$. Compute the value of k for which $a_k = T$.

14. Let $T = TNYWR$. A rectangular prism has a length of 1, a width of 3, a height of h, and has a total surface area of T. Compute the value of h.

15. The zeros of $x^2 + bx + 93$ are r and s. If the zeros of $x^2 - 22x + c$ are $r + 1$ and $s + 1$, compute c.

2013 Super Relay Answers

1. 5
2. 8
3. 0
4. 15
5. 7
6. 36
7. 74
8. $\dfrac{67\pi}{3}$
9. 7
10. 184
11. 8
12. 19
13. 17
14. $\dfrac{27}{2}$
15. 114

2013 Super Relay Solutions

1. Note that $n^2 + n^0 + n^1 + n^3 = n^2 + 1 + n + n^3 = (n^2 + 1)(1 + n)$. Because 13 is prime, 13 must be a divisor of one of these factors. The smallest positive integer n such that $13 \mid 1 + n$ is $n = 12$, whereas the smallest positive integer n such that $13 \mid n^2 + 1$ is $n = \mathbf{5}$.

2. For each point of tangency of consecutive circles, drop a perpendicular from that point to $\overline{AB}$. For each of the $T-2$ circles between the first and last circles, the distance between consecutive perpendiculars is $2 \cdot 1 = 2$. Furthermore, the distance from A to the first perpendicular equals 1 (i.e., the common radius of the circles), which also equals the distance from the last perpendicular to B. Thus $AB = 1 + (T - 2) \cdot 2 + 1 = 2(T - 1)$. With $T = 5$, it follows that $AB = 2 \cdot 4 = \mathbf{8}$.

3. Let $\log_T 8 = x$. Then $T^x = 8$. Thus the given expression equals $2^x - (T^x)^{\log_T 2} = 2^x - T^{x \log_T 2} = 2^x - T^{\log_T 2^x} = 2^x - 2^x = \mathbf{0}$ (independent of T).

4. The problem requests the value of k such that $20 + k + T + 5 = 20(T + 2)$, thus $k = 19T + 15$. With $T = 0$, it follows that $k = \mathbf{15}$.

5. Let h be the distance between $\overline{AR}$ and $\overline{ML}$, and for simplicity, let $AR = ML = 15n$. Then $[ARML] = 15nh$, and $[PQWZ] = (1/2)(PQ + WZ)h$. Note that $PQ = 15n/3 = 5n$ and $WZ = 15n - 3n - 3n = 9n$. Thus $[PQWZ] = 7nh = (7/15) \cdot [ARML] = 7T/15$. With $T = 15$, the answer is $\mathbf{7}$.

6. Let $N = T + 10$. In order for k^3 ($k \in \mathbb{N}$) to be a divisor of $N!$, the largest *odd* prime factor of k (call it p) must be less than or equal to $N/3$ so that there are at least three multiples of p among the product of the first N positive integers. If $p = 3$, then the smallest possible value of N is 9, and the largest perfect cube factor of $9!$ is $2^6 \cdot 3^3$. Similarly, if $p = 5$, then the smallest possible value of N is 15, and the largest perfect cube factor of $15!$ is $2^9 \cdot 3^6 \cdot 5^3$. With $T = 7, N = 17$, and the largest perfect cube factor of $17!$ is $2^{15} \cdot 3^6 \cdot 5^3$. Thus $k^3 \mid 17!$ if and only if $k \mid 2^5 \cdot 3^2 \cdot 5^1$. Therefore $k = 2^x 3^y 5^z$, where x, y, z are nonnegative integers with $x \le 5, y \le 2, z \le 1$, yielding $6 \cdot 3 \cdot 2 = \mathbf{36}$ possible values of k.

7. Note that the x-coordinates of A and M correspond to the two roots r_1, r_2 of $x^2 + 2x - T$. If s is the side length of square $ARML$, then $AM = s\sqrt{2} = |r_1 - r_2| = \sqrt{(r_1 - r_2)^2} = \sqrt{(r_1 + r_2)^2 - 4r_1r_2} = \sqrt{(-2)^2 - 4(-T)} = 2\sqrt{1 + T}$. Thus $[ARML] = s^2 = 2(1 + T)$. With $T = 36, [ARML] = \mathbf{74}$.

8. The given information implies that triangles HEX, XAG, and GOH are congruent, hence triangle HXG is equilateral. If $HX = s$, then the radius of the circle circumscribing $\triangle HXG$ is $s/\sqrt{3}$ so that the circle's area is $\pi s^2/3$. It remains to compute s. With $m\angle HEX = 120°$, use the Law of Cosines to find

$$\begin{aligned} HX^2 &= HE^2 + EX^2 - 2HE \cdot EX \cdot \cos 120° \\ &= p^2 + q^2 - 2pq(-1/2) \\ &= p^2 + q^2 + pq. \end{aligned}$$

Using the answers 74 and 7 from positions 7 and 9, respectively, conclude that $S = \{2, 7, 37\}$ and that $(p, q) = (2, 7)$. Hence the foregoing yields $HX^2 = 4 + 49 + 14 = 67$. Thus the area of circle ω is $\boldsymbol{\frac{67\pi}{3}}$.

9. There are $\binom{n}{2}$ ways of choosing the two people to set up and $\binom{n-2}{2}$ ways of choosing the two people to take down the campsite, so there are $\frac{n(n-1)}{2} \cdot \frac{(n-2)(n-3)}{2}$ ways of choosing the four people, or $\frac{n(n-1)(n-2)(n-3)}{4}$ ways total; call this function $C(n)$. For the least n such that $\frac{n(n-1)(n-2)(n-3)}{4} \geq T$, as a rough approximation, note that $n - 3 < \sqrt[4]{4T} < n$. With $T = 184$, the approximation becomes $n - 3 < \sqrt[4]{736} < n$. Now $5^4 = 625$ while $6^4 = 1296$, so $5 < n < 9$. Try values starting from $n = 6$:

$$\begin{aligned} C(6) &= \frac{6 \cdot 5 \cdot 4 \cdot 3}{4} = 90, \\ C(7) &= \frac{7 \cdot 6 \cdot 5 \cdot 4}{4} = 210. \end{aligned}$$

Thus $n = \mathbf{7}$.

10. In this case, the two parabolas are tangent exactly when the system of equations has a unique solution. (Query: Is this the case for every pair of equations representing parabolas?) So set the right sides equal to each other: $x^2 + Tx = -(x - 2T)^2 + b$. Then $x^2 + Tx = -x^2 + 4Tx - 4T^2 + b$, or equivalently, $2x^2 - 3Tx + 4T^2 - b = 0$. The equation has a double root when the discriminant is 0, so set $(-3T)^2 - 4(4T^2 - b)(2) = 0$ and solve: $9T^2 - 32T^2 + 8b = 0$ implies $-23T^2 + 8b = 0$, or $b = 23T^2/8$. Using $T = 8$ yields $b = \mathbf{184}$.

11. Using the identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$, notice that $a_{2n+1}^2 + a_{2n+2}^2 = (a_{2n}^2 - a_{2n-1}^2)^2 + (2a_{2n}a_{2n-1})^2 = (a_{2n}^2 + a_{2n-1}^2)^2$. So surprisingly, for all $n \in \mathbb{N}$, $a_{2n+1}^2 + a_{2n+2}^2 = 1$. Thus if n is even, the sum of the squares of the first n terms is $n/2$. With $T = 19$, $T - 3 = 16$, and the sum is $\mathbf{8}$.

12. Using the formula $D(n) = \frac{n(n-3)}{2}$ twice yields $D(n) - D(n-1) = \frac{n^2-3n}{2} - \frac{n^2-5n+4}{2} = \frac{2n-4}{2} = n-2$. So $T = n - 2$, thus $n = T + 2$, and with $T = 17$, $n = \mathbf{19}$.

13. If d is the common difference of the sequence, then the n^{th} term of the sequence is $a_n = a_{16} + d(n-16)$. The values $a_{16} = 13$ and $a_{30} = 20$ yield $d = (20-13)/(30-16) = 1/2$, hence $a_n = 13 + (1/2)(n-16)$. If $a_n = T$, then $n = 2(T-13) + 16 = 2T - 10$. With $T = 27/2$, it follows that $n = \mathbf{17}$.

14. The surface area is given by the expression $2 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot h + 2 \cdot 3 \cdot h = 6 + 8h$. Because $6 + 8h = T$, $h = \frac{T-6}{8}$. With $T = 114$, $h = 108/8 = \mathbf{27/2}$.

15. Use sums and products of roots formulas: the desired quantity $c = (r+1)(s+1) = rs+r+s+1$. From the first equation, $rs = 93$, while from the second equation, $(r+1)+(s+1) = r+s+2 = 22$. So $rs + r + s + 1 = 93 + 22 - 1 = \mathbf{114}$.

2014 Contest

Contents

2014 Team Problems

T-1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

T-2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

T-3. The square $ARML$ is contained in the xy-plane with $A = (0, 0)$ and $M = (1, 1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

T-4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

T-5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\begin{cases} \log_x y^2 + \log_y x^5 &= 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 &= k - 3 \end{cases}$$

T-6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area $1/2$ and vertices at lattice points on the boundary of $WASH$.

T-7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

T-8. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

T-9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

T-10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2 z - x = 14$.

2014 Team Answers

T-1. 4

T-2. $\left(\frac{1}{4}, \frac{1}{27}\right)$

T-3. $\frac{2}{7}\sqrt{53}$

T-4. 326

T-5. $\frac{43}{48}$

T-6. 3432

T-7. $\frac{45}{2}$

T-8. $2 + \pi$

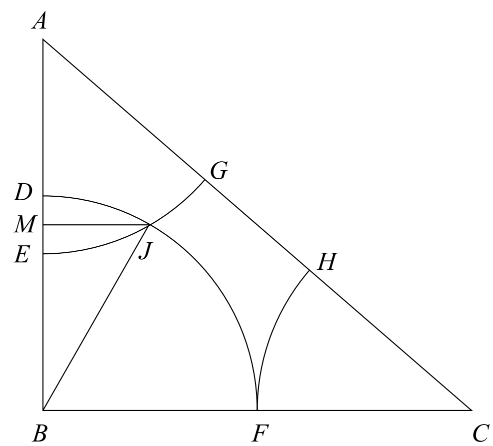
T-9. 10

T-10. $(-266, -3, -28)$

2014 Team Solutions

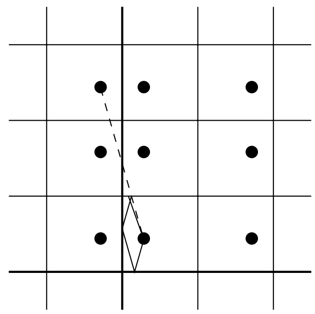
T-1. Consider the ordered pairs of digits (X, Y) for which $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ **is** a multiple of 11. Recall that a number is a multiple of 11 if and only if the alternating sum of the digits is a multiple of 11. Because $1 + 3 + 5 + 7 = 16$, the sum of the remaining digits, namely $2 + X + Y$, must equal 5 or 16. Thus $X + Y$ must be either 3 or 14, making $X = 3 - Y$ (if $Y = 0$, 1, 2, or 3) or $14 - Y$ (if $Y = 5$, 6, 7, 8, or 9). Thus a solution (X, Y) exists unless $Y = \mathbf{4}$.

T-2. Label the triangle as $\triangle ABC$, with $AB = 2\sqrt{3}$ and $BC = 4$. Let D and E lie on $\overline{AB}$ such that $DB = AE = 2$. Let F be the midpoint of $\overline{BC}$, so that $BF = FC = 2$. Let G and H lie on $\overline{AC}$, with $AG = HC = 2$. Now draw the arcs of radius 2 between E and G, D and F, and F and H. Let the intersection of arc DF and arc EG be J. Finally, let M be the midpoint of $\overline{AB}$. The completed diagram is shown below.



The region R consisting of all points within $\triangle ABC$ that lie within 2 units of any vertex is the union of the three sectors EAG, DBF, and FCH. The angles of these sectors, being the angles $\angle A, \angle B$, and $\angle C$, sum to $180°$, so the sum of their areas is 2π. Computing the area of R requires subtracting the areas of all intersections of the three sectors that make up R.
The only sectors that intersect are EAG and DBF. Half this area of intersection, the part above $\overline{MJ}$, equals the difference between the areas of sector DBJ and of $\triangle MBJ$. Triangle MBJ is a $1 : \sqrt{3} : 2$ right triangle because $BM = \sqrt{3}$ and $BJ = 2$, so the area of $\triangle MBJ$ is $\frac{\sqrt{3}}{2}$. Sector DBJ has area $\frac{1}{12}(4\pi) = \frac{\pi}{3}$, because $\mathrm{m}\angle DBJ = 30°$. Therefore the area of intersection of the sectors is $2(\frac{\pi}{3} - \frac{\sqrt{3}}{2}) = \frac{2\pi}{3} - \sqrt{3}$. Hence the total area of R is $2\pi - (\frac{2\pi}{3} - \sqrt{3}) = \frac{4\pi}{3} + \sqrt{3}$. The total area of $\triangle ABC$ is $4\sqrt{3}$, therefore the desired probability is $\frac{\frac{4\pi}{3}+\sqrt{3}}{4\sqrt{3}} = \frac{\pi}{3\sqrt{3}} + \frac{1}{4}$. Then $a = \frac{1}{4}$ and $b = \left(\frac{1}{3\sqrt{3}}\right)^2 = \frac{1}{27}$, hence the answer is $\left(\mathbf{\frac{1}{4}, \frac{1}{27}}\right)$.

T-3. Consider repeatedly reflecting square $ARML$ over its sides so that the entire plane is covered by copies of $ARML$. A path starting at $(2/7, 3/7)$ that touches one or more sides and returns to $(2/7, 3/7)$ corresponds to a straight line starting at $(2/7, 3/7)$ and ending at the image of $(2/7, 3/7)$ in one of the copies of $ARML$. To touch three sides, the path must cross three lines, at least one of which must be vertical and at least one of which must be horizontal.



If the path crosses two horizontal lines and the line $x = 0$, it will have traveled a distance of 2 units vertically and 4/7 units vertically for a total distance of $\sqrt{2^2 + (4/7)^2}$ units. Similarly, the total distance traveled when crossing two horizontal lines and $x = 1$ is $\sqrt{2^2 + (10/7)^2}$, the total distance traveled when crossing two vertical lines and $y = 0$ is $\sqrt{2^2 + (6/7)^2}$, and the total distance traveled when crossing two vertical lines and $y = 1$ is $\sqrt{2^2 + (8/7)^2}$. The least of these is

$$\sqrt{2^2 + (4/7)^2} = \mathbf{\frac{2}{7}\sqrt{53}}.$$

T-4. If 306 is an element of S_k, then there exists an integer $m \geq 0$ such that $306 = k + mk^2$. Thus $k \mid 306$ and $k^2 \mid 306 - k$. The second relation can be rewritten as $k \mid 306/k - 1$, which implies that $k \leq \sqrt{306}$ unless $k = 306$. The prime factorization of 306 is $2 \cdot 3^2 \cdot 17$, so the set of factors of 306 less than $\sqrt{306}$ is $\{1, 2, 3, 6, 9, 17\}$. Check each in turn:

$$\begin{aligned} 306 - 1 &= 305, && 1^2 \mid 305 \\ 306 - 2 &= 304, && 2^2 \mid 304 \\ 306 - 3 &= 303, && 3^2 \nmid 303 \\ 306 - 6 &= 300, && 6^2 \nmid 300 \\ 306 - 9 &= 297, && 9^2 \nmid 297 \\ 306 - 17 &= 289, && 17^2 \mid 289. \end{aligned}$$

Thus the set of possible k is $\{1, 2, 17, 306\}$, and the sum is $1 + 2 + 17 + 306 = \mathbf{326}$.

T-5. Let $\log_x y = a$. Then the first equation is equivalent to $2a + \frac{5}{a} = 2k - 1$, and the second equation is equivalent to $\frac{5a}{2} - \frac{3}{2a} = k - 3$. Solving this system by eliminating k yields the quadratic equation $3a^2 + 5a - 8 = 0$, hence $a = 1$ or $a = -\frac{8}{3}$. Substituting each of these values of a into either of the original equations and solving for k yields $(a,k)=(1,4)$ or $\left(-\frac{8}{3},-\frac{149}{48}\right)$. Adding the values of k yields the answer of $\mathbf{43/48}$.

Alternate Solution: In terms of $a=\log_x y$, the two equations become $2a+\frac{5}{a}=2k-1$ and $\frac{5a}{2}-\frac{3}{2a}=k-3$. Eliminate $\frac{1}{a}$ to obtain $31a=16k-33$; substitute this into either of the original equations and clear denominators to get $96k^2-86k-1192=0$. The sum of the two roots is $86/96=\mathbf{43/48}$.

T-6. Define a *fault line* to be a side of a tile other than its base. Any tiling of $WASH$ can be represented as a sequence of tiles $t_1,t_2,\ldots,t_{14}$, where t_1 has a fault line of $\overline{WH}$, t_{14} has a fault line of $\overline{AS}$, and where t_k and t_{k+1} share a fault line for $1\leq k\leq 13$. Also note that to determine the position of tile t_{k+1}, it is necessary and sufficient to know the fault line that t_{k+1} shares with t_k, as well as whether the base of t_{k+1} lies on $\overline{WA}$ (abbreviated "B" for "bottom") or on $\overline{SH}$ (abbreviated "T" for "top"). Because rectangle $WASH$ has width 7, precisely 7 of the 14 tiles must have their bases on $\overline{WA}$. Thus any permutation of 7 B's and 7 T's determines a unique tiling $t_1,t_2,\ldots,t_{14}$, and conversely, any tiling $t_1,t_2,\ldots,t_{14}$ corresponds to a unique permutation of 7 B's and 7 T's. Thus the answer is $\binom{14}{7}=\mathbf{3432}$.

Alternate Solution: Let $T(a,b)$ denote the number of ways to triangulate the polygon with vertices at $(0,0)$, $(b,0)$, $(a,1)$, $(0,1)$, where each triangle has area $1/2$ and vertices at lattice points. The problem is to compute $T(7,7)$. It is easy to see that $T(a,0)=T(0,b)=1$ for all a and b. If a and b are both positive, then either one of the triangles includes the edge from $(a-1,1)$ to $(b,0)$ or one of the triangles includes the edge from $(a,1)$ to $(b-1,0)$, but not both. (In fact, as soon as there is an edge from $(a,1)$ to $(x,0)$ with $x<b$, there must be edges from $(a,1)$ to $(x',0)$ for all $x\leq x'<b$.) If there is an edge from $(a-1,1)$ to $(b,0)$, then the number of ways to complete the triangulation is $T(a-1,b)$; if there is an edge from $(a,1)$ to $(b-1,0)$, then the number of ways to complete the triangulation is $T(a,b-1)$; thus $T(a,b)=T(a-1,b)+T(a,b-1)$. The recursion and the initial conditions describe Pascal's triangle, so $T(a,b)=\binom{a+b}{a}$. In particular, $T(7,7)=\binom{14}{7}=\mathbf{3432}$.

T-7. Because $\cos 2x=1-2\sin^2 x$, $\sin^2 x=\dfrac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 8^\circ}{2}+\frac{1-\cos 16^\circ}{2}+\cdots+\frac{1-\cos 352^\circ}{2}=\frac{44}{2}-\frac{1}{2}\left(\cos 8^\circ+\cos 16^\circ+\cdots+\cos 352^\circ\right).$$

If $\alpha=\cos 8^\circ+i\sin 8^\circ$, then α is a primitive 45^{th} root of unity, and $1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{44}=0$. Hence $\alpha+\alpha^2+\cdots+\alpha^{44}=-1$, and because the real part of α^n is simply $\cos 8n^\circ$,

$$\cos 8^\circ+\cos 16^\circ+\cdots+\cos 352^\circ=-1.$$

Thus the desired sum is $22-(1/2)(-1)=\mathbf{45/2}$.

Alternate Solution: The problem asks to simplify the sum

$$\sin^2 a+\sin^2 2a+\sin^2 3a+\cdots+\sin^2 na$$

where $a = 4°$ and $n = 44$. Because $\cos 2x = 1 - 2\sin^2 x$, $\sin^2 x = \dfrac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 2a}{2} + \frac{1-\cos 4a}{2} + \cdots + \frac{1-\cos 2na}{2} = \frac{n}{2} - \frac{1}{2}\left(\cos 2a + \cos 4a + \cdots + \cos 2na\right).$$

Let $Q = \cos 2a + \cos 4a + \cdots + \cos 2na$. By the sum-to-product identity,

$$\begin{aligned} \sin 3a - \sin a &= 2\cos 2a \sin a, \\ \sin 5a - \sin 3a &= 2\cos 4a \sin a, \\ &\vdots \\ \sin(2n+1)a - \sin(2n-1)a &= 2\cos 2na \sin a. \end{aligned}$$

Thus

$$\begin{aligned} Q \cdot 2\sin a &= (\sin 3a - \sin a) + (\sin 5a - \sin 3a) + \cdots + (\sin(2n+1)a - \sin(2n-1)a) \\ &= \sin(2n+1)a - \sin a. \end{aligned}$$

With $a = 4°$ and $n = 44$, the difference on the right side becomes $\sin 356° - \sin 4°$; note that the terms in this difference are opposites, because of the symmetry of the unit circle. Hence

$$\begin{aligned} Q \cdot 2\sin 4° &= -2\sin 4°, \text{ and} \\ Q &= -1. \end{aligned}$$

Thus the original sum becomes $44/2 - (1/2)(-1) = \mathbf{45/2}$.

T-8. Call the region R, and let R_q be the portion of R in the q^{th} quadrant. Noting that the point (x, y) is in R if and only if $(\pm x, \pm y)$ is in R, it follows that $[R_1] = [R_2] = [R_3] = [R_4]$, and so $[R] = 4[R_1]$. So it suffices to determine $[R_1]$.

In the first quadrant, the boundary equation is just $x^2 + y^2 = x + y \Rightarrow (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$. This equation describes a circle of radius $\frac{\sqrt{2}}{2}$ centered at $(\frac{1}{2}, \frac{1}{2})$. The portion of the circle's interior which is inside the first quadrant can be decomposed into a right isosceles triangle with side length 1 and half a circle of radius $\frac{\sqrt{2}}{2}$. Thus $[R_1] = \frac{1}{2} + \frac{\pi}{4}$, hence $[R] = \mathbf{2 + \boldsymbol{\pi}}$.

T-9. Write $a_n = a_1 + r(n-1)$ and $b_n = b_1 + s(n-1)$. Then $a_{20} + b_{14} = a_1 + b_1 + 19r + 13s$, while $b_{20} + a_{14} = a_1 + b_1 + 13r + 19s = a_{20} + b_{14} + 6(s - r)$. Because both sequences consist only of integers, r and s must be integers, so $b_{20} + a_{14} \equiv a_{20} + b_{14} \bmod 6$. Thus the least possible value of $b_{20} + a_{14}$ is 4. If $b_{20} = 3$ and $a_{14} = 1$, then $\{a_n\}$ must be a decreasing sequence (else a_{13} would not be positive) and $a_{20} \le -5$, which is impossible. The case $b_{20} = a_{14} = 2$ violates the requirement that the terms be distinct, and by reasoning analogous to the first case, $b_{20} = 1, a_{14} = 3$ is also impossible. Hence the sum $b_{20} + a_{14}$ is at least 10. To show that 10 is attainable, make $\{a_n\}$ decreasing and b_{20} as small as possible: set $b_{20} = 1$, $a_{14} = 9$, and $a_n = 23 - n$. Then $a_{20} = 3$, yielding $b_{14} = 997$. Hence $s = \dfrac{997-1}{14-20} = \dfrac{996}{-6} = -166$ and

$b_1 = 997 - (13)(-166) = 3155$, yielding $b_n = 3155 - 166(n-1)$. Because $b_{20} = 1 \le a_{20}$ and $b_{19} = 167 \ge a_1$, the sequences $\{b_n\}$ and $\{a_n\}$ are distinct for $1 \le n \le 20$, completing the proof. Hence the minimum possible value of $b_{20} + a_{14}$ is **10**. [Note: This solution, which improves on the authors' original solution, is due to Ravi Jagadeesan of Phillips Exeter Academy.]

T-10. First, eliminate x: $y(y^2z - x) + (xy - z^2) = 14(y+1) \Rightarrow z^2 - y^3z + 14(y+1) = 0$. Viewed as a quadratic in z, this equation implies $z = \dfrac{y^3 \pm \sqrt{y^6 - 56(y+1)}}{2}$. In order for z to be an integer, the discriminant must be a perfect square. Because $y^6 = (y^3)^2$ and $(y^3-1)^2 = y^6 - 2y^3 + 1$, it follows that $|56(y+1)| \ge 2|y^3| - 1$. This inequality only holds for $|y| \le 5$. Within that range, the only values of y for which $y^6 - 56y - 56$ is a perfect square are -1 and -3. If $y = -1$, then $z = -1$ or $z = 0$. If $y = -3$, then $z = 1$ or $z = -28$. After solving for the respective values of x in the various cases, the four lattice points satisfying the system are $(-15,-1,-1), (-14,-1,0), (-5,-3,1)$, and $(-266,-3,-28)$. The farthest solution point from the origin is therefore $\mathbf{(-266, -3, -28)}$.

2014 Individual Problems

I-1. Charlie was born in the twentieth century. On his birthday in the present year (2014), he notices that his current age is twice the number formed by the rightmost two digits of the year in which he was born. Compute the four-digit year in which Charlie was born.

I-2. Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A+B\cdot C$ by two different methods. Pat follows the proper order of operations, computing $A+(B\cdot C)$. Chris ignores order of operations, choosing instead to compute $(A+B)\cdot C$. Compute the probability that Pat and Chris get the same answer.

I-3. Bobby, Peter, Greg, Cindy, Jan, and Marcia line up for ice cream. In an *acceptable lineup*, Greg is ahead of Peter, Peter is ahead of Bobby, Marcia is ahead of Jan, and Jan is ahead of Cindy. For example, the lineup with Greg in front, followed by Peter, Marcia, Jan, Cindy, and Bobby, in that order, is an acceptable lineup. Compute the number of acceptable lineups.

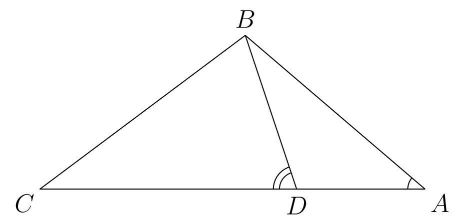
I-4. In triangle ABC, $a=12$, $b=17$, and $c=13$. Compute $b\cos C - c\cos B$.

I-5. The sequence of words $\{a_n\}$ is defined as follows: $a_1 = X$, $a_2 = O$, and for $n \geq 3$, a_n is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX$, $a_4 = OXO$, $a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

I-6. Compute the smallest positive integer n such that $214\cdot n$ and $2014\cdot n$ have the same number of divisors.

I-7. Let N be the least integer greater than 20 that is a palindrome in both base 20 and base 14. For example, the three-digit base-14 numeral $(13)5(13)_{14}$ (representing $13\cdot 14^2+5\cdot 14^1+13\cdot 14^0$) is a palindrome in base 14, but not in base 20, and the three-digit base-14 numeral $(13)31_{14}$ is not a palindrome in base 14. Compute the base-10 representation of N.

I-8. In triangle ABC, $BC = 2$. Point D is on $\overline{AC}$ such that $AD = 1$ and $CD = 2$. If $m\angle BDC = 2m\angle A$, compute $\sin A$.



I-9. Compute the greatest integer $k \leq 1000$ such that $\binom{1000}{k}$ is a multiple of 7.

I-10. An integer-valued function f is called *tenuous* if $f(x) + f(y) > x^2$ for all positive integers x and y. Let g be a tenuous function such that $g(1) + g(2) + \cdots + g(20)$ is as small as possible. Compute the minimum possible value for $g(14)$.

2014 Individual Answers

I-1. 1938

I-2. $\dfrac{9}{25}$

I-3. 20

I-4. 10

I-5. 667

I-6. 19133

I-7. 105

I-8. $\dfrac{\sqrt{6}}{4}$

I-9. 979

I-10. 136

2014 Individual Solutions

I-1. Let N be the number formed by the rightmost two digits of the year in which Charlie was born. Then his current age is $100 - N + 14 = 114 - N$. Setting this equal to $2N$ and solving yields $N = 38$, hence the answer is **1938**.

Alternate Solution: Let N be the number formed by the rightmost two digits of the year in which Charlie was born. The number of years from 1900 to 2014 can be thought of as the number of years before Charlie was born plus the number of years since he was born, or N plus Charlie's age. Thus $N + 2N = 114$, which leads to $N = 38$, so the answer is **1938**.

I-2. If Pat and Chris get the same answer, then $A+(B\cdot C) = (A+B)\cdot C$, or $A+BC = AC+BC$, or $A = AC$. This equation is true if $A = 0$ or $C = 1$; the equation places no restrictions on B. There are 25 triples (A, B, C) where $A = 0$, 25 triples where $C = 1$, and 5 triples where $A = 0$ and $C = 1$. As all triples are equally likely, the answer is $\frac{25+25-5}{5^3} = \frac{45}{125} = \mathbf{\frac{9}{25}}$.

I-3. There are 6 people, so there are $6! = 720$ permutations. However, for each arrangement of the boys, there are $3! = 6$ permutations of the girls, of which only one yields an acceptable lineup. The same logic holds for the boys. Thus the total number of permutations must be divided by $3! \cdot 3! = 36$, yielding $6!/(3! \cdot 3!) = \mathbf{20}$ acceptable lineups.

Alternate Solution: Once the positions of Greg, Peter, and Bobby are determined, the entire lineup is determined, because there is only one acceptable ordering of the three girls. Because the boys occupy three of the six positions, there are $\binom{6}{3} = \mathbf{20}$ acceptable lineups.

I-4. Using the Law of Cosines, $a^2 + b^2 - 2ab\cos C = c^2$ implies

$$b\cos C = \frac{a^2 + b^2 - c^2}{2a}.$$

Similarly,

$$c\cos B = \frac{a^2 - b^2 + c^2}{2a}.$$

Thus

$$\begin{aligned} b\cos C - c\cos B &= \frac{a^2+b^2-c^2}{2a} - \frac{a^2-b^2+c^2}{2a} \\ &= \frac{2b^2 - 2c^2}{2a} \\ &= \frac{b^2 - c^2}{a}. \end{aligned}$$

With the given values, the result is $(17^2 - 13^2)/12 = 120/12 = \mathbf{10}$.

Alternate Solution: Let H be the foot of the altitude from A to $\overline{BC}$; let $BH = x$, $CH = y$, and $AH = h$. Then $b\cos C = y$, $c\cos B = x$, and the desired quantity is $Q = y - x$. However, $y + x = a$, so $y^2 - x^2 = aQ$. By the Pythagorean Theorem, $y^2 = b^2 - h^2$ and $x^2 = c^2 - h^2$, so $y^2 - x^2 = (b^2 - h^2) - (c^2 - h^2) = b^2 - c^2$. Thus $aQ = b^2 - c^2$, and $Q = \frac{b^2-c^2}{a}$ as in the first solution.

I-5. Let P denote a palindromic word, let Q denote any word, and let $\overline{R}$ denote the reverse of word R. Note that if two consecutive terms of the sequence are $a_n = P$, $a_{n+1} = Q$, then $a_{n+2} = Q\overline{P} = QP$ and $a_{n+3} = QP\overline{Q}$. Thus if a_n is a palindrome, so is a_{n+3}. Because a_1 and a_2 are both palindromes, then so must be all terms in the subsequences $a_4, a_7, a_{10}, \ldots$ and $a_5, a_8, a_{11}, \ldots$.

To show that the other terms are not palindromes, note that if P' is *not* a palindrome, then $QP'\overline{Q}$ is also not a palindrome. Thus if a_n is not a palindrome, then a_{n+3} is not a palindrome either. Because $a_3 = OX$ is not a palindrome, neither is any term of the subsequence $a_6, a_9, a_{12}, \ldots$. (Alternatively, counting the number of X's in each word a_i shows that the number of X's in a_{3k} is odd. So if a_{3k} were to be a palindrome, it would have to have an odd number of letters, with an X in the middle. However, it can be shown that the length of a_{3k} is even. Thus a_{3k} cannot be a palindrome.)

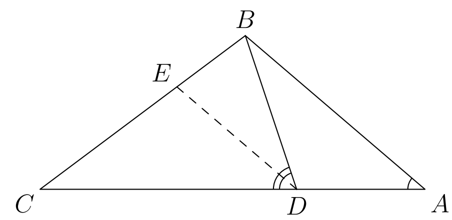
In total there are $1000 - 333 = \mathbf{667}$ palindromes among the first 1000 terms.

I-6. Let $D(n)$ be the number of divisors of the integer n. Note that if $D(214n) = D(2014n)$ and if some p divides n and is relatively prime to both 214 and 2014, then $D(\frac{214n}{p}) = D(\frac{2014n}{p})$. Thus any prime divisor of the smallest possible positive n will be a divisor of $214 = 2 \cdot 107$ or $2014 = 2 \cdot 19 \cdot 53$. For the sake of convenience, write $n = 2^{a-1}19^{b-1}53^{c-1}107^{d-1}$, where $a, b, c, d \geq 1$. Then $D(214n) = (a+1)bc(d+1)$ and $D(2014n) = (a+1)(b+1)(c+1)d$. Divide both sides by $a+1$ and expand to get $bcd + bc = bcd + bd + cd + d$, or $bc - bd - cd - d = 0$.

Because the goal is to minimize n, try $d = 1$: $bc - b - c - 1 = 0 \Rightarrow (b-1)(c-1) = 2$, which has solutions $(b, c) = (2, 3)$ and $(3, 2)$. The latter gives the smaller value for n, namely $19^2 \cdot 53 = 19133$. The only quadruples (a, b, c, d) that satisfy $2^{a-1}19^{b-1}53^{c-1}107^{d-1} < 19133$ and $d > 1$ are $(1, 1, 2, 2), (1, 2, 1, 2)$, and $(1, 1, 1, 3)$. None of these quadruples satisfies $bc - bd - cd - d = 0$, so the minimum value is $n = \mathbf{19133}$.

I-7. Because N is greater than 20, the base-20 and base-14 representations of N must be at least two digits long. The smallest possible case is that N is a two-digit palindrome in both bases. Then $N = 20a + a = 21a$, where $1 \leq a \leq 19$. Similarly, in order to be a two-digit palindrome in base 14, $N = 14b + b = 15b$, with $1 \leq b \leq 13$. So N would have to be a multiple of both 21 and 15. The least common multiple of 21 and 15 is 105, which has the base 20 representation of $105 = 55_{20}$ and the base-14 representation of $105 = 77_{14}$, both of which are palindromes. Thus the answer is $\mathbf{105}$.

I-8. Let $[ABC] = K$. Then $[BCD] = \frac{2}{3} \cdot K$. Let $\overline{DE}$ be the bisector of $\angle BDC$, as shown below.



Notice that $\mathrm{m}\angle DBA = \mathrm{m}\angle BDC - \mathrm{m}\angle A = \mathrm{m}\angle A$, so triangle ADB is isosceles, and $BD = 1$. (Alternately, notice that $\overline{DE} \parallel \overline{AB}$, and by similar triangles, $[CDE] = \frac{4}{9} \cdot K$, which means $[BDE] = \frac{2}{9} \cdot K$. Because $[CDE] : [BDE] = 2$ and $\angle BDE \cong \angle CDE$, conclude that $\frac{CD}{BD} = 2$, thus $BD = 1$.) Because BCD is isosceles, it follows that $\cos \angle BDC = \frac{1}{2} BD/CD = \frac{1}{4}$. By the half-angle formula,

$$\sin A = \sqrt{\frac{1 - \cos \angle BDC}{2}} = \sqrt{\frac{3}{8}} = \boldsymbol{\frac{\sqrt{6}}{4}}.$$

I-9. The ratio of binomial coefficients $\binom{1000}{k} / \binom{1000}{k+1} = \frac{k+1}{1000-k}$. Because 1000 is 1 less than a multiple of 7, namely $1001 = 7 \cdot 11 \cdot 13$, either $1000 - k$ and $k + 1$ are both multiples of 7 or neither is. Hence whenever the numerator is divisible by 7, the denominator is also. Thus for the largest value of k such that $\binom{1000}{k}$ is a multiple of 7, $\frac{k+1}{1000-k}$ must equal $7 \cdot \frac{p}{q}$, where p and q are relatively prime integers and $7 \nmid q$. The only way this can happen is when $k + 1$ is a multiple of 49, the greatest of which less than 1000 is 980. Therefore the greatest value of k satisfying the given conditions is $980 - 1 = \mathbf{979}$.

Alternate Solution: Rewrite 1000 in base 7: $1000 = 2626_7$. Let $k = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}_7$. By Lucas's Theorem, $\binom{1000}{k} \equiv \binom{2}{a}\binom{6}{b}\binom{2}{c}\binom{6}{d} \bmod 7$. The binomial coefficient $\binom{p}{q} = 0$ only when $q > p$. Base 7 digits cannot exceed 6, and $k \leq 1000$, thus the greatest value of k that works is $2566_7 = \mathbf{979}$. (Alternatively, the least value of k that works is $30_7 = 21$; because $\binom{n}{k} = \binom{n}{n-k}$, the greatest such k is $1000 - 21 = 979$.)

I-10. For a tenuous function g, let $S_g = g(1) + g(2) + \cdots + g(20)$. Then:

$$\begin{aligned} S_g &= (g(1) + g(20)) + (g(2) + g(19)) + \cdots + (g(10) + g(11)) \\ &\geq (20^2 + 1) + (19^2 + 1) + \cdots + (11^2 + 1) \\ &= 10 + \sum_{k=11}^{20} k^2 \\ &= 2495. \end{aligned}$$

The following argument shows that if a tenuous function g attains this sum, then $g(1) = g(2) = \cdots = g(10)$. First, if the sum equals 2495, then $g(1) + g(20) = 20^2 + 1, g(2) + g(19) = 19^2 + 1, \ldots, g(10) + g(11) = 11^2 + 1$. If $g(1) < g(2)$, then $g(1) + g(19) < 19^2 + 1$, which contradicts the tenuousness of g. Similarly, if $g(2) > g(1)$, then $g(2) + g(20) < 20^2 + 1$. Therefore $g(1) = g(2)$. Analogously, comparing $g(1)$ and $g(3)$, $g(1)$ and $g(4)$, etc. shows that $g(1) = g(2) = g(3) = \cdots = g(10)$.

Now consider all functions g for which $g(1) = g(2) = \cdots = g(10) = a$ for some integer a. Then $g(n) = n^2 + 1 - a$ for $n \geq 11$. Because $g(11) + g(11) > 11^2 = 121$, it is the case that $g(11) \geq 61$. Thus $11^2 + 1 - a \geq 61 \Rightarrow a \leq 61$. Thus the smallest possible value for $g(14)$ is $14^2 + 1 - 61 =$ **136**.

Power Question 2014: Power of Potlucks

In each town in ARMLandia, the residents have formed groups, which meet each week to share math problems and enjoy each others' company over a potluck-style dinner. Each town resident belongs to exactly one group. Every week, each resident is required to make one dish and to bring it to his/her group.

It so happens that each resident knows how to make precisely two dishes. Moreover, no two residents of a town know how to make the same pair of dishes. Shown below are two example towns. In the left column are the names of the town's residents. Adjacent to each name is the list of dishes that the corresponding resident knows how to make.

ARMLton

Resident	Dishes
Paul	pie, turkey
Arnold	pie, salad
Kelly	salad, broth

ARMLville

Resident	Dishes
Sally	steak, calzones
Ross	calzones, pancakes
David	steak, pancakes

The *population* of a town T, denoted $\operatorname{pop}(T)$, is the number of residents of T. Formally, the town itself is simply the set of its residents, denoted by $\{r_1, \ldots, r_{\operatorname{pop}(T)}\}$ unless otherwise specified. The set of dishes that the residents of T collectively know how to make is denoted $\operatorname{dish}(T)$. For example, in the town of ARMLton described above, $\operatorname{pop}(\text{ARMLton}) = 3$, and $\operatorname{dish}(\text{ARMLton}) = \{\text{pie, turkey, salad, broth}\}$.

A town T is called *full* if for every pair of dishes in $\operatorname{dish}(T)$, there is exactly one resident in T who knows how to make those two dishes. In the examples above, ARMLville is a full town, but ARMLton is not, because (for example) nobody in ARMLton knows how to make both turkey and salad.

Denote by $\mathcal{F}_d$ a full town in which collectively the residents know how to make d dishes. That is, $|\operatorname{dish}(\mathcal{F}_d)| = d$.

1a. Compute $\operatorname{pop}(\mathcal{F}_{17})$. [1 pt]

1b. Let $n = \operatorname{pop}(\mathcal{F}_d)$. In terms of n, compute d. [1 pt]

1c. Let T be a full town and let $D \in \operatorname{dish}(T)$. Let T' be the town consisting of all residents of T who do not know how to make D. Prove that T' is full. [2 pts]

In order to avoid the embarrassing situation where two people bring the same dish to a group dinner, if two people know how to make a common dish, they are forbidden from participating in the same group meeting. Formally, a *group assignment* on T is a function $f : T \to \{1, 2, \ldots, k\}$, satisfying the condition that if $f(r_i) = f(r_j)$ for $i \neq j$, then r_i and r_j do not know any of the same recipes. The *group number* of a town T, denoted $\operatorname{gr}(T)$, is the least positive integer k for which there exists a group assignment on T.

For example, consider once again the town of ARMLton. A valid group assignment would be $f(\text{Paul}) = f(\text{Kelly}) = 1$ and $f(\text{Arnold}) = 2$. The function which gives the value 1 to each resident of ARMLton is **not** a group assignment, because Paul and Arnold must be assigned to different groups.

2a. Show that gr(ARMLton) = 2. [1 pt]

2b. Show that gr(ARMLville) = 3. [1 pt]

3a. Show that $\mathrm{gr}(\mathcal{F}_4) = 3$. [1 pt]

3b. Show that $\mathrm{gr}(\mathcal{F}_5) = 5$. [2 pts]

3c. Show that $\mathrm{gr}(\mathcal{F}_6) = 5$. [2 pts]

4. Prove that the sequence $\mathrm{gr}(\mathcal{F}_2), \mathrm{gr}(\mathcal{F}_3), \mathrm{gr}(\mathcal{F}_4), \ldots$ is a non-decreasing sequence. [2 pts]

For a dish D, a resident is called a D-*chef* if he or she knows how to make the dish D. Define $\mathrm{chef}_T(D)$ to be the set of residents in T who are D-chefs. For example, in ARMLville, David is a steak-chef and a pancakes-chef. Further, $\mathrm{chef}_{\mathrm{ARMLville}}(\mathrm{steak}) = \{\mathrm{Sally, David}\}$.

5. Prove that

$$\sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)| = 2\,\mathrm{pop}(T).$$ [2 pts]

6. Show that for any town T and any $D \in \mathrm{dish}(T)$, $\mathrm{gr}(T) \geq |\mathrm{chef}_T(D)|$. [2 pts]

If $\mathrm{gr}(T) = |\mathrm{chef}_T(D)|$ for some $D \in \mathrm{dish}(T)$, then T is called *homogeneous*. If $\mathrm{gr}(T) > |\mathrm{chef}_T(D)|$ for each dish $D \in \mathrm{dish}(T)$, then T is called *heterogeneous*. For example, ARMLton is homogeneous, because gr(ARMLton) = 2 and exactly two chefs make pie, but ARMLville is heterogeneous, because even though each dish is only cooked by two chefs, gr(ARMLville) = 3.

7. For $n = 5, 6$, and 7, find a heterogeneous town T of population 5 for which $|\mathrm{dish}(T)| = n$. [3 pts]

A *resident cycle* is a sequence of distinct residents $r_1, \ldots, r_n$ such that for each $1 \leq i \leq n-1$, the residents r_i and r_{i+1} know how to make a common dish, residents r_n and r_1 know how to make a common dish, and no other pair of residents r_i and r_j, $1 \leq i, j \leq n$ know how to make a common dish. Two resident cycles are *indistinguishable* if they contain the same residents (in any order), and distinguishable otherwise. For example, if r_1, r_2, r_3, r_4 is a resident cycle, then r_2, r_1, r_4, r_3 and r_3, r_2, r_1, r_4 are indistinguishable resident cycles.

8a. Compute the number of distinguishable resident cycles of length 6 in $\mathcal{F}_8$. [1 pt]

8b. In terms of k and d, find the number of distinguishable resident cycles of length k in $\mathcal{F}_d$. [1 pt]

9. Let T be a town with at least two residents that has a single resident cycle that contains every resident. Prove that T is homogeneous if and only if $\mathrm{pop}(T)$ is even. [3 pts]

10. Let T be a town such that, for each $D \in \operatorname{dish}(T)$, $|\operatorname{chef}_T(D)| = 2$.

    a. Prove that there are finitely many resident cycles $C_1, C_2, \ldots, C_j$ in T so that each resident belongs to exactly one of the C_i. [3 pts]

    b. Prove that if $\operatorname{pop}(T)$ is odd, then T is heterogeneous. [3 pts]

11. Let T be a town such that, for each $D \in \operatorname{dish}(T)$, $|\operatorname{chef}_T(D)| = 3$.

    a. Either find such a town T for which $|\operatorname{dish}(T)|$ is odd, or show that no such town exists. [2 pts]

    b. Prove that if T contains a resident cycle such that for every dish $D \in \operatorname{dish}(T)$, there exists a chef in the cycle that can prepare D, then $\operatorname{gr}(T) = 3$. [3 pts]

12. Let k be a positive integer, and let T be a town in which $|\operatorname{chef}_T(D)| = k$ for every dish $D \in \operatorname{dish}(T)$. Suppose further that $|\operatorname{dish}(T)|$ is odd.

    a. Show that k is even. [1 pt]

    b. Prove the following: for every group in T, there is some dish $D \in \operatorname{dish}(T)$ such that no one in the group is a D-chef. [3 pts]

    c. Prove that $\operatorname{gr}(T) > k$. [3 pts]

13a. For each odd positive integer $d \geq 3$, prove that $\operatorname{gr}(\mathcal{F}_d) = d$. [3 pts]

13b. For each even positive integer d, prove that $\operatorname{gr}(\mathcal{F}_d) = d - 1$. [4 pts]

Solutions to 2014 Power Question

1. a. There are $\binom{17}{2} = 136$ possible pairs of dishes, so $\mathcal{F}_{17}$ must have 136 people.

   b. With d dishes there are $\binom{d}{2} = \frac{d^2-d}{2}$ possible pairs, so $n = \frac{d^2-d}{2}$. Then $2n = d^2 - d$, or $d^2 - d - 2n = 0$. Using the quadratic formula yields $d = \frac{1+\sqrt{1+8n}}{2}$ (ignoring the negative value).

   c. The town T' consists of all residents of T who do not know how to make D. Because T is full, every pair of dishes $\{d_i, d_j\}$ in $\operatorname{dish}(T)$ can be made by some resident r_{ij} in T. If $d_i \neq D$ and $d_j \neq D$, then $r_{ij} \in T'$. So every pair of dishes in $\operatorname{dish}(T) \setminus \{D\}$ can be made by some resident of T'. Hence T' is full.

2. a. Paul and Arnold cannot be in the same group, because they both make pie, and Arnold and Kelly cannot be in the same group, because they both make salad. Hence there must be at least two groups. But Paul and Kelly make none of the same dishes, so they can be in the same group. Thus a valid group assignment is

$$\begin{aligned} \text{Paul} &\mapsto 1 \\ \text{Kelly} &\mapsto 1 \\ \text{Arnold} &\mapsto 2. \end{aligned}$$

   Hence $\operatorname{gr}(\text{ARMLton}) = 2$.

   b. Sally and Ross both make calzones, Ross and David both make pancakes, and Sally and David both make steak. So no two of these people can be in the same group, and $\operatorname{gr}(\text{ARMLville}) = 3$.

3. a. Let the dishes be d_1, d_2, d_3, d_4 and let resident r_{ij} make dishes d_i and d_j, where $i < j$. There are six pairs of dishes, which can be divided into nonoverlapping pairs in three ways: $\{1,2\}$ and $\{3,4\}$, $\{1,3\}$ and $\{2,4\}$, and $\{1,4\}$ and $\{2,3\}$. Hence the assignment $r_{12}, r_{34} \mapsto 1$, $r_{13}, r_{24} \mapsto 2$, and $r_{14}, r_{23} \mapsto 3$ is valid, hence $\operatorname{gr}(\mathcal{F}_4) = 3$.

   b. First, $\operatorname{gr}(\mathcal{F}_5) \geq 5$: there are $\binom{5}{2} = 10$ people in $\mathcal{F}_5$, and because each person cooks two different dishes, any valid group of three people would require there to be six different dishes—yet there are only five. So each group can have at most two people. A valid assignment using five groups is shown below.

Residents	Group
r_{12}, r_{35}	1
r_{13}, r_{45}	2
r_{14}, r_{23}	3
r_{15}, r_{24}	4
r_{25}, r_{34}	5

   c. Now there are $\binom{6}{2} = 15$ people, but there are six different dishes, so it is possible (if done carefully) to place three people in a group. Because four people in a single group would require there to be eight different dishes, no group can have more than three people, and so $15/3 = 5$ groups is minimal. (Alternatively, there are five different residents who can cook dish d_1, and no two of these can be in the same group, so there must be at least five groups.) The assignment below attains that minimum.

Residents	Group
r_{12}, r_{34}, r_{56}	1
r_{13}, r_{25}, r_{46}	2
r_{14}, r_{26}, r_{35}	3
r_{15}, r_{24}, r_{36}	4
r_{16}, r_{23}, r_{45}	5

4. Pick some $n \geq 2$ and a full town $\mathcal{F}_n$ whose residents prepare dishes $d_1, \ldots, d_n$, and let $\mathrm{gr}(\mathcal{F}_n) = k$. Suppose that $f_n : \mathcal{F}_n \to \{1, 2, \ldots, k\}$ is a valid group assignment for $\mathcal{F}_n$. Then remove from $\mathcal{F}_n$ all residents who prepare dish d_n; by problem 1c, this operation yields the full town $\mathcal{F}_{n-1}$. Define $f_{n-1}(r) = f_n(r)$ for each remaining resident r in $\mathcal{F}_n$. If r and s are two (remaining) residents who prepare a common dish, then $f_n(r) \neq f_n(s)$, because f_n was a valid group assignment. Hence $f_{n-1}(r) \neq f_{n-1}(s)$ by construction of f_{n-1}. Therefore f_{n-1} is a valid group assignment on $\mathcal{F}_{n-1}$, and the set of groups to which the residents of $\mathcal{F}_{n-1}$ are assigned is a (not necessarily proper) subset of $\{1, 2, \ldots, k\}$. Thus $\mathrm{gr}(\mathcal{F}_{n-1})$ is at most k, which implies the desired result.

5. Because each chef knows how to prepare exactly two dishes, and no two chefs know how to prepare the same two dishes, each chef is counted exactly twice in the sum $\Sigma\,|\mathrm{chef}_T(D)|$. More formally, consider the set of "resident-dish pairs":

$$S = \{(r, D) \in T \times \mathrm{dish}(T) \mid r \text{ makes } D\}.$$

Count $|S|$ in two different ways. First, every dish D is made by $|\mathrm{chef}_T(D)|$ residents of T, so

$$|S| = \sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)|\,.$$

Second, each resident knows how to make exactly two dishes, so

$$|S| = \sum_{r \in T} 2 = 2\,\mathrm{pop}(T).$$

6. Let $D \in \mathrm{dish}(T)$. Suppose that f is a valid group assignment on T. Then for $r, s \in \mathrm{chef}_T(D)$, if $r \neq s$, it follows that $f(r) \neq f(s)$. Hence there must be at least $|\mathrm{chef}_T(D)|$ distinct groups in the range of f, i.e., $\mathrm{gr}(T) \geq |\mathrm{chef}_T(D)|$.

7. For $n = 5$, this result is attained as follows:

Resident	Dishes
Amy	d_1, d_2
Benton	d_2, d_3
Carol	d_3, d_4
Devin	d_4, d_5
Emma	d_5, d_1

For each dish D, note that $\text{chef}_T(D) = 2$. But $\text{gr}(T) > 2$, because if T had at most two groups, at least one of them would contain three people, and choosing any three people will result in a common dish that two of them can cook. Hence T is heterogeneous.

For $n \geq 6$, it suffices to assign dishes to residents so that there are three people who must be in different groups and that no dish is cooked by more than two people, which guarantees that $\text{gr}(T) \geq 3$ and $\text{chef}_T(D) \leq 2$ for each dish D.

Resident	Dishes
Amy	d_1, d_2
Benton	d_1, d_3
Carol	d_2, d_3
Devin	d_4, d_5
Emma	d_5, d_6

Note that Devin's and Emma's dishes are actually irrelevant to the situation, so long as they do not cook any of d_1, d_2, d_3, which already have two chefs each. Thus we can adjust this setup for $n = 7$ by setting Devin's dishes as d_4, d_5 and Emma's dishes as d_6, d_7. (In this last case, Devin and Emma are extremely compatible: they can both be put in a group with anyone else in the town!)

8. a. Because the town is full, each pair of dishes is cooked by exactly one resident, so it is simplest to identify residents by the pairs of dishes they cook. Suppose the first resident cooks (d_1, d_2), the second resident (d_2, d_3), the third resident (d_3, d_4), and so on, until the sixth resident, who cooks (d_6, d_1). Then there are 8 choices for d_1 and 7 choices for d_2. There are only 6 choices for d_3, because $d_3 \neq d_1$ (otherwise two residents would cook the same pair of dishes). For $k > 3$, the requirement that no two intermediate residents cook the same dishes implies that d_{k+1} cannot equal any of $d_1, \ldots, d_{k-1}$, and of course d_k and d_{k+1} must be distinct dishes. Hence there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20{,}160$ six-person resident cycles, not accounting for different starting points in the cycle and the two different directions to go around the cycle. Taking these into account, there are $20{,}160/(6 \cdot 2) = 1{,}680$ distinguishable resident cycles.

   b. Using the logic from 8a, there are $d(d-1)\cdots(d-k+1)$ choices for $d_1, d_2, \ldots, d_k$. To account for indistinguishable cycles, divide by k possible starting points and 2 possible directions, yielding $\frac{d(d-1)\cdots(d-k+1)}{2k}$ or $\frac{d!}{2k(d-k)!}$ distinguishable resident cycles.

9. Note that for every $D \in \text{dish}(T), \text{chef}_T(D) \leq 2$, because otherwise, $r_1, r_2, \ldots, r_n$ could not be a resident cycle. Without loss of generality, assume the cycle is $r_1, r_2, \ldots, r_n$. If n is even, assign resident r_i to group 1 if i is odd, and to group 2 if i is even. This is a valid group assignment, because the only pairs of residents who cook the same dish are (r_i, r_{i+1}) for $i = 1, 2, \ldots, n-1$ and (r_n, r_1). In each case, the residents are assigned to different groups. This proves $\text{gr}(T) = 2$, so T is homogeneous.

   On the other hand, if n is odd, suppose for the sake of contradiction that there are only two groups. Then either r_1 and r_n are in the same group, or for some i, r_i and r_{i+1} are in the same group. In either case, two residents in the same group share a dish, contradicting the requirement that no members of a group have a common dish. Hence $\operatorname{gr}(T) \geq 3$ when n is odd, making T heterogeneous.

10. a. First note that the condition $|\operatorname{chef}_T(D)| = 2$ for all D implies that $\operatorname{pop}(T) = |\operatorname{dish}(T)|$, using the equation from problem 5. So for the town in question, the population of the town equals the number of dishes in the town. Because no two chefs cook the same pair of dishes, it is impossible for such a town to have exactly two residents, and because each dish is cooked by exactly two chefs, it is impossible for such a town to have only one resident.

    The claim is true for towns of three residents satisfying the conditions: such towns must have one resident who cooks dishes d_1 and d_2, one resident who cooks dishes d_2 and d_3, and one resident who cooks dishes d_3 and d_1, and those three residents form a cycle. So proceed by (modified) strong induction: assume that for some $n > 3$ and for all positive integers k such that $3 \leq k < n$, every town T with k residents and $|\operatorname{chef}_T(D)| = 2$ for all $D \in \operatorname{dish}(T)$ can be divided into a finite number of resident cycles such that each resident belongs to exactly one of the cycles. Let T_n be a town of n residents, and arbitrarily pick resident r_1 and dishes d_1 and d_2 cooked by r_1. Then there is exactly one other resident r_2 who also cooks d_2 (because $|\operatorname{chef}_{T_n}(d_2)| = 2$). But r_2 also cooks another dish, d_3, which is cooked by another resident, r_3. Continuing in this fashion, there can be only two outcomes: either the process exhausts all the residents of T_n, or there exists some resident r_m, $m < n$, who cooks the same dishes as r_{m-1} and r_ℓ for $\ell < m - 1$.

    In the former case, r_n cooks another dish; but every dish besides d_1 is already cooked by two chefs in T_n, so r_n must also cook d_1, closing the cycle. Because every resident is in this cycle, the statement to be proven is also true for T_n.

    In the latter case, the same logic shows that r_m cooks d_1, also closing the cycle, but there are other residents of T_n who have yet to be accounted for. Let $C_1 = \{r_1, \ldots, r_m\}$, and consider the town T' whose residents are $T_n \setminus C_1$. Each of dishes $d_1, \ldots, d_m$ is cooked by two people in C_1, so no chef in T' cooks any of these dishes, and no dish in T' is cooked by any of the people in C_1 (because each person in C_1 already cooks two dishes in the set $\operatorname{dish}(C_1)$). Thus $|\operatorname{chef}_{T'}(D)| = 2$ for each D in $\operatorname{dish}(T')$. It follows that $\operatorname{pop}(T') < \operatorname{pop}(T)$ but $\operatorname{pop}(T') > 0$, so by the inductive hypothesis, the residents of T' can be divided into disjoint resident cycles.
    Thus the statement is proved by strong induction.

    b. In order for T to be homogeneous, it must be possible to partition the residents into exactly two dining groups. First apply 10a to divide the town into finitely many resident cycles C_i, and assume towards a contradiction that such a group assignment $f : T \to \{1, 2\}$ exists. If $\operatorname{pop}(T)$ is odd, then at least one of the cycles C_i must contain an odd number of residents; without loss of generality, suppose this cycle to be C_1, with residents $r_1, r_2, \ldots, r_{2k+1}$. (By the restrictions noted in part a, $k \geq 1$.) Now because r_i and r_{i+1} cook a dish in common, $f(r_i) \neq f(r_{i+1})$ for all i. Thus if $f(r_1) = 1$, it follows that $f(r_2) = 2$, and that $f(r_3) = 1$, etc. So $f(r_i) = f(r_1)$ if i is odd and $f(r_i) = f(r_2)$ if i is even; in particular, $f(r_{2k+1}) = f(1)$. But that equation would imply that r_1 and r_{2k+1} cook no dishes in common, which is impossible if they are the first and last residents in a resident cycle. So no such group assignment can exist, and $\mathrm{gr}(T) \geq 3$. Hence T is heterogeneous.

11. a. In problem 5, it was shown that

$$2\,\mathrm{pop}(T) = \sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)|\,.$$

Therefore $\sum_{D\in\mathrm{dish}(T)} |\mathrm{chef}_T(D)|$ is even. But if $|\mathrm{chef}_T(D)| = 3$ for all $D \in \mathrm{dish}(T)$, then the sum is simply $3\,|\mathrm{dish}(T)|$, so $|\mathrm{dish}(T)|$ must be even.

b. By problem 6, it must be the case that $\mathrm{gr}(T) \geq 3$. Let $C = \{r_1, r_2, \ldots, r_n\}$ denote a resident cycle such that for every dish $D \in \mathrm{dish}(T)$, there exists a chef in C that can prepare D. Each resident is a chef for two dishes, and every dish can be made by two residents in C (although by three in T). Thus the number of residents in the resident cycle C is equal to $|\mathrm{dish}(T)|$, which was proved to be even in the previous part.

Define a group assignment by setting

$$f(r) = \begin{cases} 1 & \text{if } r \notin C \\ 2 & \text{if } r = r_i \text{ and } i \text{ is even} \\ 3 & \text{if } r = r_i \text{ and } i \text{ is odd.} \end{cases}$$

For any $D \in \mathrm{dish}(T)$, there are exactly three D-chefs, and exactly two of them belong to the resident cycle C. Hence exactly one of the D-chefs r will have $f(r) = 1$. The remaining two D-chefs will be r_i and r_{i+1} for some i, or r_1 and r_n. In either case, the group assignment f will assign one of them to 2 and the other to 3. Thus any two residents who make a common dish will be assigned different groups by f, so f is a valid group assignment, proving that $\mathrm{gr}(T) = 3$.

12. a. From problem 5,

$$2\,\mathrm{pop}(T) = \sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)|\,.$$

Because $|\mathrm{chef}_T(D)| = k$ for all $D \in \mathrm{dish}(T)$, the sum is $k \cdot \mathrm{dish}(T)$. Thus $2\,\mathrm{pop}\,T = k \cdot \mathrm{dish}(T)$, and so $k \cdot \mathrm{dish}(T)$ must be even. By assumption, $|\mathrm{dish}(T)|$ is odd, so k must be even.

b. Suppose for the sake of contradiction that there is some n for which the group $R = \{r \in T \mid f(r) = n\}$ has a D-chef for every dish D. Because f is a group assignment and f assigns every resident of R to group n, no two residents of R make the same dish. Thus for every $D \in \mathrm{dish}(T)$, exactly one resident of R is a D-chef; and each D-chef cooks exactly one other dish, which itself is not cooked by anyone else in R. Thus the dishes come in pairs: for each dish D, there is another dish D' cooked by the D-chef in R and no one else in R. However, if the dishes can be paired off, there must be an even number of dishes, contradicting the assumption that $|\mathrm{dish}(T)|$ is odd. Thus for every n, the set $\{r \in T \mid f(r) = n\}$ must be missing a D-chef for some dish D.

c. Let f be a group assignment for T, and let $R = \{r \in T \mid f(r) = 1\}$. From problem 12b, there must be some $D \in \text{dish}(T)$ with no D-chefs in R. Moreover, f cannot assign two D-chefs to the same group, so there must be at least k other groups besides R. Hence there are at least $1 + k$ different groups, so $\text{gr}(T) > k$.

13. a. Fix $D \in \text{dish}(\mathcal{F}_d)$. Then for every other dish $D' \in \text{dish}(\mathcal{F}_d)$, there is exactly one chef who makes both D and D', hence $|\text{chef}_{\mathcal{F}_d}(D)| = d - 1$, which is even because d is odd. Thus for each $D \in \text{dish}(\mathcal{F}_d)$, $|\text{chef}_{\mathcal{F}_d}(D)|$ is even. Because $|\text{dish}(\mathcal{F}_d)| = d$ is odd and $|\text{chef}_{\mathcal{F}_d}(D)| = d - 1$ for every dish in $\mathcal{F}_d$, problem 12c applies, hence $\text{gr}(\mathcal{F}_d) > d - 1$.

Label the dishes $D_1, D_2, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \leq i < j \leq d$ so that $r_{i,j}$ is a D_i-chef and a D_j-chef. Define $f : \mathcal{F}_d \to \{0, 1, \ldots, d-1\}$ by letting $f(r_{i,j}) \equiv i + j \bmod d$.

Suppose that $f(r_{i,j}) = f(r_{k,\ell})$, so $i + j \equiv k + \ell \bmod d$. Then $r_{i,j}$ and $r_{k,\ell}$ are assigned to the same group, which is a problem if they are different residents but are chefs for the same dish. This overlap occurs if and only if one of i and j is equal to one of k and ℓ. If $i = k$, then $j \equiv \ell \bmod d$. As j and ℓ are both between 1 and d, the only way they could be congruent modulo d is if they were in fact equal. That is, $r_{i,j}$ is the same resident as $r_{k,\ell}$. The other three cases ($i = \ell$, $j = k$, and $j = \ell$) are analogous. Thus f is a valid group assignment, proving that $\text{gr}(\mathcal{F}_d) \leq d$. Therefore $\text{gr}(\mathcal{F}_d) = d$.

b. In problem 4, it was shown that the sequence $\text{gr}(\mathcal{F}_2), \text{gr}(\mathcal{F}_3), \ldots$ is nondecreasing. If d is even, $\text{gr}(\mathcal{F}_d) \geq \text{gr}(\mathcal{F}_{d-1})$, and because $d-1$ is odd, problem 13a applies: $\text{gr}(\mathcal{F}_{d-1}) = d-1$. Hence $\text{gr}(\mathcal{F}_d) \geq d - 1$. Now it suffices to show that $\text{gr}(\mathcal{F}_d) \leq d - 1$ by exhibiting a valid group assignment $f : \mathcal{F}_d \to \{1, 2, \ldots, d - 1\}$.

Label the dishes $D_1, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \leq i < j \leq d$ so that $r_{i,j}$ is a D_i-chef and a D_j-chef. Let $R = \{r_{i,j} \mid i, j \neq d\}$. That is, R is the set of residents who are not D_d-chefs. Using 1c, R is a full town with $d-1$ dishes, so from 12a, it has a group assignment $f : R \to \{1, 2, \ldots, d-1\}$. For each $D_i \in \text{dish}(\mathcal{F}_d)$, $i \neq d$, $|\,\text{chef}_R(D_i)| = d-2$. Because there are $d - 1$ groups and $|\text{chef}_R(D_i)| = d - 2$, exactly one group n_i must not contain a D_i-chef for each dish D_i.

It cannot be the case that $n_i = n_j$ for $i \neq j$. Indeed, suppose for the sake of contradiction that $n_i = n_j$. Without loss of generality, assume that $n_i = n_j = 1$ (by perhaps relabeling the dishes). Then any resident $r \in R$ assigned to group 1 (that is, $f(r) = 1$) would be neither a D_i-chef nor a D_j-chef. The residents in R who are assigned to group 1 must all be chefs for the remaining $d - 3$ dishes. Because each resident cooks two dishes, and no two residents of group 1 can make a common dish,

$$|\{r \in R \mid f(r) = 1\}| \leq \frac{d-3}{2}.$$

For each of the other groups $2, 3, \ldots, d - 1$, the number of residents of R in that group is no more than $(d - 1)/2$, because there are $d - 1$ dishes in R, each resident cooks two dishes, and no two residents in the same group can make a common dish. However, because $d-1$ is odd, the size of any group is actually no more than $(d-2)/2$. Therefore

$$\begin{aligned}
|R| &= \sum_{k=1}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&= |\{r \in R \mid f(r) = 1\}| + \sum_{k=2}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&\leq \frac{d-3}{2} + \sum_{k=2}^{d-1} \frac{d-2}{2} \\
&= \frac{d-3}{2} + \frac{(d-2)^2}{2} \\
&= \frac{d^2-3d+1}{2} < \frac{d^2-3d+2}{2} = |R|\,.
\end{aligned}$$

This is a contradiction, so it must be that $n_i \neq n_j$ for all $i \neq j$, making f a valid group assignment on $\mathcal{F}_d$. Hence $\operatorname{gr}(\mathcal{F}_d) = d-1$.

2014 Relay Problems

R1-1. Let $T = (0, 0)$, $N = (2, 0)$, $Y = (6, 6)$, $W = (2, 6)$, and $R = (0, 2)$. Compute the area of pentagon $TNYWR$.

R1-2. Let $T = TNYWR$. The lengths of the sides of a rectangle are the zeroes of the polynomial $x^2 - 3Tx + T^2$. Compute the length of the rectangle's diagonal.

R1-3. Let $T = TNYWR$. Let $w > 0$ be a real number such that T is the area of the region above the x-axis, below the graph of $y = \lceil x \rceil^2$, and between the lines $x = 0$ and $x = w$. Compute $\lceil 2w \rceil$.

R2-1. Compute the least positive integer n such that $\gcd(n^3, n!) \geq 100$.

R2-2. Let $T = TNYWR$. At a party, everyone shakes hands with everyone else exactly once, except Ed, who leaves early. A grand total of $20T$ handshakes take place. Compute the number of people at the party who shook hands with Ed.

R2-3. Let $T = TNYWR$. Given the sequence u_n such that $u_3 = 5$, $u_6 = 89$, and $u_{n+2} = 3u_{n+1} - u_n$ for integers $n \geq 1$, compute u_T.

2014 Relay Answers

R1-1. 20

R1-2. $20\sqrt{7}$

R1-3. 10

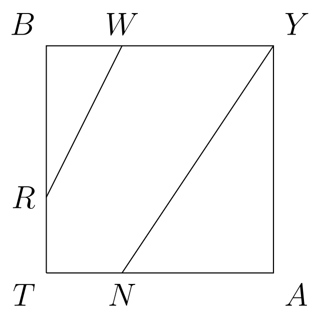
R2-1. 8

R2-2. 7

R2-3. 233

2014 Relay Solutions

R1-1. Pentagon $TNYWR$ fits inside square $TAYB$, where $A = (6, 0)$ and $B = (0, 6)$. The region of $TAYB$ not in $TNYWR$ consists of triangles $\triangle NAY$ and $\triangle WBR$, as shown below.



Thus

$$\begin{aligned}[TNYWR] &= [TAYB] - [NAY] - [WBR] \\ &= 6^2 - \frac{1}{2} \cdot 4 \cdot 6 - \frac{1}{2} \cdot 2 \cdot 4 \\ &= \mathbf{20}.\end{aligned}$$

R1-2. Let r and s denote the zeros of the polynomial $x^2 - 3Tx + T^2$. The rectangle's diagonal has length $\sqrt{r^2 + s^2} = \sqrt{(r + s)^2 - 2rs}$. Recall that for a quadratic polynomial $ax^2 + bx + c$, the sum of its zeros is $-b/a$, and the product of its zeros is c/a. In this particular instance, $r + s = 3T$ and $rs = T^2$. Thus the length of the rectangle's diagonal is $\sqrt{9T^2 - 2T^2} = T \cdot \sqrt{7}$. With $T = 20$, the rectangle's diagonal is $\mathbf{20\sqrt{7}}$.

R1-3. Write $w = k + \alpha$, where k is an integer, and $0 \le \alpha < 1$. Then

$$T = 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 \cdot \alpha.$$

Computing $\lceil 2w \rceil$ requires computing w to the nearest half-integer. First obtain the integer k. As $\sqrt{7} > 2$, with $T = 20\sqrt{7}$, one obtains $T > 40$. As $1^2 + 2^2 + 3^2 + 4^2 = 30$, it follows that $k \ge 4$. To obtain an upper bound for k, note that $700 < 729$, so $10\sqrt{7} < 27$, and $T = 20\sqrt{7} < 54$. As $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$, it follows that $4 < w < 5$, and hence $k = 4$.

It now suffices to determine whether or not $\alpha > 0.5$. To this end, one must determine whether $T > 1^2 + 2^2 + 3^2 + 4^2 + 5^2/2 = 42.5$. Indeed, note that $2.5^2 = 6.25 < 7$, so $T > (20)(2.5) = 50$. It follows that $\alpha > 0.5$, so $4.5 < w < 5$. Thus $9 < 2w < 10$, and $\lceil 2w \rceil = \mathbf{10}$.

Alternate Solution: Once it has been determined that $4 < w < 5$, the formula for T yields $1 + 4 + 9 + 16 + 25 \cdot \alpha = 20\sqrt{7}$, hence $\alpha = \frac{4\sqrt{7}-6}{5}$. Thus $2\alpha = \frac{8\sqrt{7}-12}{5} = \frac{\sqrt{448}-12}{5} > \frac{21-12}{5} = 1.8$. Because $2w = 2k + 2\alpha$, it follows that $\lceil 2w \rceil = \lceil 8 + 2\alpha \rceil = \mathbf{10}$, because $1.8 < 2\alpha < 2$.

R2-1. Note that if p is prime, then $\gcd(p^3, p!) = p$. A good strategy is to look for values of n with several (not necessarily distinct) prime factors so that n^3 and $n!$ will have many factors in common. For example, if $n = 6$, $n^3 = 216 = 2^3 \cdot 3^3$ and $n! = 720 = 2^4 \cdot 3^2 \cdot 5$, so $\gcd(216, 720) = 2^3 \cdot 3^2 = 72$. Because 7 is prime, try $n = 8$. Notice that $8^3 = 2^9$ while $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Thus $\gcd(512, 8!) = 2^7 = 128 > 100$, hence the smallest value of n is **8**.

R2-2. If there were n people at the party, including Ed, and if Ed had not left early, there would have been $\binom{n}{2}$ handshakes. Because Ed left early, the number of handshakes is strictly less than that, but greater than $\binom{n-1}{2}$ (everyone besides Ed shook everyone else's hand). So find the least number n such that $\binom{n}{2} \geq 160$. The least such n is 19, because $\binom{18}{2} = 153$ and $\binom{19}{2} = 171$. Therefore there were 19 people at the party. However, $171 - 160 = 11$ handshakes never took place. Therefore the number of people who shook hands with Ed is $19 - 11 - 1 = \mathbf{7}$.

R2-3. By the recursive definition, notice that $u_6 = 89 = 3u_5 - u_4$ and $u_5 = 3u_4 - u_3 = 3u_4 - 5$. This is a linear system of equations. Write $3u_5 - u_4 = 89$ and $-3u_5 + 9u_4 = 15$ and add to obtain $u_4 = 13$. Now apply the recursive definition to obtain $u_5 = 34$ and $u_7 = \mathbf{233}$.

Alternate Solution: Notice that the given values are both Fibonacci numbers, and that in the Fibonacci sequence, $f_1 = f_2 = 1$, $f_5 = 5$, and $f_{11} = 89$. That is, 5 and 89 are six terms apart in the Fibonacci sequence, and only three terms apart in the given sequence. This relationship is not a coincidence: alternating terms in the Fibonacci sequence satisfy the given recurrence relation for the sequence $\{u_n\}$, that is, $f_{n+4} = 3f_{n+2} - f_n$. Proof: if $f_n = a$ and $f_{n+1} = b$, then $f_{n+2} = a+b$, $f_{n+3} = a+2b$, and $f_{n+4} = 2a+3b = 3(a+b) - b = 3f_{n+2} - f_n$. To compute the final result, continue out the Fibonacci sequence to obtain $f_{12} = 144$ and $u_7 = f_{13} = \mathbf{233}$.

2014 Tiebreaker Problems

TB-1. A student computed the repeating decimal expansion of $\frac{1}{N}$ for some integer N, but inserted six extra digits into the repetend to get $.0\overline{0231846597}$. Compute the value of N.

TB-2. Let n be a four-digit number whose square root is three times the sum of the digits of n. Compute n.

TB-3. Compute the sum of the reciprocals of the positive integer divisors of 24.

2014 Tiebreaker Answers

TB-1. 606

TB-2. 2916

TB-3. $\dfrac{5}{2}$

2014 Tiebreaker Solutions

TB-1. Because the given repetend has ten digits, the original had four digits. If $\dfrac{1}{N} = .0\overline{\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}} = \dfrac{\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}}{99990}$, then the numerator must divide $99990 = 10 \cdot 99 \cdot 101 = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 101$.

Note that all 3- and 4-digit multiples of 101 contain at least one digit which appears twice. Because the 10-digit string under the vinculum (i.e., 0231846597) contains no repeated digits, $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ cannot be a multiple of 101. So $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ divides $2 \cdot 3^2 \cdot 5 \cdot 11 = 990$. The only divisor of 990 that can be formed from four of the given digits (taken in order) is 0165, that is, 165. Hence $\dfrac{1}{N} = \dfrac{165}{99990} = \dfrac{1}{606} \Rightarrow N = \mathbf{606}$.

TB-2. Because $\sqrt{n}$ is a multiple of 3, n must be a multiple of 9. Therefore the sum of the digits of n is a multiple of 9. Thus $\sqrt{n}$ must be a multiple of 27, which implies that n is a multiple of 27^2. The only candidates to consider are 54^2 $(= 2916)$ and 81^2 $(= 6561)$, and only **2916** satisfies the desired conditions.

TB-3. The map $n \mapsto 24/n$ establishes a one-to-one correspondence among the positive integer divisors of 24. Thus

$$\begin{aligned} \sum_{\substack{n|24 \\ n>0}} \frac{1}{n} &= \sum_{\substack{n|24 \\ n>0}} \frac{1}{24/n} \\ &= \frac{1}{24} \sum_{\substack{n|24 \\ n>0}} n. \end{aligned}$$

Because $24 = 2^3 \cdot 3$, the sum of the positive divisors of 24 is $(1+2+2^2+2^3)(1+3) = 15 \cdot 4 = 60$. Hence the sum is $60/24 = \mathbf{5/2}$.

Alternate Solution: Because $24 = 2^3 \cdot 3$, any positive divisor of 24 is of the form $2^a 3^b$ where $a = 0, 1, 2,$ or 3, and $b = 0$ or 1. So the sum of the positive divisors of 24 can be represented as the product $(1 + 2 + 4 + 8)(1 + 3)$. Similarly, the sum of their reciprocals can be represented as the product $\left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8}\right)\left(\dfrac{1}{1} + \dfrac{1}{3}\right)$. The first sum is $\dfrac{15}{8}$ and the second is $\dfrac{4}{3}$, so the product is $\mathbf{5/2}$.

2014 Super Relay Problems

1. The sequence $a_1, a_2, a_3, \ldots$ is a geometric sequence with $a_{20} = 8$ and $a_{14} = 2^{21}$. Compute a_{21}.

2. Let $T = TNYWR$. Circles L and O are internally tangent and have radii T and $4T$, respectively. Point E lies on circle L such that $\overline{OE}$ is tangent to circle L. Compute OE.

3. Let $T = TNYWR$. In a right triangle, one leg has length T^2 and the other leg is 2 less than the hypotenuse. Compute the triangle's perimeter.

4. Let $T = TNYWR$. If $x + 9y = 17$ and $Tx + (T + 1)y = T + 2$, compute $20x + 14y$.

5. Let $T = TNYWR$. Let $f(x) = ax^2 + bx + c$. The product of the roots of f is T. If $(-2, 20)$ and $(1, 14)$ lie on the graph of f, compute a.

6. Let $T = TNYWR$. Let $z_1 = 15 + 5i$ and $z_2 = 1 + Ki$. Compute the smallest positive integral value of K such that $|z_1 - z_2| \geq 15T$.

7. Let $T = TNYWR$. Suppose that T people are standing in a line, including three people named Charlie, Chris, and Abby. If the people are assigned their positions in line at random, compute the probability that Charlie is standing next to at least one of Chris or Abby.

8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. Let $\alpha = \sin^{-1} A$ and let $\beta = \cos^{-1} B$. Compute $\sin(\alpha + \beta) + \sin(\alpha - \beta)$.

9. Let $T = TNYWR$. If r is the radius of a right circular cone and the cone's height is $T - r^2$, let V be the maximum possible volume of the cone. Compute π/V.

10. Let $T = TNYWR$. If $\log T = 2 - \log 2 + \log k$, compute the value of k.

11. Let $T = TNYWR$. Nellie has a flight from Rome to Athens that is scheduled to last for $T + 30$ minutes. However, owing to a tailwind, her flight only lasts for T minutes. The plane's speed is 1.5 miles per minute faster than what it would have been for the originally scheduled flight. Compute the distance (in miles) that the plane travels.

12. Let $T = TNYWR$. Compute $\sqrt{\sqrt{\sqrt[T]{10^{T^2-T}}}}$.

13. Let $T = TNYWR$. Regular hexagon $SUPERB$ has side length $\sqrt{T}$. Compute the value of $BE \cdot SU \cdot RE$.

14. Let $T = TNYWR$. Chef Selma is preparing a burrito menu. A burrito consists of: (1) a choice of chicken, beef, turkey, or no meat, (2) exactly one of three types of beans, (3) exactly one of two types of rice, and (4) exactly one of K types of cheese. Compute the smallest value of K such that Chef Selma can make at least T different burrito varieties.

15. Compute the smallest positive integer N such that $20N$ is a multiple of 14 and $14N$ is a multiple of 20.

2014 Super Relay Answers

1. 1

2. $2\sqrt{2}$

3. 40

4. 8

5. $\dfrac{8}{5}$

6. 25

7. $\dfrac{47}{300}$

8. $\dfrac{94}{4225}$

9. $\dfrac{12}{169}$

10. 13

11. 650

12. 100

13. 9

14. 3

15. 70

2014 Super Relay Solutions

1. Let r be the common ratio of the sequence. Then $a_{20} = r^{20-14} \cdot a_{14}$, hence $8 = r^6 \cdot 2^{21} \Rightarrow r^6 = \frac{2^3}{2^{21}} = 2^{-18}$, so $r = 2^{-3} = \frac{1}{8}$. Thus $a_{21} = r \cdot a_{20} = \frac{1}{8} \cdot 8 = \mathbf{1}$.

2. Because $\overline{OE}$ is tangent to circle L, $\overline{LE} \perp \overline{OE}$. Also note that $LO = 4T - T = 3T$. Hence, by the Pythagorean Theorem, $OE = \sqrt{(3T)^2 - T^2} = 2T\sqrt{2}$ (this also follows from the Tangent-Secant Theorem). With $T = 1$, $OE = \mathbf{2\sqrt{2}}$.

3. Let c be the length of the hypotenuse. Then, by the Pythagorean Theorem, $(T^2)^2 + (c-2)^2 = c^2 \Rightarrow c = \frac{T^4}{4} + 1$. With $T = 2\sqrt{2}$, $T^4 = 64$, and $c = 17$. So the triangle is a $8-15-17$ triangle with perimeter $\mathbf{40}$.

4. Multiply each side of the first equation by T to obtain $Tx + 9Ty = 17T$. Subtract the second equation to yield $9Ty - Ty - y = 16T - 2 \Rightarrow y(8T - 1) = 2(8T - 1)$. Hence either $T = \frac{1}{8}$ (in which case, the value of y is not uniquely determined) or $y = 2$. Plug $y = 2$ into the first equation to obtain $x = -1$. Hence $20x + 14y = -20 + 28 = \mathbf{8}$.

5. Using Vièta's Formula, write $f(x) = ax^2 + bx + Ta$. Substituting the coordinates of the given points yields the system of equations: $4a - 2b + Ta = 20$ and $a + b + Ta = 14$. Multiply each side of the latter equation by 2 and add the resulting equation to the former equation to eliminate b. Simplifying yields $a = \dfrac{16}{T+2}$. With $T = 8$, $a = \mathbf{8/5}$.

6. Note that $z_1 - z_2 = 14 + (5-K)i$, hence $|z_1 - z_2| = \sqrt{14^2 + (5-K)^2}$. With $T = 8/5, 15T = 24$, hence $14^2 + (5-K)^2 \geq 24^2$. Thus $|5 - K| \geq \sqrt{24^2 - 14^2} = \sqrt{380}$. Because K is a positive integer, it follows that $K - 5 \geq 20$, hence the desired value of K is $\mathbf{25}$.

7. First count the number of arrangements in which Chris stands next to Charlie. This is $(T-1) \cdot 2! \cdot (T-2)! = 2 \cdot (T-1)!$ because there are $T - 1$ possible leftmost positions for the pair $\{\text{Charlie}, \text{Chris}\}$, there are $2!$ orderings of this pair, and there are $(T-2)!$ ways to arrange the remaining people. There are equally many arrangements in which Abby stands next to Charlie. However, adding these overcounts the arrangements in which Abby, Charlie, and Chris are standing next to each other, with Charlie in the middle. Using similar reasoning as above, there are $(T-2) \cdot 2! \cdot (T-3)! = 2 \cdot (T-2)!$ such arrangements. Hence the desired probability is $\dfrac{2 \cdot 2 \cdot (T-1)! - 2 \cdot (T-2)!}{T!} = \dfrac{2 \cdot (T-2)!(2T-2-1)}{T!} = \dfrac{2(2T-3)}{T(T-1)}$. With $T = 25$, the fraction simplifies to $\mathbf{\dfrac{47}{300}}$.

8. The given conditions are equivalent to $\sin\alpha = A$ and $\cos\beta = B$. Using either the sum-to-product or the sine of a sum/difference identities, the desired expression is equivalent to $2(\sin\alpha)(\cos\beta) = 2 \cdot A \cdot B$. With $A = \frac{47}{300}$ and $B = \frac{12}{169}$, $2 \cdot A \cdot B = \frac{2 \cdot 47}{25 \cdot 169} = \mathbf{\frac{94}{4225}}$.

9. The cone's volume is $\frac{1}{3}\pi r^2(T - r^2)$. Maximizing this is equivalent to maximizing $x(T - x)$, where $x = r^2$. Using the formula for the vertex of a parabola (or the AM-GM inequality), the maximum value occurs when $x = \frac{T}{2}$. Hence $V = \frac{1}{3}\pi \cdot \frac{T}{2} \cdot \frac{T}{2} = \frac{\pi T^2}{12}$, and $\pi/V = 12/T^2$. With $T = 13$, $V = \mathbf{\frac{12}{169}}$.

10. Write $2 = \log 100$ and use the well-known properties for the sum/difference of two logs to obtain $\log T = \log\left(\frac{100k}{2}\right)$, hence $k = \frac{T}{50}$. With $T = 650$, $k = \mathbf{13}$.

11. Let D be the distance in miles traveled by the plane. The given conditions imply that $\frac{D}{T} - \frac{D}{T+30} = 1.5 \Rightarrow \frac{30D}{T(T+30)} = 1.5 \Rightarrow D = \frac{T(T+30)}{20}$. With $T = 100, D = 5 \cdot 130 = \mathbf{650}$.

12. The given radical equals $\left(\left(\left(10^{T^2-T}\right)^{\frac{1}{T}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 10^{(T-1)/4}$. With $T = 9$, this simplifies to $10^2 = \mathbf{100}$.

13. Because $\overline{SU}$ and $\overline{RE}$ are sides of the hexagon, $SU = RE = \sqrt{T}$. Let H be the foot of the altitude from R to $\overline{BE}$ in $\triangle BRE$ and note that each interior angle of a regular hexagon is 120°. Thus $BE = BH + HE = 2\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{T}\right) = \sqrt{3T}$. Thus $BE \cdot SU \cdot RE = \sqrt{3T} \cdot \sqrt{T} \cdot \sqrt{T} = T\sqrt{3T}$. With $T = 3$, the answer is $\mathbf{9}$.

14. Using the Multiplication Principle, Chef Selma can make $4 \cdot 3 \cdot 2 \cdot K = 24K$ different burrito varieties. With $T = 70$, the smallest integral value of K such that $24K \geq 70$ is $\left\lceil \frac{70}{24} \right\rceil = \mathbf{3}$.

15. Because $\gcd(14, 20) = 2$, the problem is equivalent to computing the smallest positive integer N such that $7 \mid 10N$ and $10 \mid 7N$. Thus $7 \mid N$ and $10 \mid N$, and the desired value of N is $\operatorname{lcm}(7, 10) = \mathbf{70}$.

Part II

ARML Local Contests

ARML Local 2009

Contents

Team Problems

1. Suppose that $3 + 2i$ is one root of a quadratic function $f(x) = x^2 + Ax + B$, where A and B are real numbers. Compute the ordered pair (A, B).

2. If $f(x)$ is a line of slope -3, compute the slope of the line $f(f(f(x))) + f(f(x)) + f(x)$.

3. A six-sided die has these markings on its sides: \$12, \$12, \$16, \$16, \$26, and \$26. The following game is played. The die is rolled and the player wins whatever amount of money comes up. The player keeps rolling the die and winning money until a money amount comes up a second time, in which case the game ends. A sample game would be rolls of \$16, \$12, and \$16, in which case the player wins \$44. Compute the expected winnings by the player in this game.

4. If $f(x) = \log_x(x + 2)$, compute $3^{f(3)+f(9)}$.

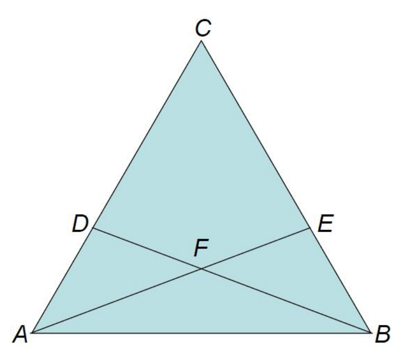
5. In an equiangular hexagon $ABCDEF$, $AB = BC = 3$, $CD = 8$, and $EF = 5$. Compute the area of $ABCDEF$.

6. Compute the number of values of θ with $0 \le \theta \le 2\pi$ for which $|\sin\theta| + |\cos\theta| = \dfrac{4}{3}$.

7. You are playing a game. Your opponent has distributed five red balls into five boxes randomly. All arrangements are equally likely; that is, the left-to-right placements $[0, 0, 5, 0, 0]$, $[0, 2, 0, 3, 0]$, and $[1, 1, 1, 1, 1]$ are equally likely. You place one blue ball into each box. The player with the most balls in a box wins the box (neither player wins a box with the same number of balls of each color). Whoever wins the most boxes wins the game. Compute the probability you win the game.

8. When $(x + y + z)^{2009}$ is expanded and like terms are grouped, there are k terms with coefficients that are *not* multiples of five. Compute k.

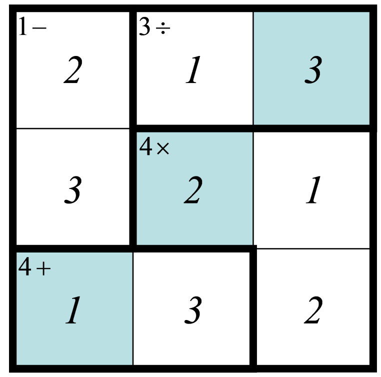
9. On equilateral triangle ABC, points D and E are on sides $\overline{AC}$ and $\overline{BC}$ such that $AD = BE = 1$. $\overline{BD}$ and $\overline{AE}$ meet at F. Given $AB = 3$, compute the area of triangle ADF.



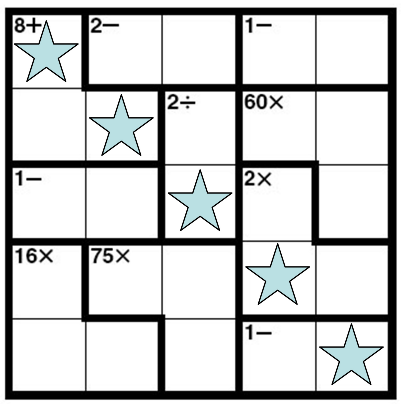
10. To solve a KenKen puzzle, you fill in an $n \times n$ grid with the digits $1, \ldots, n$ according to the following two rules:

    1. Each row and column contains exactly one of each digit.

    2. Each bold-outlined group of cells is a cage containing digits which achieve the specified result using the specified mathematical operation: addition (+), subtraction (-), multiplication (×), and division (÷) on the digits in some order. Digits may repeat inside a cage.



A solved 3×3 KenKen appears to the right. Below is a 5×5 KenKen puzzle. On your answer sheet, enter the digits in the starred squares in the correctly solved KenKen puzzle, *in order from left to right*. In the example, you would enter $1, 2, 3$. Digits may appear more than once in the starred squares.



Team Solutions

1. The other root of this quadratic equation is $3 - 2i$. A is -1 times the sum of the roots (6), B is the product of the roots (13), so the ordered pair is $\boxed{(-6, 13)}$.

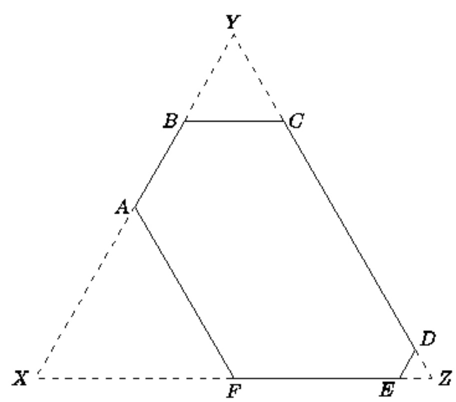
2. As the line is unspecified, let $f(x) = -3x$. $f(f(x)) = f(-3x) = -3(-3x) = 9x$ and $f(f(f(x))) = f(9x) = -3(9x) = -27x$. The sum of these three functions is $-21x$, so the slope is $\boxed{-21}$.

3. The expected amount of money earned per throw is \$18 (the average of the values on the faces), so the problem reduces to determining the expected number of throws. The game will end in 2 to 4 throws, with 2 throws occurring 1/3 of the time, 3 throws occurring 4/9 of the time (the probability the second throw is different than the first is 2/3, the probability the third throw is the same as either the first or second is 2/3), and 4 throws is $1-1/3-4/9 = 2/9$. Thus, the expected number of throws is $2\times(1/3)+3\times(4/9)+4\times(2/9) = 26/9$, so the expected winnings is $(26/9) \times \$18 = \boxed{\$52}$.

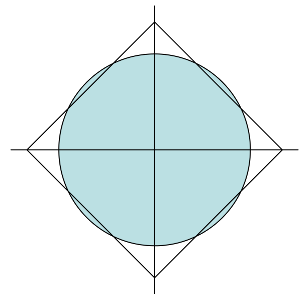
4. $3^{f(3)+f(9)} = 3^{\log_3 5+\log_9 11} = 3^{\log_3 5} \times 3^{\log_9 11} = 5 \times \left(9^{1/2}\right)^{\log_9 11} = 5 \times \left(9^{\log_9 11}\right)^{1/2} = \boxed{5\sqrt{11}}$.

5. Extend the sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ to obtain an equilateral triangle XYZ as shown. If we set $DE = u$ and $AF = v$, then we have $3 + 3 + v = 3 + 8 + u = u + 5 + v$, from which we conclude that $u = 1$ and $v = 6$. By subtraction of areas, we get:

$$[ABCDEF] = [XYZ] - [AFX] - [BYC] - [DZE] = \frac{\sqrt{3}}{4}\left(12^2 - 6^2 - 3^2 - 1^2\right) = \boxed{\frac{49\sqrt{3}}{2}}.$$



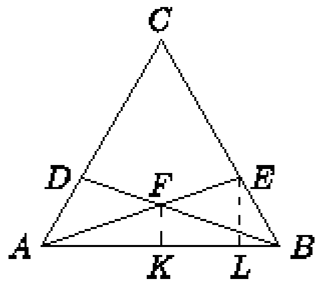
6. Points on the unit circle have coordinates $(\cos\theta, \sin\theta)$ for some value of θ, $0 \le \theta \le 2\pi$. So, we can view these points on the plane. $|x| + |y| = k$ describes a square with corners at $(\pm k, 0)$ and $(0, \pm k)$. Each edge of this square intersects the unit circle twice, provided $1 < k < \sqrt{2}$, so there are $\boxed{8}$ points of intersection.



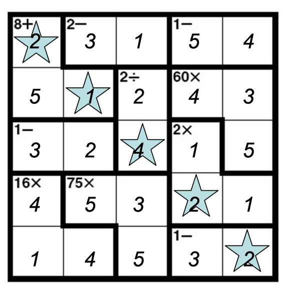
7. There are $\binom{5+5-1}{5-1} = 126$ arrangements of the red balls, all equally likely. You win if and only if your opponent puts three or more red balls in any single box. For example: [3,1,1,0,0] means that your opponent wins the first box and you win the last two. There are five arrangements with five balls in one box, twenty arrangements with four in one box and a single ball in the other, and fifty arrangements with three in one box, and either two in one other box or a single ball in two boxes. The probability of winning is $\frac{50+20+5}{126} = \frac{75}{126} = \boxed{\frac{25}{42}}$.

8. The coefficient of the term $x^a y^b z^c$ is $\frac{2009!}{a!b!c!}$, where we assume $a + b + c = 2009$. The number of powers of five in the numerator is $\left\lfloor \frac{2009}{5} \right\rfloor + \left\lfloor \frac{2009}{5^2} \right\rfloor + \left\lfloor \frac{2009}{5^3} \right\rfloor + \left\lfloor \frac{2009}{5^4} \right\rfloor = 401 + 80 + 16 + 3$. In order for this to equal the number of powers of five in the denominator, we need $\left\lfloor \frac{a}{5^k} \right\rfloor + \left\lfloor \frac{b}{5^k} \right\rfloor + \left\lfloor \frac{c}{5^k} \right\rfloor = \left\lfloor \frac{2009}{5^k} \right\rfloor$ for all k. Note that the floor function is sub-additive, so we will always have $LHS \le RHS$. Noting that $2009_{10} = 31014_5$, we first distribute the three powers of 5^4 among a, b, and c. There are $\binom{3+3-1}{3-1} = 10$ ways to place them. After doing this, we have placed 15 of the 16 powers of 5^3, so we only get to distribute one more, and there are $\binom{1+3-1}{3-1} = 3$ places for it to go. After this, all 80 powers of 5^2 have been taken care of, and we have 1 leftover power of 5 to place (3 places for it to go as well). After this, we have coefficients summing to 2005, so we can place the remaining 4 powers anywhere: there are $\binom{4+3-1}{3-1} = 15$ places for these. Overall, there are $10 \times 3 \times 1 \times 3 \times 15 = \boxed{1350}$ terms whose coefficients are not multiples of 5.

9. The area equals $\dfrac{AD \cdot AF \cdot \sin \angle DAF}{2} = \dfrac{AF \cdot \sin \angle CAE}{2}$. Dropping perpendiculars from E and F to $\overline{AB}$ at points K and L we get $BL = \frac{1}{2}$ and $\frac{AF}{AE} = \frac{AK}{AL} = \frac{3/2}{5/2} = \frac{3}{5}$. The area of triangle ADF is: $\frac{3}{10} \cdot AE \cdot \sin \angle CAE = \frac{3}{10} \cdot CE \cdot \sin \angle ACE = \frac{3}{10} \cdot 2 \cdot \frac{\sqrt{3}}{2} = \boxed{\dfrac{3\sqrt{3}}{10}}$ using the Law of Sines on $\triangle ACE$.



10. The answer is $\boxed{2, 1, 4, 2, 2}$, and the solved puzzle is shown below. There are many ways to get to the final solution. The key thing to note is the unique arrangements of digits in the L-shaped 75×, 16×, and 2× cages immediately give you the lower half of the puzzle, since the 1− cage in the lower right corner is uniquely determined by the other digits in the L-shaped cages.



Theme Problems

Votes That Count

***"Those who cast the votes decide nothing.
Those who count the votes decide everything."***

In this round, we will be looking at a number of different voting systems, as well as how voters choose who to vote for. In the simplest election, voters each choose a single candidate that they prefer over all of the others and the candidate with the most votes wins (this is called *plurality voting*). A slightly more complex election requires the winning candidate to receive greater than half of the votes (this is called *majority voting*), with a runoff between the two candidates receiving the most votes in the first round if no candidate has a majority (for example, the 2008 Georgia senate race).

Part 1: Preference Rankings

It is reasonable to assume that given a pair of candidates A and B, a voter either prefers one over the other ($A < B$ or $A > B$) or is indifferent between the two ($A = B$). Also, given a set of candidates, it is reasonable to assume these preference relationships are transitive ($A \leq B$ and $B \leq C \to A \leq C$). Therefore, we can determine a preference ranking for each voter. For example, if a voter prefers candidate A over all others, is indifferent between candidates B and C, but prefer both of them over candidate D, then that voters preference ranking would be $A > B, C > D$. We can also write preference rankings vertically, with candidates on the same row being equally preferable: $\begin{matrix} A \\ BC \\ D \end{matrix}$.

1. Compute the number of distinct preference rankings of four candidates.

   Note that $\begin{matrix} A \\ BC \\ D \end{matrix}$ and $\begin{matrix} A \\ CB \\ D \end{matrix}$ are *not* distinct preference rankings.

A preference ranking is called *strict* if the voter always prefers one candidate over the other. For the remainder of the questions in this round, we will assume all preference rankings are strict.

Consider an election with four voters (w, x, y, and z) and three candidates (A, B, and C). The voters' preference rankings are $\begin{matrix} w & x & y & z \\ A & A & B & B \\ B & C & C & C \\ C & B & A & A \end{matrix}$. Based on first place votes alone, candidates A and B are tied 2-2.

One way to break the tie is to perform a *Borda count*: For each candidate (for example, A), and each voter (for example, w), count the number of candidates that the voter w prefers A over (in this example, 2). Sum these values over all voters, and the candidate with the highest sum wins. In this case, B is the winner of the Borda count.

Candidate	w	x	y	z	Total
A	2	2	0	0	4
B	1	0	2	2	5
C	0	1	1	1	3

2. Consider an election with m candidates and n voters where one candidate has greater than half of the first place votes, but loses a Borda count, we'll call this situation an (m, n)-*Borda upset*. Compute the ordered pair (m, n) such that an (m, n)-Borda upset exists, and $m + n$ is minimized.

Another voting method using preference rankings is called *instant runoff voting* (IRV). In one version of IRV, all voters submit strict preference rankings of candidates. The following process is used to determine the winner:

```
Step 1:  Does any candidate have a majority of voters that have them as
           their top choice?
         If yes, then that candidate is the winner.  END
         If not, go to Step 2.

Step 2:  Is there a unique candidate with the fewest first place votes?
         If yes, then remove that candidate from ALL preference rankings
           and go to Step 1.
         If no, then some other process must be used to determine who is
           removed from the election.
```

So, for example, for three candidates A, B, and C, if there are 10 voters with preference ranking $\begin{array}{c}A\\B\\C\end{array}$, 8 with preference ranking $\begin{array}{c}B\\A\\C\end{array}$, and 6 voters with a preference ranking $\begin{array}{c}C\\B\\A\end{array}$, then in the first round of IRV, candidate C would be eliminated, leaving 10 voters with preference ranking $\begin{array}{c}A\\B\end{array}$ and 14 voters with preference ranking $\begin{array}{c}B\\A\end{array}$. Therefore, candidate B wins this IRV election, despite the fact that candidate A had more first-place votes than any other candidate when the process began. When this happens with m candidates and n voters, we call this a (m, n)-*IRV upset*. Note that if there is a tie for last place at any stage, then the IRV election does not produce a winning candidate, so there must always be a unique last place candidate at every stage for the process to conclude.

3. Let n_3 and n_4 be the minimum values of n such that there exists a $(3, n)$-IRV upset and $(4, n)$-IRV upset, respectively. Compute the ordered pair (n_3, n_4).

Part 2: Winning Coalitions and Relative Power in Elections

To win the presidential election, a candidate needs a majority of the 538 electoral votes (that is, 270). Most states distribute all of their electoral votes to the candidate that gets the most votes in their statewide election. However, is California (with 55 electoral votes) really eleven times as important as, say, West Virginia (with 5 electoral votes)? The answer is, naturally, it depends.

Given n voters (which we will denote with the set $S = \{1, \dots, n\}$), every subset of S is either a winning coalition or a losing coalition. Winning coalitions are *monotonic*, that is, if W is a winning coalition and V contains W, then V is also a winning coalition.

We define a *weighted majority vote* (WMV) as follows: given an integer threshold T and that each voter i has v_i votes (v_i all integers) a subset W of S is a winning coalition of a WMV if and only if $\sum_{i \in W} v_i \geq T$. We can represent a WMV by $(T; v_1, v_2, \dots, v_n)$. For example, the US presidential election (ignoring the fact that some states that split their electoral votes) can be represented as a WMV: $(270; 55, 34, 31, 27, \dots, 3, 3)$.

Many elections or voting processes can be represented as WMVs. On some issues, the Australian government has votes involving the six state legislatures plus the federal government. Each of the six states gets one vote, while the federal government gets two. In the case of a 4-4 tie, the federal government makes the decision. It is easy to show that this voting process is equivalent to the WMV $(5; 1, 1, 1, 1, 1, 1, 3)$, with the three votes going to the federal government: a coalition is a winning coalition if and only if it contains at least five states or the federal government and at least two states.

4. In the UN Security Council, there are five permanent members and ten non-permanent members. For any resolution to pass, it must have the support of *all five* of the permanent members and at least four of the ten non-permanent members. Consider the WMV $(T; P, P, P, P, P, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ where P denotes the integral number of votes for each of the permanent members of the council and the ones denote the votes of the non-permanent members of the council. Compute the ordered pair (T, P) such that T is minimized and W is a winning coalition in this WMV if and only if the corresponding set of council members can pass a resolution in the UN Security Council.

To measure the relative power of a voter, we consider the *Shapley-Shubik Power Index.* It is defined as follows.

Consider all $n!$ orderings of n voters. Given an ordering$(p_1, \dots, p_n)$ of the set $\{1, \dots, n\}$, let $W_i = \bigcup_{j \leq i} \{p_j\}$. A voter p_k is *pivotal* for this ordering if W_{k-1} is not a winning coalition, but W_k is a winning coalition. The Shapley-Shubik Power Index for a voter is the fraction of the orderings for which the voter is pivotal.

For example, consider the WMV $(7; 5, 4, 2)$. There are six orderings of the voters, and we will underline the pivotal voter in each case: $24\underline{5}$ $2\underline{5}4$ $42\underline{5}$ $4\underline{5}2$ $5\underline{2}4$ $5\underline{4}2$. Thus, the Shapley-Shubik Power

Index of the first voter is 4/6, or 2/3, while the Shapley-Shubik Power Index of the other two voters are 1/6 each.

5. Consider the WMV for the Australian federal government: $(5; 1, 1, 1, 1, 1, 1, 3)$. Let s be the Shapley-Shubik Power Index of any one of the states and let f be the Shapley-Shubik Power Index of the federal government. Compute the ordered pair (s, f).

There are other ways to define winning coalitions. In the US legislature, a bill is sent to the president for signature ("winning") if and only if it has a coalition of at least 218 of the 435 members of the House and at least 51 of the 101 members of the Senate (including the Vice President in the Senate count, the Vice President is not required to be a part of a coalition for it to be a winning coalition) supporting it.[1]

6. Compute the Shapley-Shubik Power Index of the Vice President.

If $W \subseteq S = \{1, 2, \ldots, n\}$ is a winning coalition, $i \in W$ is called a *swing voter* for W if $W - \{i\}$ is a losing coalition. We can define another power index called the *Banzhaf-Penrose Power Index* as follows. Let s_i be the number of winning coalitions for which voter i is a swing voter. The Banzhaf-Penrose Power Index of voter i is $B_i = \dfrac{s_i}{\sum_{j=1}^{n} s_j}$.

Going back to our earlier WMV example, $(7; 5, 4, 2)$, there are three winning coalitions, $\{5, 4\}$, $\{5, 2\}$, and $\{5, 4, 2\}$. Both voters are swing voters in the first two coalitions, but only the 5 in the third coalition is a swing voter. Accordingly, there are five swing voters in total, so the Banzhaf-Penrose Power Indices for the three voters are 3/5, 1/5, and 1/5, respectively.

7. Consider the WMV $(6; 1, 2, 3, 4)$. Compute the ordered 4-tuple (B_1, B_2, B_3, B_4), that is, the Banzhaf-Penrose Power Index for each voter.

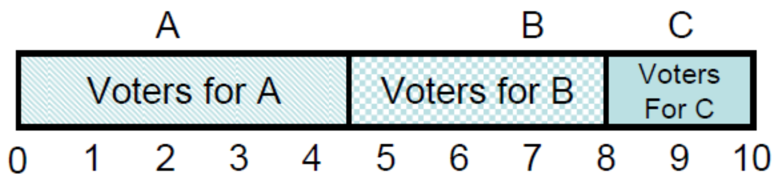
Part 3: Models of Voter Preference

There are many ways that voters come up with their preference rankings for candidates. Some people are single-issue voters. That is, their preference ranking depends on how close a candidate's view on a single issue is to their own. Say a candidate's view on an issue could be represented as a real number on a 0 to 10 scale, and there were three candidates (A, B, C) whose views on an issue are 2, 7, and 9. Then, a single-issue voter whose view on an issue was 5 would have a preference ranking of $B > A > C$ since the distance between the voter's view on the issue and those of the candidates are 3, 2, and 4, respectively.

If all voters were single issue voters and voters' views on the issue were evenly distributed throughout the 0–10 scale, then candidate A would win a plurality election, as voters with views between

[1] Technically, the Vice President does not vote, except in the case of ties. It is equivalent, however, to let the vice president vote all the time, since his vote only matters when the other 100 senators are tied 50-50.

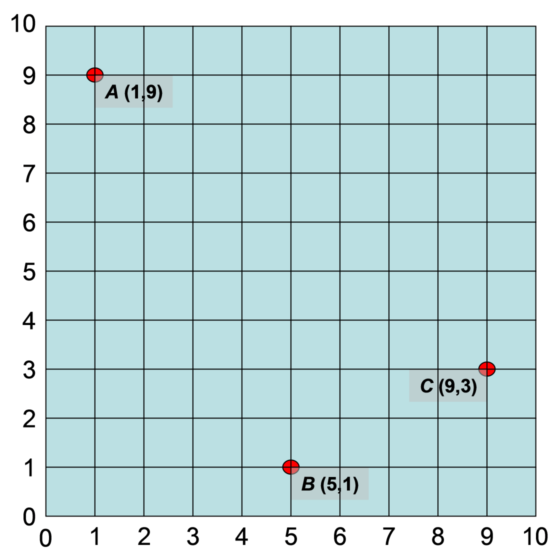
0 and 4.5 (45%) would vote for candidate A, those with views between 4.5 and 8 (35%) would vote for candidate B, and those with views between 8 and 10 (20%) would vote for candidate C. We will call this a *single-issue election*.



8. In a single-issue election, candidates A and B have randomly picked their views on the issue (all views are equally likely and the views of candidates A and B are independent). Candidate C, knowing the views of candidates A and B, can choose his view to maximize the fraction of the votes he receives. Compute the probability that it is impossible for candidate C to pick a view that results in candidate C winning the election.

We would hope that voters are a bit more nuanced than this and perhaps base their preferences on *two* issues!

Assume there are three candidates A, B, and C with views on two issues (again, on a 0–10 scale) as given in the graph below.



A voter sets their preference ranking depending on the (Cartesian) distance between their views on these two topics and those of the candidate. If voters' views on both topics were independent and evenly distributed throughout the 0–10 scales, then the fraction of the votes received by a candidate would be equal to the fraction of the square $[0, 10] \times [0, 10]$ that is closest to the point corresponding to the candidate's views.

9. Under the above assumption, if f_i is the fraction of the vote received by candidate i, compute the ordered triple (f_A, f_B, f_C). It is acceptable to give the answer as an ordered triple of fractions or percentages.

10. A voter in this election with views (x_v, y_v) is indifferent between all three candidates. Compute the ordered pair (x_v, y_v).

Theme Solutions

1. Let the *height* of a preference ranking be the number of rows in the vertical representation of a preference ranking. There are $4! = 24$ preference rankings of height 4, and one preference ranking of height 1. For height 3, there are three rows to choose where the tie should go, then $\binom{4}{2} = 6$ ways to pick the candidates in the tie, then 2 ways to pick the highest ranked remaining candidate, for a total of 36 rankings of height 3. Finally, a ranking of height two can either be two pairs or a single candidate either above or below three indifferent candidates. We know there are six arrangements of the first type and eight of the second (choose one of the four candidates then choose to put him above or below the other three). In total, there are $\boxed{75}$ preference rankings of four candidates.

2. Clearly there cannot be a Borda upset with two candidates, so $m \geq 3$. With three candidates, and an odd number of voters, a Borda upset occurs when there are k voters with preference ranking $\begin{matrix}A\\B\\C\end{matrix}$ and $k-1$ voters with preference ranking $\begin{matrix}B\\C\\A\end{matrix}$, provided that $2k < 3k-2 \to k > 2$, so there exists a $(3,5)$-Borda upset. With four candidates, there exists a three voter Borda upset: $\begin{matrix}A & A & B\\B & B & C\\C & C & D\\D & D & A\end{matrix}$, with A having a majority of first place votes, but losing a Borda count to B, 7 to 6. The answer is $\boxed{(4,3)}$.

3. With three candidates, we need a distribution of first place votes v_A, v_B, and v_C such that $v_A > v_B > v_C$ and $v_A < v_B + v_C$. Then, we have a $(3, v_A + v_B + v_C)$-IRV upset with v_A rankings of $\begin{matrix}A\\B\\C\end{matrix}$ or $\begin{matrix}A\\C\\B\end{matrix}$, v_B rankings of $\begin{matrix}B\\A\\C\end{matrix}$ or $\begin{matrix}B\\C\\A\end{matrix}$, and v_C rankings of $\begin{matrix}C\\B\\A\end{matrix}$. Then, after the first round, when candidate C is eliminated, there are v_A voters with preference ranking $\begin{matrix}A\\B\end{matrix}$ and $v_B + v_C$ voters with preference ranking $\begin{matrix}B\\A\end{matrix}$, an upset. The ordered triple of positive integers that satisfies the inequalities with the sum $v_A + v_B + v_C$ minimized is $(v_A, v_B, v_C) = (4,3,2)$, so $n_3 = 9$. With four candidates, consider three voters with preference ranking $\begin{matrix}A\\B\\C\\D\end{matrix}$, two with $\begin{matrix}B\\A\\C\\D\end{matrix}$, two with $\begin{matrix}C\\B\\A\\D\end{matrix}$ and one with $\begin{matrix}D\\B\\C\\A\end{matrix}$. Candidate D is eliminated in the first round, transferring one first place vote to candidate B. Candidate C is eliminated in the second round, transferring two first place votes to candidate B, who now has a majority of first place votes. It is easy to verify there are no smaller values of n_4 than 8 that lead to a $(4, n_4)$-IRV upset. The answer to the question is $\boxed{(9,8)}$.

4. The smallest winning coalition is all five permanent members of the council with four non-permanent members. Therefore, $5P + 4 \geq T$. To minimize T, we say that $5P + 4 = T$. There

are two maximal non-winning coalitions: one consists of the five permanent members as well as three non-permanent members, so $5P + 3 < T$. The second consists of four permanent members and all ten of the non-permanent members, so $4P + 10 < T$. From the inequalities, we can establish that $4P + 10 < 5P + 4 \to 6 < P$, so $P = 7$ in order to minimize T and $T = 39$. The answer is $\boxed{(39, 7)}$.

5. To be the pivotal voter, the federal government's three votes needs to be in either the third, fourth, or fifth position. There are $7! = 5040$ orderings of the voters, of which $3 \times 6! = 2160$ have the federal government in the third, fourth, or fifth position, so the Shapley-Shubik Power Index of the federal government is $\frac{2160}{5040} = \frac{3}{7}$. The sum of the power indices across all voters is one, and all of the states are equally powerful, so $6s + \frac{3}{7} = 1$. This simplifies to $6s = \frac{4}{7} \to s = \frac{4}{42} = \frac{2}{21}$. The answer is $\boxed{\left(\frac{2}{21}, \frac{3}{7}\right)}$.

6. We first claim that the sum of the power indices over all of the members of the House is equal to the sum of the power indices over all of the members of the Senate, namely $\frac{1}{2}$. For every ordering of the members of the House and Senate where a House member is pivotal, the reverse ordering of the House and Senate members has a Senate member as pivotal, and vice versa. Since all Senate members have the same Shapley-Shubik Power Index, the power index of the vice president is $\frac{1/2}{101} = \boxed{\frac{1}{202}}$.

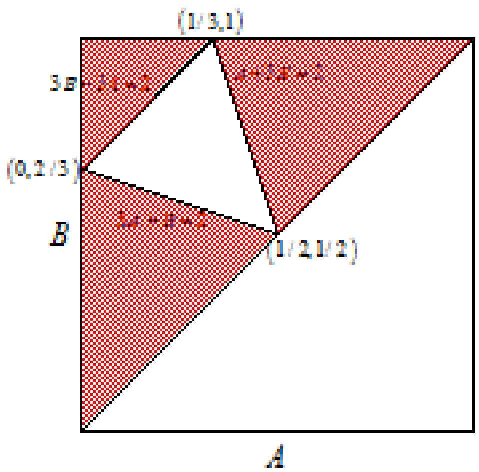
7. We list the winning coalitions and underline all swing voters: $\underline{2}\underline{4}$, $\underline{3}\underline{4}$, $\underline{1}\underline{2}\underline{3}$, $1\underline{2}\underline{4}$, $1\underline{3}\underline{4}$, $23\underline{4}$, and 1234. There are $2 + 2 + 3 + 2 + 2 + 1 = 12$ swing voters in total, so $(B_1, B_2, B_3, B_4) = \left(\frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{5}{12}\right) = \boxed{\left(\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{5}{12}\right)}$.

8. We can simplify calculations by setting the range of views on the single-issue election to be from 0 to 1. Then the probability that it is impossible for candidate C to pick a winning view is equal to the area of the set of points (A, B) such that it is impossible for candidate C to pick a winning view, given the other two views are A and B. It is easier to find the set of points where C can pick a winning view. Assume without loss of generality that $0 \le A \le B \le 1$. There are three cases to consider:

Cases	Strategy for C	Votes for A	Votes for B	Votes for C
$C < A$	Pick just less than A	$\frac{B-A}{2}$	$\frac{B-A}{2} + (1 - B)$	A
$A < C < B$	See below	$A + \frac{C-A}{2}$	$\frac{B-C}{2} + (1 - B)$	$\frac{B-A}{2}$
$B < C$	Pick just more than B	$A + \frac{B-A}{2}$	$\frac{B-A}{2}$	$(1 - B)$

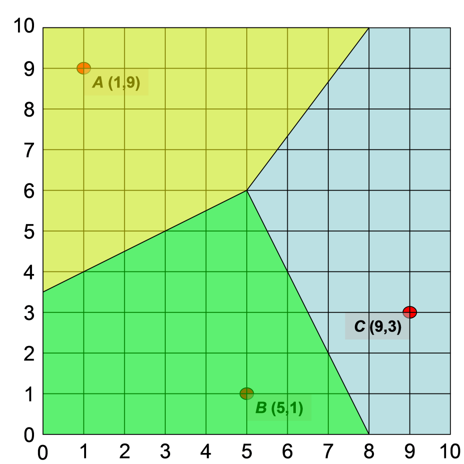
In the first case, candidate C wins if $\frac{B-A}{2} + (1 - B) < A$ or $2 < 3A + B$. In the third case, candidate C wins if $A + \frac{B-A}{2} < 1 - B$ or $2 > A + 3B$. In the second case, candidate C wins if there is a number between A and B such that $A + \frac{C-A}{2} < \frac{B-A}{2}$ and $\frac{B-C}{2} + (1 - B) < \frac{B-A}{2}$, or $C < B - 2A$ and $C > A + 2(1 - B)$. Since we know that $B - 2A \le 1$ and $0 \le A + 2(1 - B)$, there is a winning pick for C provided $A + 2(1 - B) < B - 2A$ or $2 < 3B - 3A$.

Plotting these three regions on the $AB-$plane, we see the region where C cannot win (the clear section). The triangle has a base and height of $\frac{\sqrt{2}}{3}$, and area $\frac{1}{9}$. By symmetry, there is an identical triangle on the other side of the line $A = B$, so the answer is $\boxed{\dfrac{2}{9}}$.



9. The set of points equidistant from two candidates views forms the perpendicular bisector of the segment between the two views. Voters on either side of the bisector favor the respective candidate. When we show all three perpendicular bisectors on the graph, we get the partition of the voter space as seen to the right. Adding up areas of rectangles and triangles, we get the voter distribution $(32.25\%, 32.75\%, 35\%)$ or $\boxed{\left(\dfrac{129}{400}, \dfrac{131}{400}, \dfrac{7}{20}\right)}$.

10. Look at the graphic and note the bisectors all meet at $\boxed{(5, 6)}$. Alternately, you can find the center of the circle that goes through the three candidates' views using the method of your choice and get the same answer.



Individual Problems

1. Given $A \wedge B = \dfrac{A+B}{AB}$, compute $(2 \wedge 6) \wedge (3 \wedge 4)$.

2. Compute the smallest value of n for which the mean and median of the set $\{13, 21, 26, 28, 41, n\}$ are equal.

3. Compute the number of subsets S of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the smallest element of S is equal to the size of S.

4. $A_1 A_2 \ldots A_{2009}$ is a regular 2009-gon of area K. If $A_1 A_{1005} = 6$, compute $\lfloor K \rfloor$.

5. Let x, y, and z be real numbers randomly chosen between 0 and 100. Compute the probability that $\lceil x + y + z \rceil = \lceil x \rceil + \lceil y \rceil + \lceil z \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

6. An integer is called "rotatable" if it only uses the digits 0, 1, 6, 8, or 9. Compute the sum of all three-digit rotatable numbers (note that the first digit cannot be a zero).

7. Compute the number of ways to place the integers 1 through 7 in the blanks below so that the chain of inequalities is satisfied.

$$\_\_ < \_\_ < \_\_ > \_\_ < \_\_ > \_\_ > \_\_$$

$$(\text{Example}: \ 1 < 2 < 5 > 4 < 7 > 6 > 3)$$

8. The *period* of a sequence $s_0, s_1, \ldots$ is the smallest positive integer k such that $s_n = s_{n+k}$ for all $n \geq 0$. Compute the period of the sequence defined by $s_0 = \tan 4^\circ$, $s_n = \dfrac{2s_{n-1}}{1 - s_{n-1}^2}$ for $n > 0$.

9. For a positive integer n, define a_n to be the smallest positive integer with exactly n^2 positive factors. Compute the smallest n for which $a_n > a_{n+1}$.

10. Let θ be an acute angle for which $\sin \theta$, $\sin 2\theta$, and $\sin 4\theta$ form a strictly increasing arithmetic sequence. Compute $\cos^3 \theta - \cos \theta$.

Individual Solutions

1. $(2 \wedge 6) \wedge (3 \wedge 4) = \dfrac{2+6}{2 \times 6} \wedge \dfrac{3+4}{3 \times 4} = \dfrac{8}{12} \wedge \dfrac{7}{12} = \dfrac{\frac{8}{12} + \frac{7}{12}}{\frac{8}{12} \times \frac{7}{12}} = \dfrac{\frac{15}{12}}{\frac{56}{144}} = \dfrac{15 \times 12}{56} = \boxed{\dfrac{45}{14}}$.

2. The mean of the set is $\frac{129+n}{6}$; the middle two elements of the set are either 21 and 26 (if $n \leq 21$), n and 26 (if $21 \leq n \leq 28$), or 26 and 28 (if $n \geq 28$). The median in these cases are $\frac{47}{2}$, $\frac{n+26}{2}$, and 27, respectively. Starting with the smallest possibility (the first one) and solving for n, we get $\frac{129+n}{6} = \frac{47}{2} \to \boxed{n = 12}$.

3. Condition off of k, the smallest element of S, with $1 \leq k \leq 5$. Then there are $k-1$ elements left to pick from $\{k+1, \ldots, 9\}$ giving us $\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4} = 1+7+15+10+1 = \boxed{34}$.

4. Consider the circle circumscribing the 2009-gon. Note that $A_1A_{1004} = 6$ as well, and $\angle A_{1004}A_1A_{1005} = \dfrac{\pi}{2009} = \alpha$. The diameter of the circle circumscribing the 2009-gon is $\dfrac{6}{\cos(\alpha/2)}$, so its radius is $r = \dfrac{3}{\cos(\alpha/2)} \approx 3$. The area of the polygon is $\dfrac{2009r^2}{2} \sin\left(\dfrac{2\pi}{2009}\right) \approx \dfrac{2009r^2}{2} \times \dfrac{2\pi}{2009} = \pi r^2 \approx 28.27$, so the answer is $\boxed{28}$. We did play a bit fast and loose with the explanation here, but basically, the area of the 2009-gon is so close to the area of its circumscribed circle, the effect on the area would not be noted in the first or second decimal place.

5. Write each number as $x = x_I + x_R$ where x_I is an integer and $0 < x_R \leq 1$, the integer part and the remainder. Then $\lceil x \rceil = \lceil x_I + x_R \rceil = x_I + \lceil x_R \rceil = x_I + 1$ and $\lceil x \rceil + \lceil y \rceil + \lceil z \rceil = x_I + y_I + z_I + 3$. $\lceil x+y+z \rceil = \lceil (x_I + y_I + z_I) + (x_R + y_R + z_R) \rceil = (x_I + y_I + z_I) + \lceil (x_R + y_R + z_R) \rceil$. Therefore, the two sides of the equation are equal if and only if the sum of the remainders is greater than two. The probability that the sum of three numbers randomly drawn from between 0 and 1 is greater than two is $\boxed{\dfrac{1}{6}}$. To see this, consider the volume of the intersection of the set $x + y + z \geq 2$ and the unit cube $(0, 1] \times (0, 1] \times (0, 1]$ in three-dimensional space.

6. There are 100 rotatable numbers, since there are four choices for the first digit and five choices apiece for the latter digits. For each of 1, 6, 8, and 9, there are 25 three-digit rotatable numbers that begin with that digit. The contribution to the sum of the hundreds place alone is $25\,(100 + 600 + 800 + 900) = 60000$. Similarly, for each of the digits 0, 1, 6, 8, and 9, there are 20 three-digit rotatable numbers that have that digit in the tens place and the ones place, contributing $20\,(00 + 10 + 60 + 80 + 90) = 4800$ and $20\,(0 + 1 + 6 + 8 + 9) = 480$ to the sum, respectively. The total is $\boxed{65280}$.

7. The 7 must go in the 3rd or 5th blank. Assume it goes in the 3rd blank. All of the placements with the 7 in the 5th blank can be obtained by flipping the ones for which it goes in the 3rd, so we will double our answer at the end. There are $\binom{6}{2} = 15$ ways to choose the two numbers to place in the first two blanks, and once chosen, they must be entered in increasing order. Of the 4 remaining numbers, the largest must go in the 5th blank. There are 3 integers left to

place. We have 3 choices of which one to place in the 4^{th} blank, and the remaining two must be placed in decreasing order in the last two blanks. Thus, counting the reversed placements, there are $15 \times 3 \times 2 = \boxed{90}$ valid placements.

8. The closed form of the sequence is $s_n = \tan(2^n \times 4^\circ)$, so we need to find the smallest $n > 0$ for which $\tan 4^\circ = \tan(2^n \times 4^\circ)$. Since the tangent function is periodic with period 180°, it is equivalent to find the smallest positive n for which 180 divides $4(2^n - 1)$. Dividing out the 4, we seek the smallest positive n such that 45 divides $2^n - 1$.

   We could brute-force this, but we note that $45 = 5 \times 9$, and $2^n - 1$ is a multiple of five when 4 divides n (this can be shown with Fermat's Little Theorem or listing the first few values of $2^n - 1$). Similarly, $2^n - 1$ is a multiple of nine when 6 divides n. Thus, the period is $\operatorname{lcm}\{4, 6\} = \boxed{12}$.

9. Recall that if $p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$ is the factorization of n into powers of distinct primes, then n has $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ positive factors. We start computing the first few values of a_n:

n	n^2	a_n
1	1	1
2	4	$2 \cdot 3 = 6$
3	9	$2^2 \cdot 3^2 = 36$
4	16	$2^3 \cdot 3 \cdot 5 = 120$
5	25	$2^4 \cdot 3^4 = 36^2$
6	36	$2^2 \cdot 3^2 \cdot 5 \cdot 7 = 36 \cdot 35$

   Which is good enough to show that the smallest n for which $a_n > a_{n+1}$ is $\boxed{5}$.

10. We have $\sin 4x - \sin 2x = \sin 2x - \sin x$. We will use double-angle identities to rewrite both sides of this equality:

$$\sin 4x - \sin 2x = \sin 2x - \sin x$$
$$\sin 4x - 2\sin 2x = -\sin x$$
$$4\cos 2x \cos x \sin x - 4\cos x \sin x = -\sin x$$
$$4(2\cos^2 x - 1)\cos x - 4\cos x = -1$$
$$8\cos^3 x - 8\cos x = -1$$
$$\Rightarrow \cos^3 x - \cos x = \boxed{-\frac{1}{8}} \text{ or } \boxed{-0.125}.$$

Relay Problems

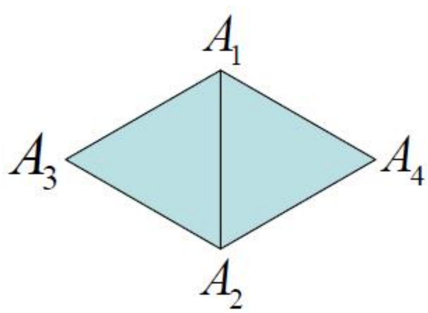
R1-1. Let $(47)_a = (74)_b$, where $(x)_c$ denotes the number represented by x written in base-c. Compute the smallest value for $a + b$ where a and b are both positive integers.

R1-2. Let $T = TNYWR$. Let y be the product of the digits of T and let z be the sum of the digits of T. The roots of the quadratic function $f(x) = x^2 + Ax + B$ have arithmetic mean y and geometric mean z. Compute the ordered pair (A, B).

R2-1. Let $f(x) = x^2 - 4^{\lceil \log_{10} x \rceil}$. Compute $f(f(7))$.

R2-2. Let $T = TNYWR$. $a_1, a_2, \ldots, a_7$ is an arithmetic sequence of increasing positive integers whose terms sum to T. Compute the smallest possible value for a_2.

R2-3. Let $T = TNYWR$. Compute the side length of the largest square that can be completely covered without overlap by at most T 2×3 tiles. Tiles may not be broken, but may be rotated, and not all T tiles must be used.



R3-1. Let M and N be distinct positive numbers such that $\log_M N = \log_N M$. Compute MN.

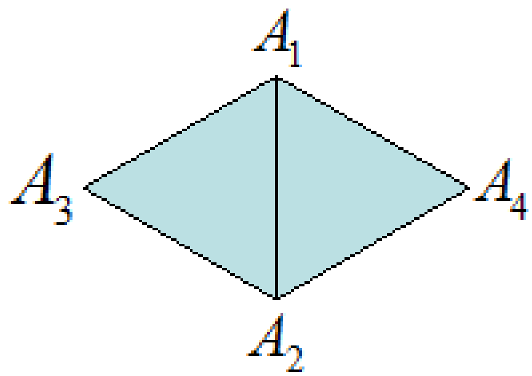
R3-2. Let $T = TNYWR$. The area bounded by the x-axis, the lines $x = T$, $x = 4$, and $y = mx$ is 15. Compute the largest value of m.

R3-3. Let $T = TNYWR$. An equilateral triangle and regular hexagon have equal perimeters. If the area of the triangle is T, compute the area of the hexagon.

R3-4. Let $T = TNYWR$. Let $A \wedge B = \dfrac{A+B}{AB}$ and $A \vee B = \dfrac{AB}{A+B}$. If $k \vee (k \wedge k) = \dfrac{k}{T^2}$, compute the greatest value of k.

R3-5. Let $T = TNYWR$. The sum of the two three-digit numbers $\underline{2}\,\underline{T}\,\underline{3}$ and $\underline{A}\,\underline{6}\,\underline{B}$ is a multiple of eleven. Compute the smallest possible value of $A + B$.

R3-6. Let $T = TNYWR$. $A_1A_2A_3 \ldots A_T$ and $A_1A_2A_{T+1} \ldots A_{2T-2}$ are distinct regular T-gons in the plane. Let S be the set of all real numbers that are distances between distinct vertices. Compute the number of elements in S. (In the example to the left, there are two distinct non-zero distances, A_1A_2 and A_3A_4.)



Relay Solutions

R1-1. $4a+7=7b+4 \to 4a+3=7b$. We need to find integral values of a and b, both greater than 7, such that this equality holds. b must be odd, and $(a,b)=(15,9)$ satisfies the equation, so $a+b=\boxed{24}$.

R1-2. $-A$ is the sum of the roots, which is twice the arithmetic mean of the digits (or $2y$). B is the product of the roots of $f(x)$, which is the square of the geometric mean (or z^2). As $y=2\times 4=8$ and $z=2+4=6$, the ordered pair $(A,B)=(-2\times 8, 6^2)=\boxed{(-16,36)}$.

R2-1. $f(f(7))=f\left(7^2-4^{\lceil \log_{10} 7\rceil}\right)=f\left(7^2-4^1\right)=f(45)=45^2-4^{\lceil \log_{10} 45\rceil}=45^2-4^2=2025-16=\boxed{2009}$.

R2-2. If $a_2=a_1+d$, then $T=a_1+\cdots+a_7=7a_1+21d=7(a_1+3d)$. As $T=2009$, $a_1+3d=287$. To minimize a_2, you minimize a_1. The smallest positive value for a_1 is 2, so $d=95$, and $a_2=2+95=\boxed{97}$.

R2-3. Since the area of the tile is six, the area of the square must be divisible by six, which also means that the side length of the square must be divisible by six. If the square has side length $L=6k$, the area of the square is $36k^2$, requiring $6k^2$ 2×3 tiles to cover. Since $T=97$, the largest value of k such that $6k^2\le 97$ is $k=4$, so $L=\boxed{24}$.

R3-1. Let $k=\log_M N=\log_N M$. Then $M^k=N$ and $N^k=M$. Combining the two equalities gives $(MN)^k=MN\to (MN)^{k-1}=1\to MN=\boxed{1}$.

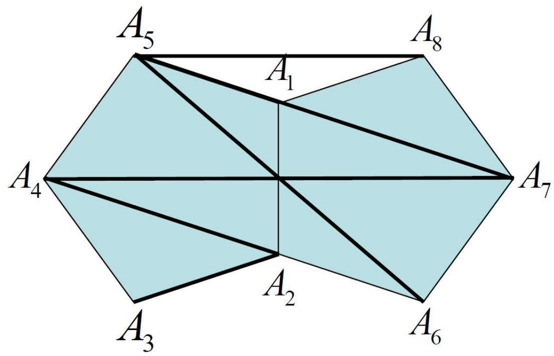
R3-2. The region bounded by the lines (assuming $T>0$) is a trapezoid with area $|4-T|\,\frac{(4+T)m}{2}$. As $T=1$, we get $|4-1|\,\frac{(4+1)m}{2}=15\to \frac{15m}{2}=15\to m=\boxed{2}$.

R3-3. The hexagon has sides with lengths equal to half that of the triangle. H be the side length of the hexagon. The area of the triangle is $\frac{(2H)^2\sqrt{3}}{4}=H^2\sqrt{3}$. The area of the hexagon is $6\frac{H^2\sqrt{3}}{4}=\frac{3}{2}H^2\sqrt{3}=\frac{3}{2}T=\boxed{3}$.

R3-4. $k\vee(k\wedge k)=k\vee\left(\dfrac{k+k}{k\times k}\right)=k\vee\dfrac{2}{k}=\dfrac{k\times\frac{2}{k}}{k+\frac{2}{k}}=\dfrac{2}{k+\frac{2}{k}}=\dfrac{2k}{k^2+2}=\dfrac{k}{T^2}\to T^2=\dfrac{k^2+2}{2}$. As $T=3\to k=\pm 4$. The greatest value of k is $\boxed{4}$.

R3-5. Note that $\underline{2}\,\underline{T}\,\underline{3} \equiv 3 - T + 2 \pmod{11}$ and $\underline{A}\,\underline{6}\,\underline{B} \equiv B - 6 + A \pmod{11}$, so $\underline{2}\,\underline{T}\,\underline{3} + \underline{A}\,\underline{6}\,\underline{B} \equiv 3 - T + 2 + B - 6 + A \equiv A + B - (T+1) \pmod{11}$. As $T = 4$, then $A + B \equiv 5 \pmod{11}$. The smallest possible value of $A + B$ is $\boxed{5}$.

R3-6. There are $\left\lfloor \frac{T}{2} \right\rfloor$ distinct distances between points inside a regular T-gon. There are $\left\lfloor \frac{T-2}{2} \right\rfloor (T-2)$ distinct distances between vertices in the separate T-gons when T is even, $\left\lfloor \frac{T-2}{2} \right\rfloor (T-2) + 1$ if T is odd. As $T = 5$, there are two intra-pentagon distances (A_2A_3 and A_2A_4), and four inter-pentagon distances (A_4A_7, A_5A_6, A_5A_7, and A_5A_8), the distances in the two sets are distinct, so the total is $\boxed{6}$.



Final note: The authors note that the sets of intra- and inter-polygon distances do not always contain distinct elements. We welcome any proofs or comments regarding this problem for other values of T.

Tiebreaker

There was no tiebreaker problem in 2009. Instead, the top individual was determined by a game of round-robin Blotto (see: `http://en.wikipedia.org/wiki/Blotto_games`).

Answers to ARML Local 2009

Team Round:

1. $(-6, 13)$	2. -21	3. 52	4. $5\sqrt{11}$	5. $\dfrac{49\sqrt{3}}{2}$
6. 8	7. $\dfrac{25}{42}$	8. 1350	9. $\dfrac{3\sqrt{3}}{10}$	10. $2, 1, 4, 2, 2$

Theme Round:

1. 75	2. $(4, 3)$	3. $(9, 8)$	4. $(39, 7)$	5. $\left(\dfrac{2}{21}, \dfrac{3}{7}\right)$
6. $\dfrac{1}{202}$	7. $\left(\dfrac{1}{12}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{5}{12}\right)$	8. $\dfrac{2}{9}$	9. $\left(\dfrac{129}{400}, \dfrac{131}{400}, \dfrac{7}{20}\right)$	10. $(5, 6)$

Individual Round:

1. $\dfrac{45}{14}$	2. 12	3. 34	4. 28	5. $\dfrac{1}{6}$
6. 65280	7. 90	8. 12	9. 5	10. $-\dfrac{1}{8}$

Relay Round:

Relay 1:	1. 24	2. $(-16, 36)$				
Relay 2:	1. 2009	2. 97	3. 24			
Relay 3:	1. 1	2. 2	3. 3	4. 4	5. 5	6. 6

ARML Local 2010

Contents

Team Problems

1. Let S be the set of positive factors of 2010. Compute the median of S.

2. Point P is located in the interior of rectangle $ABCD$ and $PA = 56$, $PB = 25$ and $PC = 33$. Compute PD.

3. At a meeting of the Nuclear Powerplant Workers of America, every person has 3, 4, 5, 6, or 7 fingers on each hand, with the probability of having k fingers on a hand being $\frac{2^{2-|5-k|}}{10}$. If the number of fingers on a worker's left hand is independent of the number of fingers on the worker's right hand, compute the probability that a NPWA member has at least 10 fingers.

4. Given an unlimited set of identical 1×1, 1×2, and 1×3 tiles, compute the number of ways to cover a 1×8 rectangle with the tiles without overlapping tiles and with all tiles entirely inside the rectangle. Rotations of a set of tiles are considered distinct (for examples, two 1×3 tiles followed by a 1×2 tile is a distinct tiling from a 1×2 tile followed by two 1×3 tiles).

5. For two positive integers x and y, $x^2 - y^2 = 240$. Compute the (positive) difference between the largest and the smallest possible values of xy.

6. Walls are erected on a frictionless surface in the form of an equilateral triangle $\triangle ABC$ of side length 1. A tiny super bouncy ball is fired from vertex A at side $\overline{BC}$. The ball bounces off of 9 walls before hitting a vertex of the triangle for the first time. Compute the total distance travelled by the ball upon arrival.

7. A hexagonal pyramid has a regular hexagon of side length 1 for a base, and its apex is located a distance 1 directly above the center of the hexagon. A sphere inscribed inside the pyramid touches every face of the pyramid. Compute the radius of the inscribed sphere.

8. The function $f(x) = \frac{x^4 + 3x^3 - 4x^2}{x^2 + 4x - 4}$ is negative over two different intervals of the real numbers. Compute the sum of the lengths of the two intervals.

9. The set of points $S = \{(x, y) : |x| + |x - y| + |x + y| \leq k\}$ has area 10 in the xy-plane. Compute k.

10.
$$\begin{array}{cccc} & A & B & C \\ & \times & D & E \\ \hline & F & E & C \\ D & E & C & \\ \hline 4 & 6 & B & C \end{array}$$

In the multiplication problem above, A, B, C, D, E, and F are distinct, non-zero digits. Compute the six-digit number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}\,\underline{E}\,\underline{F}$.

Team Solutions

1. $2010 = 2 \times 3 \times 5 \times 67$, so 2010 has 16 factors. Eight of them are going to be multiples of 67, eight are not. The smallest of the former is 67, the largest of the latter is 30. Their average is $\boxed{97/2}$ or 48.5.

2. In any rectangle, the sum of the squares of the distances from an interior point to opposite vertices is invariant, i.e., this sum is the same for both pairs of opposite vertices. Thus, $PA^2 + PC^2 = PB^2 + PD^2$ and we have $56^2 + 33^2 = 25^2 + PD^2 \to 3136 + 1089 = 625 + PD^2 \to PD^2 = 3600 \to PD = \boxed{60}$.

   Alternate Solution: Suppose the point P were located on the side $\overline{AB}$. Then $\triangle PBC$ is a right triangle and, using the Pythagorean Theorem, $BC^2 = 464$. Now $\triangle PAD$ is also a right triangle. Therefore, $PD^2 = 56^2 + 464 = 3600 \to PD = 60$. This solution assumes that the solution is invariant for points P located in the interior of the rectangle <u>OR</u> on the rectangle. A nice topic for discussion!

3. The probability distribution on number of fingers per hand is $(0.1, 0.2, 0.4, 0.2, 0.1)$. By symmetry, the probability of having $10 + k$ fingers is equal to the probability of having $10 - k$ fingers. The probability of having 10 fingers is $0.1^2 + 0.2^2 + 0.4^2 + 0.2^2 + 0.1^2 = 0.26$. The probability of having greater than 10 fingers is $(1 - 0.26)/2 = 0.37$, so the probability of having at least 10 fingers is $0.26 + 0.37 = \boxed{0.63}$.

   Note: We apologize to any nuclear powerplant workers offended by this question. Please don't come after us. After all, we can see you glowing from a mile away.

4. Let $T(n)$ denote the number of ways a $1 \times n$ rectangle can be tiled using these tiles. $T(n)$ satisfies the recursion relation $T(n) = T(n-1) + T(n-2) + T(n-3)$ for all positive n with the initial condition $T(0) = 1$, $T(n) = 0$ for $n < 0$ (consider the size of the rightmost tile of any tiling). The first few values of $T(n)$ for non-negative n are 1, 1, 2, 4, 7, 13, 24, 44, $\boxed{81}$.

5. We want the values of x and y to be alternately as close and as far apart as possible, i.e., maximize and minimize $|x - y|$. Clearly, the maximum occurs when $x = 240$ and $y = 1$ (or vice versa).

   Factoring, $240 = 2^4 \times 3 \times 5$ and $x = 16$ and $y = 15$.

   $x^2 - y^2 = (x+y)(x-y) \to \begin{cases} x + y = 240 \\ x - y = 1 \end{cases}$ or $\begin{cases} x + y = 16 \\ x - y = 15 \end{cases}$, but each of these systems produce non-integer values of x and y. Clearly, for integer values of x and y, the parity of $x+y$ and $x-y$ must be the same. Revising our choices for x and y $\begin{cases} x + y = 120 \\ x - y = 2 \end{cases}$ or $\begin{cases} x + y = 20 \\ x - y = 12 \end{cases} \to$ $(x, y) = (61, 59)$ or $(16, 4)$.

   $61(59) - 16(4) = (60 + 1)(60 - 1) - 64 = 3600 - 1 - 64 = \boxed{3535}$.

6. Since angles are preserved upon hitting walls, we can use a series of reflections to express the ball's path as a straight line through a triangular tiling of the plane. In Figure 1 below,

$\triangle ABC$ is at the top and the triangles below contain the number of walls the ball must pass through (bounce off of, in the original triangle) in order to enter:

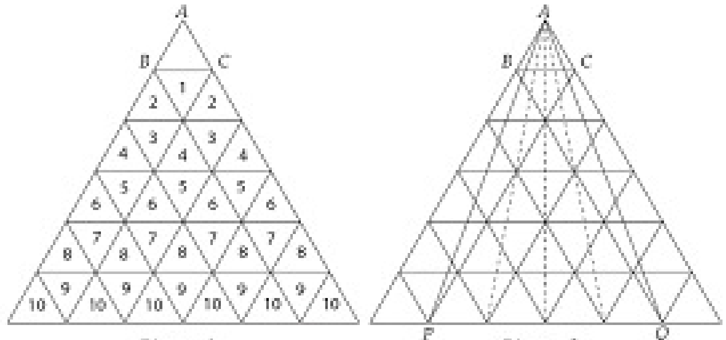
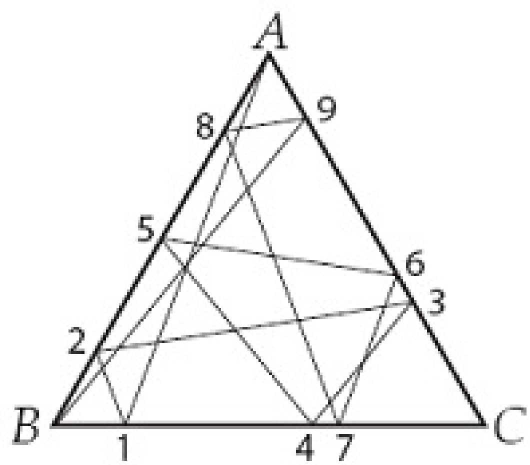


Figure 1 Figure 2

Of the vertices in the '9-bounce' row of triangles, all but two of them are impossible to reach (see the dotted lines in Figure 2), since they correspond to paths that pass through at least one corner of $\triangle ABC$ previously.

The two vertices that are reachable, P and Q, are both equidistant from A (the solid lines in Figure 2); hence the total distance travelled is $AP = AQ = \sqrt{\left(3\sqrt{3}\right)^2 + 2^2} = \boxed{\sqrt{31}}$.

A picture of the ball's path (in the direction towards P) is shown below. The path towards Q is its mirror image:

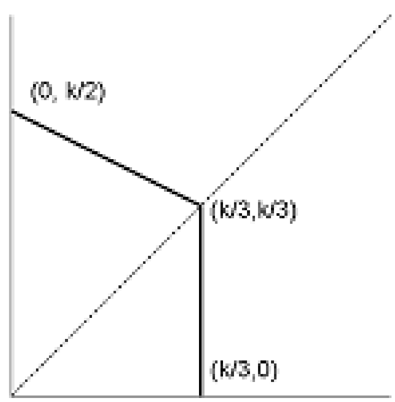


7. The volume of a pyramid is $\frac{Bh}{3}$, where B is the area of the base and h is the height. The hexagon has area $6 \times \frac{\sqrt{3}}{4}$, so the volume of the pyramid is $\frac{\sqrt{3}}{2}$. We can also dissect the pyramid into 7 smaller pyramids whose bases are the 7 faces of the original pyramid, and whose apexes (apices?) are the center of the inscribed sphere. Since the height of each of these pyramids is r, the radius of the sphere, we have $\frac{\sqrt{3}}{2} = \frac{1}{3} \times r \times \left(6 \times \frac{\sqrt{3}}{4} + 6L\right)$, where L is the area of one of the 6 lateral faces of the pyramid. Each of these faces is an isosceles triangle with

base 1 and leg $\sqrt{2}$. We then have a height of $\frac{\sqrt{7}}{2}$, and so $L = \frac{\sqrt{7}}{4}$. Solving for r we get $\frac{r}{3}\left(6 \cdot \frac{\sqrt{3}}{4} + 6 \cdot \frac{\sqrt{7}}{4}\right) = \frac{\sqrt{3}}{2} \to r = \frac{\sqrt{3}}{\sqrt{3}+\sqrt{7}} = \boxed{\frac{\sqrt{21}-3}{4}}$.

8. $f(x) = \frac{x^4+3x^3-4x^2}{x^2+4x-4} = \frac{x^2(x+4)(x-1)}{\left(x+(2-2\sqrt{2})\right)\left(x+\left(2+2\sqrt{2}\right)\right)}$. $f(x)$ has zeroes at 0, 1, and -4 and asymptotes at $-2-2\sqrt{2}$ and $-2+2\sqrt{2}$ (approximately -4.8 and 0.8). Testing the sign of a single point in the interval between two zeroes or asymptotes give you the sign of the function over the entire interval. The function is negative between $-2-2\sqrt{2}$ and -4 as well as between $-2+2\sqrt{2}$ and 1. The sum of the lengths of these intervals is $\left(1-\left(2\sqrt{2}-2\right)\right) + \left(-4-\left(-2\sqrt{2}-2\right)\right) = 1-2\sqrt{2}+2-4+2\sqrt{2}+2 = \boxed{1}$.

9. Noting that the function $f(x,y) = |x|+|x-y|+|x+y|$ is symmetric about both the x- and y-axis, it suffices to compute value of k for which the area in the first quadrant is $5/2$. When $x \ge y$, $|x|+|x-y|+|x+y| = x+x-y+x+y = 3x$. When $x \le y$, $|x|+|x-y|+|x+y| = x+y-x+x+y = x+2y$. For a fixed value of k, this function looks like the graphic below in the first quadrant. The area of the region is $\frac{1}{2}\left(\frac{k}{2}+\frac{k}{3}\right)\frac{k}{3} = \frac{5k^2}{36} = \frac{5}{2} \to k = \boxed{3\sqrt{2}}$.



10. $E \times C$ and $D \times C$ both end in C, and the only non-zero C for which that is possible for two different digits is 5. E and D must be odd, so by looking in the thousands column, we conclude that D is 3. From that same column we conclude that $D \times A = D$, so A must be 1. $1\underline{B}5 \times 3 = 3\underline{E}5$, and the only distinct non-zero B and odd E that hold are $B = 2$ and $E = 7$. $125 \times 7 = \underline{F}75$, so $F = 8$. Therefore, $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}\,\underline{E}\,\underline{F} = \boxed{125378}$. The completed multiplication is given below.

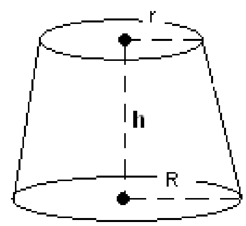
$$\begin{array}{cccc} & 1 & 2 & 5 \\ & \times & 3 & 7 \\ & 8 & 7 & 5 \\ 3 & 7 & 5 & \\ \hline 4 & 6 & 2 & 5 \end{array}$$

Theme Problems

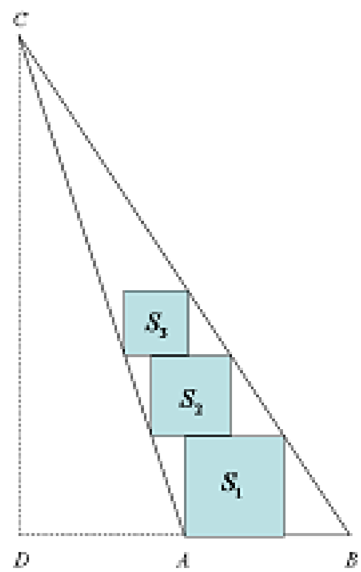
The Winter Olympics

"It is a beautiful day here in Vancouver for the start of the 2010 Winter Olympics. If only we had some snow."

1. The Vancouver Olympic Committee decides to build a conical mountain of snow with a radius 8 kilometers (R in the picture below). Because they run out of snow on the way up, they decide instead to shape the mountain instead into the frustum of a cone 4 kilometers high (h in the picture) with the top of the frustum having radius 5 kilometers (r in the picture). Compute the *exposed* surface area (in sq km) of the resulting snow "mountain".



"Now over to the Opening Ceremonies, where they seem to be having trouble with the torch pillars. . . "



The cross section of the pillar is an obtuse triangle $\triangle ABC$ with a base $\overline{AB}$ 5 meters long and a height of 15 meters. D is the foot of the altitude from C to $\overleftrightarrow{AB}$ and $\overline{AD}$ is also 5 meters long. The pillar is supported internally by beams with a square cross section. The first supporting beam (denoted S_1 in the graphic) has one side on $\overline{AB}$ and has one corner on $\overline{AC}$ and the opposite corner on $\overline{BC}$.

2. Compute (in meters) the side length of S_1.

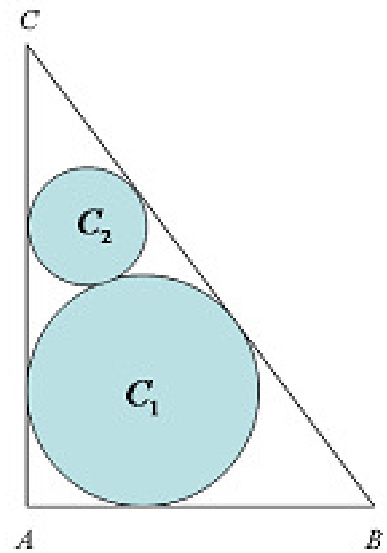
A second support beam (denoted S_2 in the above graphic) is placed on top of the first beam. This second beam also has one corner on $\overline{AC}$ and the opposite corner on $\overline{BC}$.

3. Compute (in meters) the side length of S_2.

This process of stacking support beams is repeated ad-infinitum: beam S_{n+1} is placed on top of beam S_n, and each beam has one corner on $\overline{AC}$ and the opposite corner on $\overline{BC}$.

4. Compute the fraction of $\triangle ABC$ that is filled by cross-sections of support beams.

"They seem to have made things right with the pillar."



After further fixes, the cross section of the pillar is now a right triangle $\triangle ABC$ with a base $\overline{AB}$ 3 meters long and a height of 4 meters. The pillar is supported internally by beams with a circular cross section. The first supporting beam (denoted C_1 in the graphic) is tangent to $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

5. Compute (in meters) the radius of C_1.

A second circular support beam (denoted C_2 in the graphic) is placed on top of the first beam. This second beam is tangent to $\overline{AC}$ and $\overline{BC}$, and is externally tangent to C_1. Surprisingly (?), the centers of both beams lie along the angle bisector of $\angle ACB$.

6. Compute (in meters) the radius of C_2.

"And now over to the hockey rink where the Finnish team is holding practice."

7. The Finnish hockey team wishes to practice playing 5-on-5, but have 11 players. They use the following substitution system: the players are numbered 1 to 11. For the first three minutes, players 2–6 play against players 7–11, with player 1 sitting out. Then player 1 replaces player 2 and that group plays for three minutes. In general, player n replaces player $n+1$ after three minutes (except for player 11, who replaces player 1). The practice continues until players 2–6 are going to play 7–11, and the practice stops (before 2–6 and 7–11 play again). Compute the length (in minutes) of the practice.

"An exciting hockey tournament so far. I have the results right here, well, some of them, oh dear."

For the last two questions, hockey teams will be playing in a round robin tournament. Every team will play every other team once. A win results in two *match points* for the winning team, a draw results in one match point for each team. Teams are ranked by total number of match points; teams with the same number of match points will be ranked in decreasing order by *goal difference*: the total number of goals scored by the team minus the number of goals scored against the team (the goal difference may be negative).

8. The United States, Finland, Canada, and Russia are playing in a round-robin hockey tournament. Below is an incomplete chart reporting some of the results from matches that have already been played. One team defeated another by a score of 5-1, and no other team has scored as many as five goals in one match. Also, one team won one of its matches by one more goal than it won another game. With this information, complete the chart on your answer sheet (also shown below).

Teams	Matches Played	Wins	Losses	Draws	Goals Scored By Team	Goals Scored By Opposing Team	Match Points
US		2		1		0	
Canada				0			
Russia	2				2		2
Finland	3					11	1

"On the women's side, a fifth hockey team has joined the tournament! This will be a prime opportunity."

9. This question is worth up to 8 points. If 1 to 4 errors are made in filling out the table, your team will receive 4 points.

   The United States, Finland, Canada, Slovakia, and Russia are playing in a round-robin hockey tournament. Our absent-minded reporter has lost the scores, yet has somehow managed to recall the following information:

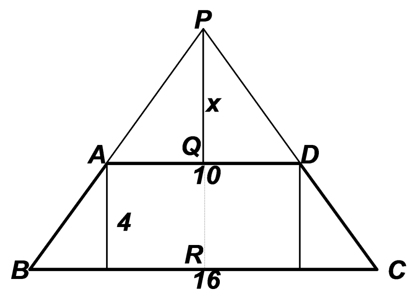
   - In each match, the total number of goals scored has been prime and at most five.
   - A prime number of matches have been played, and a prime number of matches are yet to be played.
   - All five teams currently have a prime number of match points, have scored a prime number of goals and have had a prime number of goals scored against them.
   - The total number of goals scored by all teams so far, 17, is prime.
   - The winning margin in all of the non-drawn matches is prime.
   - One match was won 5-0.
   - One team has drawn all of its matches so far.
   - There is a group of three teams such that each team has won a match to one team in the group and lost a match to the other team in the group, all by a score of 2-0.
   - The United States is currently ranked in first place, despite losing to Finland. Canada is second, and Slovakia is third. Russia and Finland, who have the same number of match points, have not played each other yet.

   This is sufficient information to completely reconstruct the match results so far on your answer sheet. In the table below (and on your answer sheet), for the entries in the first five columns, the number in row I and column J denotes the number of goals scored by team I against team J. The entries for games not played yet should be left blank. Fill in your answer sheet clearly and carefully.

Goals Scored	United States	Canada	Slovakia	Russia	Finland	Total Goals For	Total Goals Against	Goal Difference	Total Wins	Total Losses	Total Draws	Total Match Points
United States	■											
Canada		■										
Slovakia			■									
Russia				■								
Finland					■							

Theme Solutions

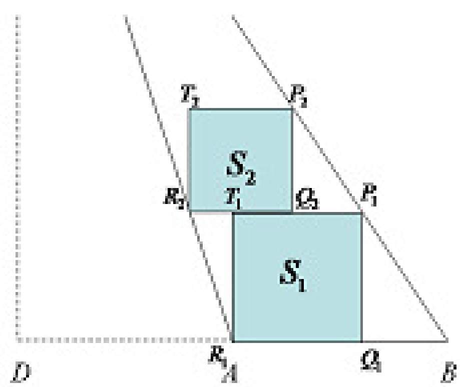
1. Consider the cross-section of the frustum, a trapezoid. Extend the non-parallel sides of the trapezoid until they intersect. The triangle formed is a cross-section of the cone from which the frustum was formed. $\triangle PQD \sim \triangle PRC \to \frac{x}{x+4} = \frac{5}{8} \to x = \frac{20}{3}$.



Since the lateral surface area of a cone is given by $\pi r \ell$, where $\ell$is the lateral height, the lateral surface area of the frustum is $\pi \cdot 8 \cdot \frac{40}{3} - \pi \cdot 5 \cdot \frac{25}{3} = \pi \left(\frac{320}{3} - \frac{125}{3} \right) = 65\pi$ and, adding the surface area of the upper circular bases (25π), we have the total exposed surface area of the frustum is $\boxed{90\pi}$.

2. For square S_i, let s_i be the side length of S_i and denote the vertices $P_iQ_iR_iT_i$, where $P_i$is the upper right vertex, and Q_i is the lower right vertex. $\triangle P_1Q_1B \sim \triangle CDB$, so $\frac{s_1}{5-s_1} = \frac{15}{10} \to 75 - 15s_1 = 10s_1 \to \boxed{s_1 = 3}$.

3. $\triangle T_1AR_2 \sim \triangle ADB \to \frac{T_1R_2}{s_1} = \frac{5}{15} \to T_1R_2 = 1$.



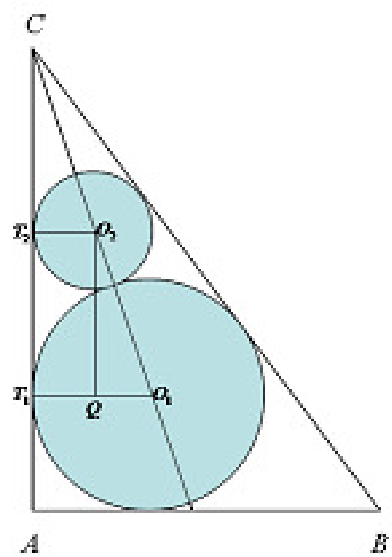
Similarly, $\triangle P_2Q_2P_1 \sim \triangle CDB \to \frac{Q_2P_1}{s_2} = \frac{10}{15} \to Q_2P_1 = \frac{2}{3}s_2$. $P_1R_2 = s_1 + T_1R_2 = s_2 + Q_2P_1 \to 3 + 1 = s_2 + \frac{2}{3}s_2 \to \boxed{s_2 = \frac{12}{5}}$.

4. The ratio of successive side lengths $\dfrac{s_n}{s_{n-1}}$ is a constant (why?) and is $\dfrac{4}{5}$, so the sum of the

areas of the squares is $\dfrac{9}{1-\left(\frac{4}{5}\right)^2} = \dfrac{9}{9/25} = 25$, thus the ratio of the sum of the areas of the squares to the area of $\triangle ABC$ is $\dfrac{25}{37.5} = \boxed{\dfrac{2}{3}}$.

5. C_1 is the incircle of $\triangle ABC$. The inradius of a triangle is the ratio of the area of the triangle to its semiperimeter, which is $\dfrac{\frac{1}{2}(3\times 4)}{\frac{1}{2}(3+4+5)} = \boxed{1}$.

6. For this problem, let O_i be the center of circle C_i, r_i be the radius of circle C_i, and T_i be the point of tangency of circle C_i with side $\overline{AC}$. Let Q be the point on $\overline{T_1O_1}$ such that $\overline{O_2Q}\perp\overline{T_1O_1}$. $\triangle CT_1O_1 \sim \triangle O_2QO_1$ and $m(\angle QO_2O_1) = \frac{1}{2}m(\angle ACB) = \theta$. Therefore, $QO_1 = O_1O_2\sin\theta \rightarrow (r_1 - r_2) = (r_1 + r_2)\sin\theta \rightarrow r_2 = \dfrac{1-\sin\theta}{1+\sin\theta}$.



Using the half-angle formula,

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}},$$

so $r_2 = \dfrac{1-\sin\theta}{1+\sin\theta} = \dfrac{1-\sqrt{\frac{1}{10}}}{1+\sqrt{\frac{1}{10}}} = \boxed{\dfrac{11-2\sqrt{10}}{9}}$.

7. After each player has sat out for three minutes, the teams are 11, 2-5 versus 6-10. After the next cycle, one of the teams is 5-9, then 4-8, 3-7, and finally 2-6. The total practice time is $3\times 11\times 5 = \boxed{165}$.

8. Each team has played at most three games. Since the US and Finland have each played three games, we conclude that Canada must have played only two. The US' record is 2-0-1 (Wins-Losses-Draws) for 5 match points. Finland's record must be 0-2-1 to have only one match point. Since the most goals scored in any game is 5 and Finland's opponents scored 11 total goals, each of Finland's three opponents must have scored against them. This means that the US and Finland did not draw, since the US' draw had to have been 0-0 since the US has not had a goal scored against them. Therefore, Russia drew with Finland. Since Russia has two match points, it must have drawn with the US as well with a 0-0 score. So, Russia's record is 0-0-2. Since Russia scored 2 goals in total and scored 0 against the US, it must have drawn against Finland with a score of 2-2. Finland's opponents in the two losses scored a total of 9 goals, so one game was 5-1 and the other was $4-x$ for a total of $3+x$ goals scored by Finland. One team won one of its games by one more than it won another game. This must have been the US since it is the only team with more than one win. So the US wins were 4-0 against Finland and 3-0 against Canada, for a total of 7 goals scored by the US. By process of elimination, Canada lost to the US 3-0 and beat Finland 5-1, for a record of 1-1-0, 5 goals scored for, 4 goals scored against, and 2 match points. The final grid is below, with the filled-in scores in italics.

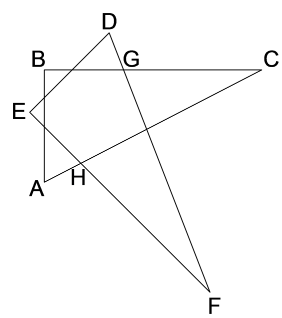
Teams	Matches Played	Wins	Losses	Draws	Goals Scored By Team	Goals Scored By Opposing Team	Match Points
US	*3*	2	*0*	1	*7*	0	*5*
Canada	*2*	*1*	*1*	0	*5*	*4*	*2*
Russia	2	*0*	*0*	*2*	2	*2*	2
Finland	3	*0*	*2*	*1*	*3*	11	1

9. The number of games already played (out of 10) must be 3, 5, or 7. 3 games played does not make sense, nor does 5, as all 5 teams would have to have 2 match points each. Three games have been won 2-0 and a fourth 5-0, so there is no one left to have a draw with the fifth team. So, 7 games have been played. There are 14 match points so they must be split 5/3/2/2/2. Accounting for the 17 goals scored so far, we can account for 11 of them in the aforementioned four games. Therefore, the remaining three games must have two goals each, and two of them must be 1-1 draws. The US has 5 match points and must have a record of 2-1-1. As the US' total goals for and against are both prime, the US could not have been involved in the 5-0 match and hence won two with a score of 2-0, lost one 0-2 and drew one 1-1. This accounts for the seventh game. Canada similarly was not in the 5-0 match and has won one 2-0 (against Finland, US-Canada-Finland is the only possible aforementioned group of three teams), drawn one 1-1 and lost one 0-2. The 5-0 win must have been by Slovakia or Russia against Finland (there's no other team left). The fact that Slovakia is ahead of Russia by goal difference and sorting out who played who gives the final answer above.

Individual Problems

1. In a triathlon, Richard swam 10 minutes at 1.5 miles per hour, biked 6 miles in 15 minutes and finished by running 15 miles at 6 miles per hour. Compute Richard's average speed (in miles per hour) over the entire course *rounded to the nearest tenth*.

2. In the picture below, $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at B and E, respectively. G is the intersection of $\overline{BC}$ and $\overline{DF}$, and H is the intersection of $\overline{AC}$ and $\overline{EF}$. If $m(\angle BAC) = 41°$, $m(\angle EFD) = 36°$ and $m(\angle AHF) = 107°$, compute (in degrees) $m(\angle FGC)$. Note that the picture is *NOT* drawn to scale.



3. Let $x = \log_{10} 2$ and $y = \log_{10} 3$. $\log_{10} 150 = ax + by + c$ for integers a, b, and c. Compute abc.

4. The product of two positive integers is 326,700. Compute the largest possible greatest common divisor of the two integers.

5. Dustin and Thomas are playing a game in which they take turns rolling a fair 10-sided die with the numbers 1 through 10 on the faces. Dustin rolls first. The game continues until either Dustin rolls a number ≤ 4 (in which case Dustin wins) or Thomas rolls a number $\leq T$ (in which case Thomas wins). Compute the smallest value of T for which Thomas' probability of winning is at least $\frac{1}{2}$.

6. The solutions of $64x^3 - 96x^2 - 52x + 42 = 0$ form an arithmetic progression. Compute the (positive) difference between the largest and smallest of the three solutions.

7. On a five-function calculator $+, -, \times, \div, \sqrt{\ }$ that can display 10 digits you perform the following procedure:

   Step 1. Enter a 5-digit number.
   Step 2. Multiply the number by 2010.
   Step 3. Take the fourth root of the result (by hitting the square root button twice).
   Step 4. Repeat Steps 2 and 3, getting a new result each time.

   After a while, you notice that the results on the screen are not changing after applying Steps 2 and 3 (that is, the results are constant, rounded to 10 digits). If N is this unchanging result, compute $\lfloor N \rfloor$, the greatest integer less than or equal to N.

8. Let x be an angle in the first quadrant. If $\tan\left(\sqrt{2}\cos x\right) = \cot\left(\sqrt{2}\sin x\right)$, compute $\sin 2x$.

9. $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is a four-digit number with *distinct* non-zero digits. $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D} - \underline{D}\,\underline{C}\,\underline{B}\,\underline{A}$ is a positive three-digit number x. Compute the (positive) difference between the largest and smallest possible values of x.

10. Regular hexagon $ABCDEF$ has side length 1. Let H be the intersection point of lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$. Compute the radius of the circle that passes through D, F, and H.

Individual Solutions

1. Richard swam $1.5\,\frac{\text{mi}}{\text{hr}} \times \frac{1}{6}\,\text{hr} = 0.25\,\text{mi}$. Similarly, he ran for $\frac{15\,\text{mi}}{6\,\text{mi/hr}} = 2.5\,\text{hr}$. The total distance traveled was $0.25 + 6 + 15 = 21.25 = \frac{85}{4}\,\text{mi}$. The total time elapsed was $\frac{1}{6} + \frac{1}{4} + \frac{5}{2} = \frac{35}{12}\,\text{hr}$. The average speed was thus $(\frac{85}{4}\,\text{mi})/(\frac{35}{12}\,\text{hr}) = \frac{255}{35}\,\text{mi/hr} = \frac{51}{7}\,\text{mi/hr} \approx 7.285\ldots$ which is $\boxed{7.3}$ rounded to the nearest tenth.

2. $\text{m}(\angle ACB) = 90^\circ - 41^\circ = 49^\circ$. Let J be the intersection of $\overline{AC}$ and $\overline{DF}$, then $\text{m}(\angle GJC) = \text{m}(\angle HJF) = 180^\circ - \text{m}(\angle EFD) - (180^\circ - \text{m}(\angle AHF)) = \text{m}(\angle AHF) - \text{m}(\angle EFD) = 71^\circ$.

   Thus, $\text{m}(\angle FGC) = 180^\circ - \text{m}(\angle GCJ) - \text{m}(\angle GJC) = 180^\circ - 49^\circ - 71^\circ = \boxed{60^\circ}$.

3. $\log_{10} 150 = \log_{10} 300 - \log_{10} 2 = \log_{10} 3 + \log_{10} 100 - \log_{10} 2 = y + 2 - x$. $abc = (-1)\,(1)\,(2) = \boxed{-2}$.

4. $326700 = 2^2 \times 3^3 \times 5^2 \times 11^2$. To maximize the greatest common divisor between the two numbers, each number should have a 2, 3, 5, and 11 as a factor, which gives a maximum possible gcd of $\boxed{330}$.

5. We instead compute the probability that Dustin wins, and let p be the probability that Thomas wins on a single throw. Either she wins on the first throw (with probability $\frac{4}{10}$), on the third after two "misses" (with probability $\left(\frac{6}{10}\,(1-p)\right)\frac{4}{10}$), or on the fifth throw after four misses (with probability $\left(\frac{6}{10}\,(1-p)\right)^2\frac{4}{10}$), and so forth. The probability that Dustin wins is $\dfrac{2/5}{1 - \frac{3(1-p)}{5}} = \dfrac{2}{5 - 3(1-p)}$. We want this probability to be less than $\dfrac{1}{2}$, so $\dfrac{2}{5 - 3(1-p)} \leq \dfrac{1}{2} \to 4 \leq 5 - 3(1-p) \to \dfrac{2}{3} \leq p$, which means that T must be at least $\boxed{7}$.

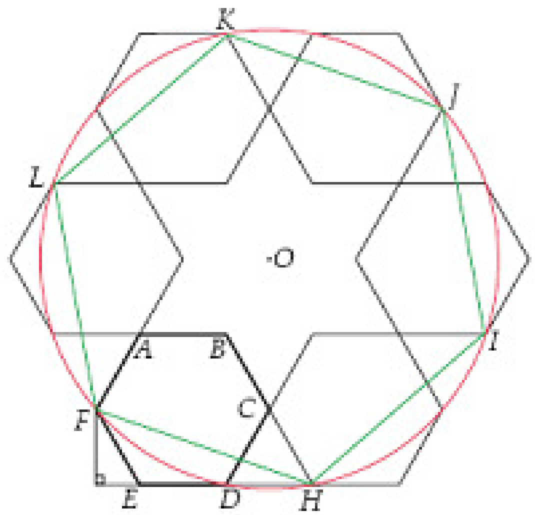
6. Assume the roots are $a - d, a$, and $a + d$. The sum of the roots is $3a = \frac{96}{64} = \frac{3}{2} \to a = \frac{1}{2}$. The product of the roots is $-\frac{42}{64} = (a-d)\,a\,(a+d) = a^3 - ad^2 = \frac{1}{8} - \frac{d^2}{2} \to \frac{d^2}{2} = \frac{50}{64} \to d^2 = \frac{100}{64} = \frac{25}{16} \to d = \frac{5}{4}$. The difference between the largest and smallest of the solutions is $2d = \boxed{\dfrac{5}{2}}$.

7. Consider the sequence $x = x_0, x_1, \ldots$, where x is the initial five-digit number, and x_1 is the result after applying the process in steps 2 and 3. In general, $x_{n+1} = \sqrt[4]{2010x_n}$. To say that this sequence converges means that for a large enough n, $x_{n+1} \approx x_n \approx x^*$, the limit of this sequence. To find it, $x_{n+1} = \sqrt[4]{2010x_n} \to x^* = \sqrt[4]{2010x^*} \to x^* = \sqrt[3]{2010}$. As $12^3 = 1728 < 2010 < 2197 = 13^3$, $\lfloor x^* \rfloor = \left\lfloor \sqrt[3]{2010} \right\rfloor = \boxed{12}$.

8. As angles, $\sqrt{2}\cos x$ and $\sqrt{2}\sin x$ must lie in the first quadrant. Since $\tan\left(\sqrt{2}\cos x\right)$ and $\tan\left(\sqrt{2}\sin x\right)$ are reciprocals of each other, we can conclude that $\sqrt{2}\sin x + \sqrt{2}\cos x = \dfrac{\pi}{2}$. Squaring both sides gives $2 + 4\sin x\cos x = \dfrac{\pi^2}{4} \to 2\sin 2x = \dfrac{\pi^2}{4} - 2 \to \sin 2x = \boxed{\dfrac{\pi^2}{8} - 1}$.

9. $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D} = 1000A + 100B + 10C + D$ and $\underline{D}\,\underline{C}\,\underline{B}\,\underline{A} = 1000D + 100C + 10B + A$. $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D} - \underline{D}\,\underline{C}\,\underline{B}\,\underline{A} = 999\,(A - D) - 90\,(C - B)$. As x is a positive three-digit number, we must conclude that $A - D = 1$. Because all of the digits are non-zero and distinct, we know that $1 \le C - B \le 8$. Therefore the largest difference is 909 (e.g., $2451 - 1542$) and the smallest is 279 (e.g. $3192 - 2931$), giving a difference of $\boxed{630}$.

10. Let O be the center of the circle through D, F, and H. By successive rotations of hexagon $ABCDEF$ about O, we get the picture below. Notice that the six smaller hexagons contribute 12 points lying on circle O, and since the hexagons were obtained by 60 degree rotations of $ABCDEF$, we have that hexagon $FHIJKL$ is regular with side length $r = FH = \sqrt{(5/2)^2 + \left(\sqrt{3}/2\right)^2} = \boxed{\sqrt{7}}$.



Alternate Solution: Position the regular hexagon with $F(0,0)$, $E(1,0)$, $D\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$. Then: $H\left(2, \sqrt{3}\right)$ and M, the midpoint of $\overline{DH}$ is $\left(\frac{7}{4}, \frac{3\sqrt{3}}{4}\right)$. O, the center of the required circle, is located at the intersection of the perpendicular bisectors of $\overline{FD}$ and $\overline{DH}$. The equations of these bisectors are, respectively,

$$\begin{cases} y = -\sqrt{3}(x - 1) \\ x + \sqrt{3}y = 4. \end{cases}$$

Thus $(x, y) = \left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$ and a radius of the circle is FO. Therefore, $r^2 = \frac{1}{4} + \frac{27}{4} = 7 \rightarrow r = \boxed{\sqrt{7}}$.

Relay Problems

R1-1. Let S be the set of prime factors of $1 + 2 + \cdots + 101$. Compute the median of S.

R1-2. Let $T = TNYWR$. Compute the sum of the areas of the distinct right triangles with integer side lengths with one side of length T.

R2-1. Compute the smallest positive integral value of n such that $\tan x \sin 2x = \cos 4x \cos nx$ has a solution at $x = \pi/6$.

R2-2. Let $T = TNYWR$. Compute $\sqrt{T + \sqrt{T + \sqrt{T + \sqrt{\cdots}}}}$.

R2-3. Let $T = TNYWR$. Compute the number of ordered T-tuples $(x_1, \ldots, x_T)$ of positive integers such that $x_1 + x_2 + \cdots + x_T = 10$.

R3-1. $7!_{10} = \underline{A}\,\underline{B}\,\underline{C}\,\underline{D}\,\underline{E}_6$ (i.e., when written in base six). Compute $A + B + C + D + E$.

R3-2. Let $T = TNYWR$. An absent-minded hat-check girl receives T different hats from T different people. At the end of the evening she hands one hat back to each person randomly. The probability that she gave back at least $T - 2$ hats to the correct person is $\dfrac{K}{T!}$. Compute K.

R3-3. Let $T = TNYWR$. $T - 3$ is the largest prime factor of $(n-1)! + (n+1)!$. Compute the smallest possible value of n.

R3-4. Let $T = TNYWR$. If $p + q = 4$ and $p^2 + q^2 = T$, compute $p^3 + q^3$.

R3-5. Let $T = TNYWR$. If $Avg(x, y)$ denotes the average of the numbers x and y, compute the (positive) difference between the largest and smallest possible value of $Avg(Avg(a, b), c) - Avg(a, Avg(b, c))$ if $-T \leq a, b, c \leq T$.

R3-6. Let $T = TNYWR$. For a positive integer n, let $s_2(n)$ denote the sum of the digits in the binary representation of n. Compute the number of positive integers less than $64 - T$ for which $s_2(n)$ is prime.

Relay Solutions

R1-1. $1 + 2 + \cdots + 101 = 5151 = 3 \times 17 \times 101$. The median of S is $\boxed{17}$.

R1-2. You can generate all of the Pythagorean triples as follows, if m and n are positive integers with $n > m$, then $n^2 - m^2$, $2mn$, and $n^2 + m^2$ is a Pythagorean triple. As $T = 17$, either $n^2 + m^2 = 17 \to (n, m) = (4, 1) \to 8 - 15 - 17$ or $n^2 - m^2 = 17 \to (n - m)(n + m) = 17 \to (n, m) = (9, 8) \to 17{-}144{-}145$.

The areas of these two triangles are 60 and 1224 respectively, giving a total of $\boxed{1284}$.

R2-1. $\tan\left(\frac{\pi}{6}\right)\sin\left(\frac{2\pi}{6}\right) = \cos\left(\frac{4\pi}{6}\right)\cos\left(\frac{n\pi}{6}\right) \to \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = -\frac{1}{2}\cos\left(\frac{n\pi}{6}\right) \to \cos\left(\frac{n\pi}{6}\right) = -1 \to n = \boxed{6}$.

R2-2. Let $x = \sqrt{T + \sqrt{T + \sqrt{T + \sqrt{\cdots}}}} \to x = \sqrt{T + x} \to x^2 = T + x \to x^2 - x - T = 0 \to x = \dfrac{1 + \sqrt{1 + 4T}}{2}$ (ignoring the negative value of x). Since $T = 6$, $x = \boxed{3}$.

R2-3. Since each of the x_i must be at least 1, that leaves $10 - T$ units to distribute amongst the x_i, which can be done $\binom{9}{T-1}$ ways. As $T = 3$, the answer is $\boxed{36}$.

R3-1. $7!_{10} = (4 \times 5 \times 7) \times (2 \times 3) \times 6 = 140 \times 6^2$. $140 = 352_6 \to 7!_{10} = 35200_6 \to A + B + C + D + E = \boxed{10}$.

R3-2. To accomplish this, the hat check girl must hand back $T - 2$ hats correctly or all T correctly. There is one way to hand them all back correctly, and $\binom{T}{2}$ ways to hand exactly $T - 2$ hats correctly (pick two people and switch their hats). As $T = 10$, $K = 45 + 1 = \boxed{46}$.

R3-3. $(n - 1)! + (n + 1)! = (n - 1)!(1 + n(n + 1)) = (n - 1)!(1 + n + n^2)$. As the factorial only has prime factors up to $n - 1$, we check whether $T - 3 = 43$ can divide $n^2 + n + 1$, which it can when $n = \boxed{6}$.

R3-4. $p^3 + q^3 = (p + q)\left(p^2 - pq + q^2\right) = 4\,(T - pq)$. $16 = 4^2 = (p + q)^2 = p^2 + 2pq + q^2 = T + 2pq \to pq = 8 - \frac{T}{2} \to p^3 + q^3 = 4\left(\frac{3T}{2} - 8\right)$. As $T = 6$, $p^3 + q^3 = \boxed{4}$.

R3-5. $Avg(Avg(a, b), c) - Avg(a, Avg(b, c)) = Avg\left(\frac{a+b}{2}, c\right) - Avg\left(a, \frac{b+c}{2}\right) = \frac{\frac{a+b}{2} + c}{2} - \frac{a + \frac{b+c}{2}}{2} = \frac{c-a}{4}$. This reaches its largest value when $c = T, a = -T$ and its smallest value when $c = -T, a = T$. The difference between these two values is $\frac{2T}{4} - \frac{-2T}{4} = T = \boxed{4}$.

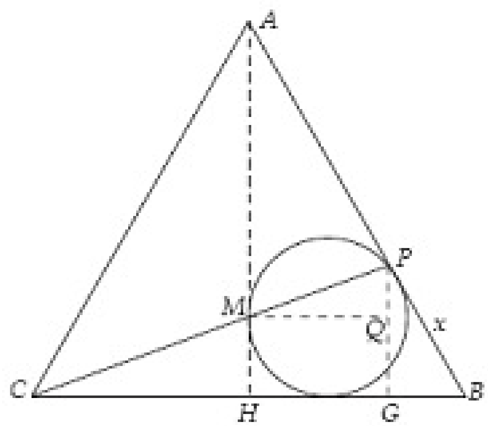
R3-6. Consider all integers n up to 63 that can be written with at most 6 digits in their binary representation. $\binom{6}{2} + \binom{6}{3} + \binom{6}{5} = 15 + 20 + 6 = 41$ of them will have a prime value of $s_2(n)$. However, since $T = 4$, we must ignore any values greater than 60 for which $s_2(n)$ is prime. There are two, 61 (111101_2) and 62 (111110_2). Therefore, the final answer is $41 - 2 = \boxed{39}$.

Tiebreaker Problem

1. Equilateral triangle $\triangle ABC$ has side length 1. Circle O is tangent to sides $\overline{AB}$ and $\overline{BC}$, and is tangent to the perpendicular bisector of $\overline{BC}$ at M. If the cevian $\overline{PC}$ passes through M, compute the area of triangle $\triangle APC$.

Tiebreaker Solution

1. Let r be the radius of circle O. Let H be the foot of the bisector to $\overline{BC}$, and drop a perpendicular from P to G, as shown. We also drop a perpendicular from M to $\overline{PG}$ at Q. We will solve for $x = \overline{BP}$.



Using 30−60−90 triangles, we have $CH = \frac{1}{2}$, $HM = GQ = r$, and $BG = \frac{x}{2}$. $\triangle CHM \sim \triangle MQP$, so we have $\frac{PQ}{MQ} = \frac{HM}{CH} = 2r$. Hence $PQ = 2r \cdot \frac{1-x}{2} = r - rx$. We now have $r + (r - rx) = PG = \frac{\sqrt{3}}{2}x$, or $x = \frac{4r}{\sqrt{3}+2r}$. Now we use the fact that $r + \sqrt{3}r = 1/2$ to get $r = \frac{\sqrt{3}-1}{4}$. Plugging this in above gives $x = \frac{8-2\sqrt{3}}{13}$. So the area of $\triangle APC$ is

$$\frac{1}{2} \cdot 1 \cdot \frac{5+2\sqrt{3}}{13} \cdot \sin 60^\circ = \boxed{\frac{6+5\sqrt{3}}{52}}.$$

Answers to ARML Local 2010

Team Round:

1. $\dfrac{97}{2}$
2. 60
3. $\dfrac{63}{100}$
4. 81
5. 3535
6. $\sqrt{31}$
7. $\dfrac{\sqrt{21}-3}{4}$
8. 1
9. $3\sqrt{2}$
10. 125378

Theme Round:

1. 90π
2. 3
3. $\dfrac{12}{5}$
4. $\dfrac{2}{3}$
5. 1
6. $\dfrac{11-2\sqrt{10}}{9}$
7. 165

8.

Teams	Matches Played	Wins	Losses	Draws	Goals Scored By Team	Goals Scored By Opposing Team	Match Points
US	*3*	2	*0*	1	*7*	0	*5*
Canada	*2*	*1*	*1*	0	*5*	*4*	*2*
Russia	2	*0*	*0*	*2*	2	*2*	2
Finland	3	*0*	*2*	*1*	*3*	11	1

9.

Goals Scored	United States	Canada	Slovakia	Russia	Finland	Total Goals For	Total Goals Against	Goal Difference	Total Wins	Total Losses	Total Draws	Total Match Points
United States		2	2	1	0	5	3	2	2	1	1	5
Canada	0		-	1	2	3	3	0	1	1	1	3
Slovakia	0	-		-	5	5	2	3	1	1	0	2
Russia	1	1	-		-	2	2	0	0	0	2	2
Finland	2	0	0	-		2	7	-5	1	2	0	2

Individual Round:

1. 7.3
2. 60°
3. -2
4. 330
5. 7
6. $\dfrac{5}{2}$
7. 12
8. $\dfrac{\pi^2}{8}-1$
9. 630
10. $\sqrt{7}$

Relay Round:

Relay 1:	1. 17	2. 1284				
Relay 2:	1. 6	2. 3	3. 36			
Relay 3:	1. 10	2. 46	3. 6	4. 4	5. 4	6. 39

Tiebreaker:

1. $\dfrac{6+5\sqrt{3}}{52}$

ARML Local 2011

Contents

Team Problems

1. Let $1.\overline{81} = 1.81818181\ldots = \dfrac{N}{D}$, where N and D are two positive integers. Compute the minimum value of $N + D$.

2. A two-digit number N is chosen at random, with all two-digit numbers equally likely. Compute the probability that at least one of $\log_2 N$, $\log_3 N$, $\log_5 N$, or $\log_7 N$ is an integer.

3. The difference between the areas of the circumcircle and incircle of an equilateral triangle is 27π. Compute the area of the triangle.

4. If Sam arranges his candies in piles of 6, there are 2 left over. If he arranges them in piles of 9, there are 5 left over. If he arranges them in piles of 15, there are 11 left over. Sam has between 100 and 200 candies. Compute how many candies Sam has.

5. Quadrilateral $JANE$ is inscribed in a circle. Diameter JN has length 50, $AN = 48$ and $NE = 30$. Compute the perimeter of $JANE$.

6. A fair coin is flipped repeatedly and the throws are recorded until three consecutive flips are (in order) Heads, Heads, and Tails *or* Heads, Tails, and Tails. Compute the probability that the sequence Heads, Heads, and Tails appears first.

7. Let a, b, c, and d be real numbers such that the graphs of $y = -|x - a| + b$ and $y = |x - c| + d$ intersect at $(2, 1)$ and $(8, -1)$. Compute $a + c$.

8. Given triangle ABC with $AB = 6, BC = 8$, and $AC = 7$. Point D is on AC such that BD bisects angle B. Circles C_1 and C_2 are drawn such that C_1 is internally tangent to each side of triangle ABD, and C_2 is internally tangent to each side of triangle CBD. Compute the sum of the areas of circles C_1 and C_2.

9. Compute $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{2007 \times 2009} + \dfrac{1}{2009 \times 2011} = \displaystyle\sum_{n=1}^{1005} \frac{1}{(2n-1)(2n+1)}$.

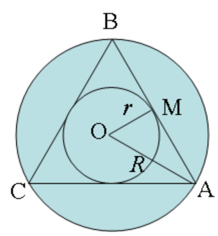
10. An *ordered partition* of an integer N is a sequence of positive integers that sum to N. As an example, $1 + 3$, $2 + 1 + 1$, and $1 + 1 + 2$ are distinct ordered partitions of 4. Compute the number of ordered partitions of 10 that consist of only odd positive integers.

Team Solutions

1. Let $x = 1.818181\ldots$, then $100x = 181.818181\ldots = 180 + x$, so $99x = 180 \to x = \frac{180}{99} = \frac{20}{11} \to N + D = \boxed{31}$.

2. There are 90 two digit numbers. Any power of 2, 3, 5, or 7 will result in one of the four logs to be an integer. There are 7 two-digit powers of these numbers: 16, 32, 64, 27, 81, 25, and 49. The answer is $\boxed{7/90}$.

3. Let R and r be circumradius and inradius of equilateral triangle ABC, respectively, and let M be the midpoint of AB (as well as the point of tangency of AB to the incircle). If O is the center of the circles, then OAM is a 30-60-90 triangle. Accordingly, the circumradius is twice that of the inradius. $\pi R^2 - \pi r^2 = 27\pi \to (2r)^2 - r^2 = 27 \to 3r^2 = 27 \to r = 3$. $AM = 3\sqrt{3} \to AB = 6\sqrt{3}$, so the area of ABC is $(AB^2)\frac{\sqrt{3}}{4} = \left(6\sqrt{3}\right)^2 \frac{\sqrt{3}}{4} = \boxed{27\sqrt{3}}$.



4. Note that in each case the left over pile is four candies short of a full pile. Accordingly, assume Sam was given four additional candies. Then the number of candies he has is a multiple of 6, 9, and 15. The least common multiple of these numbers is 90. Because he has between 100 and 200 candies, Sam must have 180 in his possession. Taking back the 4 additional candies, Sam started with $\boxed{176}$.

5. JAN and JEN are both right triangles, since JN is a diameter of the circle. Accordingly, $JA = 14$ and $JE = 40$. The perimeter of the quadrilateral is $48 + 14 + 30 + 40 = \boxed{132}$.

6. Let p be the probability that HHT appears first. The first time a heads is flipped, if the next flip is also a heads, then HHT must appear before HTT (as there will be a string of two or more heads terminated by a tails flip, forming HHT). If the next flip is tails, then if the following flip is also tails, HTT appears first. If the following flip is heads, then the process repeats itself, and the probability that HHT appears first is p. Thus, looking at the two flips after the first heads come up, we get $p = \Pr(\text{H on } 2^{\text{nd}} \text{ flip}) + \Pr(\text{T on } 2^{\text{nd}} \text{ flip, H on } 3^{\text{rd}} \text{ flip})$. Thus $p = 1/2 + (1/4)p \to (3/4)p = 1/2 \to p = \boxed{2/3}$.

7. Note that the intersection points $(2, 1)$ and $(8, -1)$ are two of the four corners of a rectangle whose other corners are the points (a, b) and (c, d). In addition, the slopes of the segments joining the corners of the rectangle all have slope $+1$ or -1. To find the unknown corners, you can solve the two systems $\{y-1 = x-2, y+1 = -(x-8)\}$ and $\{y-1 = -(x-2), y+1 = x-8\}$ to get the points $(4, 3)$ and $(6, -3)$. Therefore, $a + c = 4 + 6 = \boxed{10}$.

8. By the Angle Bisector Theorem, we have $AD = 3$ and $DC = 4$. Now we find the length of BD by the Law of Cosines. Since angles ADB and CDB are supplementary, their cosines are additive inverses of each other, so we equate $\frac{BD^2+9-36}{6BD} = \frac{64-BD^2-16}{8BD}$ and solve to obtain $BD = 6$. We find the inradius of circle C_1 by taking advantage of the formula $Area = rs$, where r is the inradius and s is the semiperimeter of the circumscribed triangle. In the case of triangle ABD, we have $\frac{9\sqrt{15}}{4} = r_1 \cdot \frac{15}{2}$, so $r_1 = \frac{3\sqrt{15}}{10}$ and the area of C_1 is $\frac{27\pi}{20}$. In the case of triangle CBD, we have $3\sqrt{15} = r_2 \cdot 9$, so $r_1 = \frac{\sqrt{15}}{3}$ and the area of C_2 is $\frac{15\pi}{9}$. The desired sum is $\boxed{181\pi/60}$.

9. Note that

$$\begin{aligned}
\sum_{n=1}^{1005} \frac{1}{(2n-1)(2n+1)} &= \sum_{n=1}^{1005} \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{1005} \frac{1}{(2n-1)} - \sum_{n=1}^{1005} \frac{1}{(2n+1)}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2011}\right) \\
&= \boxed{\frac{1055}{2011}}.
\end{aligned}$$

10. Note that there must be an even number of terms in the sum, and each term must be at least one. Let T be the number of terms in the sum. Note that by subtracting one from each term in an ordered partition of 10, there is a one-to-one correspondence between ordered partitions of 10 consisting of only odd positive integers and ordered partitions of $10-T$ that consists only of non-negative even integers (for example: $3+1+5+1 = 10$ and $2+0+4+0 = 6$). Similarly, there is a 1-to-1 correspondence between ordered partitions of $10 - T$ that consists of only non-negative even integers and ordered partitions of $\frac{10-T}{2}$ that consist of only non-negative integers (for example: $2 + 0 + 4 + 0 = 6$ and $1 + 0 + 2 + 0 = 3$). This set can be counted easily using "stars-and-bars" or your favorite balls-in-buckets counting technique: there are $\binom{\frac{10-T}{2}+(T-1)}{(T-1)}$ ways to do this. Therefore, the answer is

$$\begin{aligned}
\sum_{k=1}^{5} \binom{\frac{10-2k}{2} + (2k-1)}{(2k-1)} &= \sum_{k=1}^{5} \binom{5+k-1}{(2k-1)} \\
&= \binom{5}{1} + \binom{6}{3} + \binom{7}{5} + \binom{8}{7} + \binom{9}{9} = 5 + 20 + 21 + 8 + 1 = \boxed{55}.
\end{aligned}$$

Note: An alternate solution is to show that the number of ordered partitions of n using odd positive integers is the n^{th} Fibonacci number is an exercise left to the reader. (Hint: Split the set of ordered partitions of n into those whose last term is or is not 1.)

Theme Problems

Dice Dice Baby!

For all of the questions in this round, a n-sided die has the integers between 1 and n (inclusive) on its faces. A *roll* refers to the value that shows on the topmost face of the die after it is thrown. All values on the faces of the die are equally likely to be rolled.

1. Two ten-sided dice are thrown and the rolls are multiplied. Compute the expected value of their product.

For Questions 2 through 4, we use the following notation. For a positive integer n, let $R_1, R_2, \ldots, R_{2n}$ denote $2n$ rolls of a six-sided die. We define $P_1, P_2, \ldots, P_n$ by $P_i = R_{2i-1} \times R_{2i}$: the product of the i^{th} pair of rolls. So for rolls of 3, 5, 1, and 4, $P_1 = R_1 \times R_2 = 3 \times 5 = 15$ and $P_2 = R_3 \times R_4 = 1 \times 4 = 4$.

2. A six-sided die is thrown 10 times. If $P_2 = P_1 + 7$, $P_3 = P_2 - 4$, $P_4 = P_3 + 6$, and $P_5 = P_4 + 3$, compute P_1.

3. A six-sided die is thrown 12 times. For $3 \leq k \leq 6$, P_k satisfies the Fibonacci recurrence $P_k = P_{k-1} + P_{k-2}$. Compute the number of possible values of P_6.

For Question 4, these approximations may be useful: $\log_{10} 2 \approx 0.301$, $\log_{10} 3 \approx 0.4771$, $\log_{10} 5 \approx 0.699$, and $\log_{10} 7 \approx 0.8451$.

2. A six-sided die is thrown $2n$ times. Compute the smallest value of n such that

$$\Pr(P_i = 36 \text{ for some } i, 1 \leq i \leq n) > 1/2.$$

Consider the following two-player game. Andy chooses an integer between 2 and 12, inclusive. Bailes, knowing Andy's choice, chooses a different integer between 2 and 12. Two six-sided dice are thrown, and the rolls are summed. The player who has picked the closer number to the sum wins \$1. If both numbers are equidistant from the sum, the players split the \$1.

5. Compute the number Andy should pick to maximize his expected winnings, assuming that Bailes picks a number to maximize his expected winnings given Andy's choice.

6. Compute Andy's expected winnings, assuming an optimal strategy by both players.

Consider the following two-player game. Andy chooses an integer between 1 and 30, inclusive. Bailes, knowing Andy's choice, chooses a different integer between 1 and 30. A single 30-sided die is thrown. The player who has picked the closer number to the roll wins a number of dollars equal to the value shown on the die. For example, if Andy chooses 8 and Bailes chooses 10 and a 14 is rolled, then Bailes wins \$14. Again, in the case of a tie, the players split the money (so if a 9 was rolled in the previous example, each player would receive \$4.50).

7. Compute the number Andy should pick to maximize his expected winnings, assuming that Bailes picks a number to maximize his expected winnings given Andy's choice.

8. Compute Andy's expected winnings, assuming an optimal strategy by both players.

Consider a variant of the above game. Andy chooses an integer between 1 and 20, inclusive. Bailes, knowing Andy's choice, chooses a different integer between 1 and 20. A single 20-sided die is thrown. If the number x is rolled, the player who has picked the closer number to x wins x dollars. The other player wins $20 - x$ dollars. For example, if Andy chooses 8 and Bailes chooses 10 and a 6 is rolled, then Andy wins \$6, and Bailes wins \$14. In the case of a tie, both players receive \$10.

9. Compute the number Andy should pick to maximize his expected winnings, assuming that Bailes picks a number to maximize his expected winnings given Andy's choice.

10. Compute Andy's expected winnings, assuming an optimal strategy by both players.

Theme Solutions

1. If X and Y are random variables denoting the rolls on the two dice, then $E[XY] = E[X] \cdot E[Y]$ (as the variables are independent of each other). Since $E[X] = E[Y] = \frac{11}{2}$, $E[XY] = \left(\frac{11}{2}\right)^2 = \boxed{\dfrac{121}{4}}$.

   Alternately, since each die roll is equally likely, so is each pair (x, y) of rolls. Summing over all pairs gives $\left(\dfrac{1+2+\cdots+10}{10}\right) \times \left(\dfrac{1+2+\cdots+10}{10}\right) = \dfrac{55^2}{100} = \dfrac{11^2}{4} = \boxed{\dfrac{121}{4}}$.

2. The set of possible products is $S = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36\}$. We can write all five products in terms of P_1: for example $P_3 = P_2 - 4 = P_1 + 7 - 4 = P_1 + 3$. Similarly, $P_4 = P_1 + 9$ and $P_5 = P_1 + 12$. We need to find an element in S such that the numbers 3, 7, 9, and 12 larger are also in S. $P_1 = \boxed{3}$ is the only possibility.

3. We know that $P_6 = 3P_1 + 5P_2$ by using repeated applications of the Fibonacci recurrence. Using the same set S as in the previous solution, we have a very limited set of possible values for the starting values for (P_1, P_2), where P_1, P_2, and P_6 are in S. They are $(1, 1)$, $(1, 3)$, $(2, 2)$, $(2, 6)$, $(3, 3)$, $(5, 1)$, $(5, 2)$, and $(5, 3)$. Of these pairs, only $(1, 1)$, $(2, 2)$, and $(3, 3)$ result in a sequence of six numbers, all of which are in S: $(1, 1, 2, 3, 5, 8)$, $(2, 2, 4, 6, 10, 16)$, and $(3, 3, 6, 9, 15, 24)$. The answer is $\boxed{3}$.

4. The probability that double-sixes shows up at least once in n pairs is $1 - \Pr(P_i \neq 36)^n = 1 - \left(\frac{35}{36}\right)^n$. Consequently, we are interested in the smallest value of n such that $\left(\frac{35}{36}\right)^n < \frac{1}{2}$. Taking logarithms (base-10) of both sides, we get
   $n(\log_{10}(35) - \log_{10}(36)) = -\log_{10}(2)$
   $n(\log_{10}(5) + \log_{10}(7) - 2(\log_{10}(2) + \log_{10}(3))) < -\log_{10}(2)$
   $n(0.699 + 0.8451 - 2(0.301 + 0.4771)) < -0.301$
   $-0.0121n < -0.301 \to n < \frac{0.301}{0.0121} = \frac{3010}{121} \to n < 24.876\ldots$.
   So the answer is $\boxed{24}$.

5. It should be obvious that if Andy chooses A, Bailes should choose $A - 1$ or $A + 1$ (if possible) to maximize his chance of winning. Accordingly, given A, Bailes chance of winning is $\max\left(\sum_{k=2}^{A-1} \dfrac{6 - |7-k|}{36}, \sum_{k=A+1}^{12} \dfrac{6 - |7-k|}{36}\right)$, as $\dfrac{6 - |7-k|}{36}$ is the probability of two six-sided die rolls summing to k. For Andy to maximize his expected winnings, he should pick A to minimize Bailes' expected winnings. Note that the first term in the max function increases as A increases, while the second decreases as A increases. Therefore, the maximum winnings by Bailes is minimized when the two terms are equal, which they are at $A = \boxed{7}$.

6. Andy picks 7. Bailes picks 6 or 8 (both are equivalent, assume he picks 8). Andy wins if 2 through 7 come up on the dice, which happens with probability $21/36 = 7/12$. Therefore, his expected winnings are $\boxed{\$\dfrac{7}{12}}$.

7. Again, if Andy picks A, Bailes should pick $A-1$ or $A+1$ (if possible) to maximize his chance of winning. As all numbers are equally likely, given A, Bailes' expected winnings are

$$\frac{1}{30}\max\left(\sum_{k=1}^{A-1} k, \sum_{k=A+1}^{30} k\right) = \frac{1}{30}\max\left(\sum_{k=1}^{A-1} k, 465 - \sum_{k=1}^{A} k\right).$$

(Note that 465 is the sum of the integers from 1 to 30). Also note that

$$\max\left(\sum_{k=1}^{A-1} k, 465 - \sum_{k=1}^{A} k\right) = \max\left(\frac{(A-1)\,A}{2}, 465 - \frac{A\,(A+1)}{2}\right).$$

Treated as function of a real-valued A, the two terms in the max function cross when

$$\frac{(A-1)\,A}{2} = 465 - \frac{A\,(A+1)}{2} \rightarrow \frac{A\,(A+1)}{2} + \frac{A\,(A-1)}{2} = 465 \rightarrow A^2 = 465 \approx 21.5.$$

Therefore, the minimum of the maximum of these two terms occurs at either $A = 21$ or $A = 22$. Testing both values of A, we get $\max(210, 234) = 234$ when $A = 21$ and $\max(231, 189) = 231$ when $A = 22$. For Andy to maximize his expected winnings, he should minimize Bailes' expected winnings, which occurs at $A = \boxed{22}$.

8. Andy picks 22, Bailes picks 21. Andy wins if 22 through 30 come up, so his expected winnings are $\frac{\$22 + \$23 + \cdots + \$30}{30} = \frac{\$234}{30} = \frac{\$39}{5} = \boxed{\$7.80}$.

9. We could go through the same calculations as the previous solutions, or just observe the following: if Andy picks A, Bailes will always pick $A + 1$, as he wants to win if very high numbers come up (in which case he gets the majority of the cash) and he wants to "lose" if very low numbers come up (in which case he, again, gets the majority of the cash). Accordingly, Andy should deny Bailes the opportunity to pick $A + 1$ by choosing $\boxed{20}$.

10. Andy picks 20, Bailes picks 19. For rolls from 1 to 19, Bailes is always closer, but both players average winning \$10. If it is a 20, Andy wins \$20, Bailes wins \$0. Andy's expected winnings is $\frac{19}{20} \times \$10 + \frac{1}{20} \times \$20 = \frac{\$21}{2} = \boxed{\$10.50}$.

Individual Problems

1. Compute the value of x such that the mean and median of the set $\{2, 0, 11, x, 2011\}$ are equal.
2. When 4 is added to both the numerator and denominator of a certain fraction, the resulting fraction is equal to $3/5$. If 1 is added to both the numerator and denominator of the original fraction, the resulting fraction is equal to $1/2$. Compute this fraction.
3. The prime factorization of the three-digit number $N = \underline{A}\,\underline{0}\,\underline{B}$ is $N = 3^A \times B$. Compute N.
4. The interior angles of a convex polygon are in arithmetic progression. If the smallest interior angle is 130 degrees and the largest interior angle is 170 degrees, compute the number of sides of the polygon.
5. If $\sin x + \cos x = 5/4$, and $0 < x < \pi/4$, compute $\cos(2x)$.
6. Compute the constant term in the expansion of $\left(3x^2 + \dfrac{2}{x}\right)^6$.
7. Compute the tens digit in $1! + 2! + \cdots + 2011! = \displaystyle\sum_{n=1}^{2011} n!$.
8. $ABCD$ is a square of side length 1. E lies on $\overline{BC}$ and F lies on $\overline{CD}$ such that AEF is an equilateral triangle. Compute AE.
9. For a non-negative integer k, let $b(k)$ be the sum of the digits in the binary (base-2) representation of k. For example, $b(12) = b(1100_2) = 1 + 1 + 0 + 0 = 2$. Compute the smallest integer n such that $b(b(b(n))) > 1$.
10. Compute $\left\lceil \left(\sqrt{5} + \sqrt{3}\right)^4 \right\rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Individual Solutions

1. The median of the set is either 2, 11, or x, depending on whether x is less than 2, greater than 11, or between 2 and 11, respectively. The average of the set is $\frac{2011+11+2+0+x}{5} = \frac{2024+x}{5}$. Clearly, x must be less than 2 in order for the mean and median to be equal. Thus, the median is 2, and $\frac{2024+x}{5} = 2 \rightarrow 2024+x = 10 \rightarrow x = \boxed{-2014}$.

2. Let the fraction be $\frac{A}{B}$. From the two statements, we know that $\frac{A+4}{B+4} = \frac{3}{5}$ or $5A+20 = 3B+12$ and $\frac{A+1}{B+1} = \frac{1}{2}$ or $2A+2 = B+1$. Solving these equations simultaneously gives us $\frac{A}{B} = \boxed{\frac{5}{11}}$.

3. As B is at most 9, we know that A must be greater than 1. Accordingly, N is a multiple of nine. Therefore, we know $A+B = 9$ and $A \geq 2$. Additionally, the only way that N has B as its units digit is if $3^A \equiv 1 \mod 10$, so A is either 4 or 8. A quick check shows that $3^4 \times 5 = \boxed{405}$ is our answer.

4. The sum of the angle measurements of a convex n-gon is $180(n-2)$. Similarly, an n-term arithmetic sequence with first term 130 and last term 170 sums to $\left(\frac{130+170}{2}\right) n = 150n$. $150n = 180(n-2) \rightarrow 30n = 360 \rightarrow n = \boxed{12}$.

5. $\sin^2 x + 2\sin x \cos x + \cos^2 x = 25/16 \rightarrow 2\sin x \cos x = 9/16 \rightarrow \sin 2x = 9/16 \rightarrow \cos 2x = \sqrt{1-\left(\frac{9}{16}\right)^2} = \sqrt{\frac{175}{256}} = \boxed{\frac{5\sqrt{7}}{16}}$.

6. All terms in the expansion are of the form $\left(3x^2 + \frac{2}{x}\right)^6 = \sum_{p=0}^{6} \binom{6}{p} \left(3x^2\right)^{6-p} \left(\frac{2}{x}\right)^p$. The constant term occurs when the exponents of the x terms sum to zero, or when $2(6-p)+(-1)p = 0 \rightarrow 12-3p = 0 \rightarrow p = 4$. This term is $\binom{6}{4}\left(3x^2\right)^{6-4}\left(\frac{2}{x}\right)^4 = 15\times\left(3x^2\right)^2\times\left(2x^{-1}\right)^4 = 15\times 9\times 16 = \boxed{2160}$.

7. For $n \geq 10$ the tens digit of $n!$ is zero, as $n!$ is a multiple of 100. Accordingly, we are only interested in the last two digits of the first 9 factorials: 1 + 2 + 6 + 24 + 20 + 20 + 40 + 20 + 80 = 213, so the tens digit is $\boxed{1}$.

8. Let $AE = s$. Draw AC, which intersects EF and G. AG is an altitude of AEF, which has length $\frac{\sqrt{3}s}{2}$. GC is an altitude of CEF, a 45−45−90 triangle which has length $\frac{s}{2}$. $\sqrt{2} = AC = AG + GC = \frac{\sqrt{3}s}{2} + \frac{s}{2} \rightarrow s = \frac{2\sqrt{2}}{\sqrt{3}+1} = \boxed{\sqrt{6}-\sqrt{2}}$.

9. Working backwards, we want $b(b(b(n))) = 2$. The smallest number with 2 ones in its binary representation is 3, the smallest number with 3 ones in its binary representation is 7, and the smallest number with 7 ones in its binary representation is $\boxed{127}$.

10. Let $x = \left(\sqrt{5}+\sqrt{3}\right)^4 + \left(\sqrt{5}-\sqrt{3}\right)^4$. Note that when both terms in x are expanded and added together, the terms with odd powers of $\sqrt{5}$ and $\sqrt{3}$ will cancel, leaving only even powers of $\sqrt{5}$ and $\sqrt{3}$ (which will be integers). As $\left(\sqrt{5}-\sqrt{3}\right) < 1$, $x = \left\lceil \left(\sqrt{5}+\sqrt{3}\right)^4 \right\rceil$. $x = \left(\sqrt{5}+\sqrt{3}\right)^4 + \left(\sqrt{5}-\sqrt{3}\right)^4 = 2\left(\sqrt{5}\right)^4 + 12\left(\sqrt{15}\right)^2 + 2\left(\sqrt{3}\right)^4 = 50 + 180 + 18 = \boxed{248}$.

Relay Problems

R1-1. Farmer Martin has some chickens and some cows. He looks in his barn and counts 58 legs. He sees 17 heads on the animals in his barn. Assuming that all chickens have two legs and all cows have four legs (and all animals have one head), compute the number of chickens in the barn.

R1-2. Let $T = TNYWR$. Compute $\log_{32}\left(\sqrt{21} - \sqrt{T}\right) + \log_{32}\left(\sqrt{21} + \sqrt{T}\right)$.

R2-1. Let S be a set of nine distinct integers. Six of the elements of the set are 1, 2, 3, 5, 8, and 13. Compute the number of possible values for the median of S.

R2-2. Let $T = TNYWR$. Compute the largest integer K such that a $1 \times K$ rectangle can be completely covered by an arrangement of T disks of radius 1.

R2-3. Let $T = TNYWR$. ABC and BAD are congruent 30-60-90 triangles, both with AB as their hypotenuse of length T. If C and D are distinct points in the plane and the intersection of the triangles has positive area, compute the area of the region common to both ABC and BAD.

R3-1. Compute the number of positive integers greater than one whose fourth powers are factors of 9!.

R3-2. Let $T = TNYWR$. $\log_2 x + \log_4 x = \log_b x^T$. Compute b.

R3-3. Let $T = TNYWR$. Compute the number of ordered pairs of integers (x, y) such that $0 \leq x \leq T$, $0 \leq y \leq T$, $x + y \neq T$, and $x + y \neq T + 2$.

R3-4. Let $T = TNYWR$. Let S be the set of all positive multiples of 9 that contain at least one 2 in their base-10 representation. The smallest element of S is 27. Compute the T^{th} smallest element of S.

R3-5. Let $T = TNYWR$. Compute the minimum perimeter of a rectangle with integer side lengths and area T.

R3-6. Let $T = TNYWR$. Compute the number of three element subsets S of $\{1, 2, \ldots, T\}$ such that the elements of S can be arranged into an arithmetic sequence.

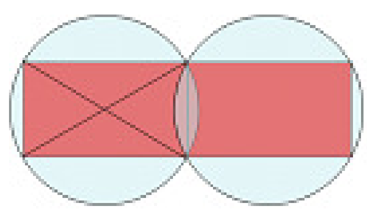
Relay Solutions

R1-1. Let H and W be the number of chickens and cows, respectively. $H+W=17$ and $2H+4W=58 \to 2H+4(17-H)=58 \to 68-2H=58 \to H=\boxed{5}$.

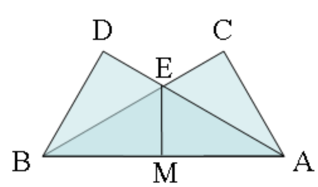
R1-2. $\log_{32}\left(\sqrt{21}-\sqrt{T}\right)+\log_{32}\left(\sqrt{21}+\sqrt{T}\right)=\log_{32}\left(\left(\sqrt{21}-\sqrt{T}\right)\left(\sqrt{21}+\sqrt{T}\right)\right)=\log_{32}(21-T)$.
As $T=5$, $\log_{32}16=\dfrac{\log_2 16}{\log_2 32}=\boxed{\dfrac{4}{5}}$.

R2-1. If all of the other elements of S are greater than 8, then the median is 8. If all of the other elements are less than 2, then the median is 2. No three integers exist such that 1, 13, or any other integer less than 2 or greater than 8 can be the median. However, any integer between 2 and 8 (inclusive) can be the median of S, so the answer is $\boxed{7}$.

R2-2. The largest $1\times n$ rectangle you can cover with a single disk of radius 1 is when $n=\sqrt{3}$. In that case, the center of the rectangle is identical to the center of the circle, and the four corners lie on the circle itself. One can quickly see that T disks can cover at most a $1\times T\sqrt{3}$ rectangle. As $T=7$, the answer is $\lfloor 7\sqrt{3}\rfloor=\boxed{12}$. Note, if unable to estimate $\sqrt{3}\approx 1.73$, alternately, consider $\left(7\sqrt{3}\right)^2=147$, which is slightly larger than 12^2.



R2-3. Let E be the intersection of AD and BC, and let M be the midpoint of AB. Triangles EBM and EAM are both $30-60-90$ triangles with bases of length $\frac{T}{2}$ and heights of $\frac{T\sqrt{3}}{6}$. The area of triangle ABE is therefore $\frac{T^2\sqrt{3}}{12}=\frac{144\sqrt{3}}{12}=\boxed{12\sqrt{3}}$.



R3-1. Note that $9!$ contains seven powers of 2 (namely, as factors of 2 and 6, as well as 4, which contains two powers of 2, and 8, which contains three powers of 2) and four powers of 3 (namely, as factors of 3 and 6, as well as 9, which contains two powers of 3). As 2^4 and 3^4 divide $9!$, so must 6^4. The answer is $\boxed{3}$.

R3-2. $\log_2 x + \log_4 x = T\log_b x \to \log_2 x + \dfrac{\log_2 x}{\log_2 4} = T\dfrac{\log_2 x}{\log_2 b} \to \dfrac{3}{2} = \dfrac{T}{\log_2 b}$. As $T = 3$, $b = \boxed{4}$.

R3-3. There are $(T+1)^2$ ordered pairs in total. Of these, $T+1$ sum to T and $T-1$ sum to $T+2$. The number of ordered pairs remaining is $(T+1)^2 + (T+1) + (T-1) = T^2 + 1$. As $T = 4$, the answer is $\boxed{17}$.

R3-4. The first two elements of the set are 27 and 72. For three digit numbers, there are two that begin with each digit besides 2 (e.g., 423 and 432), and for three-digit numbers beginning with 2, there are 11 of them (from 207 to 297). As $T = 17$, you want the second one starting with 3, which is $\boxed{342}$.

R3-5. For the perimeter to be minimized, you want the side lengths to be as close together as possible. As $342 = 2 \times 3 \times 3 \times 19$, the side lengths should be 18 and 19, resulting in a perimeter of $\boxed{74}$.

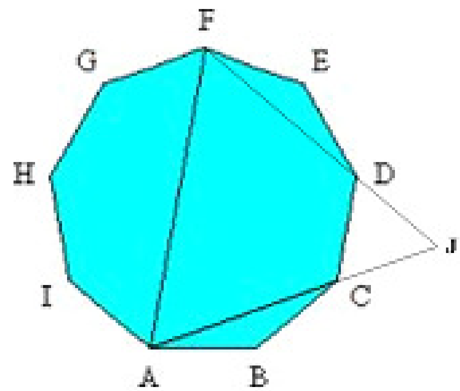
R3-6. For a fixed common difference, three elements of the sequence will be a, $a+d$, and $a+2d$ where $1 \le a \le T - 2d$. If $d = 1$, there are $T-2$ possible values of a, similarly for $d = 2$, there are $T-4$ possible values of a, and so forth. The total number of sets will be the sum of all of the numbers of the same parity as T that are less than T. As $T = 74$, we want $2 + 4 + \cdots + 72 = 74 \times \frac{36}{2} = \boxed{1332}$.

Tiebreaker Problem

1. Let $ABCDEFGHI$ be a regular nonagon of side length 3. Compute $AE - AC$.

Tiebreaker Solution

1. In the image below, note that $AF = AE$. Extending AC and DF, they intersect at a point J. The interior angles of a regular nonagon are 140 degrees. Accordingly, $\angle FDE = 20^\circ$ and $\angle FDC = 120^\circ$, so $\angle JDC = 60^\circ$. Accordingly, AFJ and JDC are equilateral triangles, so $AE = AF = AC + CJ = AC + CD$. Therefore $AE - AC = CD = \boxed{3}$.



Answers to ARML Local 2011

Team Round:

1. 31	2. $\frac{7}{90}$	3. $27\sqrt{3}$	4. 176	5. 132
6. $\frac{2}{3}$	7. 10	8. $\frac{181\pi}{60}$	9. $\frac{1055}{2011}$	10. 55

Theme Round:

1. $\frac{121}{4}$	2. 3	3. 3	4. 24	5. 7
6. $\$\frac{7}{12}$	7. 22	8. \$7.80	9. 20	10. \$10.50

Individual Round:

1. -2014	2. $\frac{5}{11}$	3. 405	4. 12	5. $\frac{5\sqrt{7}}{16}$
6. 2160	7. 1	8. $\sqrt{6}-\sqrt{2}$	9. 127	10. 248

Relay Round:

Relay 1:	1. 5	2. $\frac{4}{5}$				
Relay 2:	1. 7	2. 12	3. $12\sqrt{3}$			
Relay 3:	1. 3	2. 4	3. 17	4. 342	5. 74	6. 1332

Tiebreaker:

1. 3

ARML Local 2012

Contents

Team Problems

1. Compute the smallest positive integer for which the product of its digits is 96.

2. Compute the smallest integer $n > 1$ such that $10n$ has exactly 21 more factors than n.

3. The area of a right triangle ABC is 6 and the length of the hypotenuse $\overline{AB}$ is 10. Compute $\sin 2A$.

4. Let T be the set of all two digit positive integers, and let S_k be the sum of all elements of T *except* k. Compute the value of k such that S_k is a palindrome (that is, the digits of the number read the same forwards as backwards, such as 12321).

5. Let A, B, C, D, and E be a sequence of increasing positive integers. The set

$$\{18, 20, 21, 22, 23, 24, 25, 26, 28, 29\}$$

is the set of all pairwise sums of distinct integers in the sequence. Compute the ordered 5-tuple (A, B, C, D, E).

6. Compute the surface area of the boundary of the following region:

$$\left\{(x, y, z) : x^2 + y^2 \leq 4, |x| \leq 1, |z| \leq 4\right\}.$$

7. Compute the number of ordered 4-tuples (a, b, c, d) of positive integers such that $a+b+c+d = 45$ and no two of the integers differ by a multiple of 5.

8. A fair ten-sided die with the digits 0 to 9 (inclusive) on its faces is rolled three times. If M is the expected value of the largest possible three-digit number that can be formed using each of the three digits rolled exactly once, compute $\lfloor M \rfloor$, the greatest integer less than or equal to M. For example, if 4, 5, and 4 were rolled, the largest possible three-digit number would be 544. If three zeroes are rolled, the largest possible number is zero.

9. Compute the sum of all solutions to $\left(\log_a (2a-7)\right)\left(\log_{a-2} a\right) = -\dfrac{1}{2}$.

10. Let $\{x\}$ denote the fractional part of x. Compute the largest x such that $\{x\}^2 = x^2 - 56$.

Team Solutions

1. The correct answer will have digits in increasing order, so a strategy would be to look for the largest single digit factor of 96 (8) and iterate (left with 12, largest factor is 6, leaving 2). The answer is $\boxed{268}$.

2. Let $n = 2^a 3^b 5^c$; then $m = 10n = 2^{a+1}3^b 5^{c+1}$. Note that n has $(a+1)(b+1)(c+1)$ factors, and m has $(a+2)(b+1)(c+2)$ factors. The difference is $(b+1)(a+c+3) = 21$. Arguably, c should be zero to minimize the value of n, so $(b+1)(a+3) = 21$ has non-negative integer solutions of $(a, b) = (18, 0)$, $(4, 2)$, and $(0, 6)$. The smallest of these is the middle, which is equal to $2^4 3^2 = \boxed{144}$.

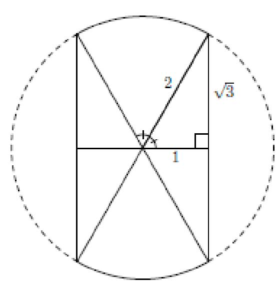
3. Let the legs of the triangle be x and y, then $xy = 12$, so:

$$\sin 2A \; = 2 \sin A \, \cos A \; = 2\left(\frac{x}{10}\right)\left(\frac{y}{10}\right) = \frac{xy}{50} = \boxed{\frac{6}{25}}.$$

4. The sum of the elements in T is 4905. The first palindrome less than 4905 is 4884, so $\boxed{21}$ has been removed from the set.

5. Note that the sum of the pairwise sums is 4 times the sum of the elements in the sequence, so $A+B+C+D+E = 59$. Note also that $A+B = 18$ and $D+E = 29$, so $C = 12$. The second smallest sum is $A+C$, so $A = 8$ and therefore $B = 10$. By similar reasoning, $C+E = 28$, so $E = 16$ and $D = 13$, so the answer is $\boxed{(8, 10, 12, 13, 16)}$.

6. The cross section of this cylindrical region looks like a circle of radius 2 with sectors cut off by the chords $x = 1$ and $x = -1$. Thus we can compute its area by breaking it into 30-60-90 triangles and $60°$ circular wedges. Thus the surface area of the of top and bottom of the solid is $2\sqrt{3} + \frac{4\pi}{3}$. The sides consist of two rectangles of height 8 and width $2\sqrt{3}$, and two curved walls of height 8 and width $\frac{2\pi}{3}$. Thus the surface area is $2\left(2\sqrt{3} + \frac{4\pi}{3}\right) + 2 \times 16\sqrt{3} + 2 \times \frac{16\pi}{3} =$ $\boxed{36\sqrt{3} + \frac{40\pi}{3}}$.



7. The only way that none of the four integers can differ by a multiple of 5 is if they all have different residues modulo 5, and these residues must sum to a multiple of 5. The only way this is possible is if the residues are 1, 2, 3, and 4. There are $4! = 24$ ways to arrange the

residues, then we are left with 7 "blocks" of five to be distributed among the four numbers, of which there are $C(10, 7) = 120$ ways to distribute, so the answer is $\boxed{2880}$.

8. Let A, B, and C be the largest, middle and smallest number rolled. By symmetry, we know that $E[A] = 9 - E[C]$ and that $E[B] = 4.5$, and the expected value of $\underline{A}\,\underline{B}\,\underline{C}$ is

$$100E[A] + 10E[B] + E[C],$$

so the challenge is to determine the expected value of the largest die roll. This is $\displaystyle\sum_{k=0}^{9} kP(A = k)$, where

$$P(A = k) = P(x_1 \le k)\, P(x_2 \le k)\, P(x_3 \le k) - P(x_1 \le k-1)\, P(x_2 \le k-1)\, P(x_3 \le k-1),$$

which is

$$\frac{(k+1)^3 - k^3}{1000} = \frac{3k^2 + 3k + 1}{1000}.$$

The expected value of A is 6.975, so the expected value of the largest possible three digit number formed is $697.5 + 45 + 2.025 = 744.525$. We are looking for the integer portion, which is $\boxed{744}$.

9. $\dfrac{\log(2a-7)}{\log a} \cdot \dfrac{\log a}{\log(a-2)} = -\dfrac{1}{2} \to \log(2a-7) = -\dfrac{1}{2}\log(a-2) \to 2a - 7 = \dfrac{1}{\sqrt{a-2}}$

Let $x = \sqrt{a-2} \to 2x^2 - 3 = \dfrac{1}{x} \to 2x^3 - 3x - 1 = 0 \to (x+1)(2x^2 - 2x - 1) = 0 \to x = \dfrac{2 \pm \sqrt{12}}{4} \to x = \dfrac{1+\sqrt{3}}{2} \to \sqrt{a-2} = \dfrac{1+\sqrt{3}}{2} \to a = 2 + \left(\dfrac{1+\sqrt{3}}{2}\right)^2 \to a = 2 + \dfrac{4+2\sqrt{3}}{4} \to$

$a = \boxed{3 + \dfrac{\sqrt{3}}{2} = \dfrac{6+\sqrt{3}}{2}}$.

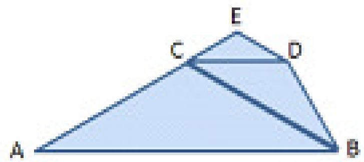
10. Since the left hand side of the equation is between 0 and 1, we know that $7 \le x \le 8$, so let $x = 7 + y$, where $y = \{x\}$ is between 0 and 1. So $y^2 = (7+y)^2 - 56 \to 0 = 49 + 14y - 56 \to y = 0.5 \to x = \boxed{7.5}$.

Theme Problems

Tussles with Triangles

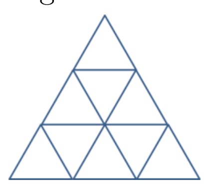
1. The lengths of the sides of a right triangle are b, $b+d$, and $b+3d$. If b and d are both positive, compute the fraction b/d.

2. ABC is an isosceles triangle with $AB = 3$ and $\mathrm{m}\angle C = 120°$. BCD is constructed such that D is external to ABC and $\triangle ABC \sim \triangle BCD$. CDE is constructed such that E is external to BCD and $\triangle BCD \sim \triangle CDE$. If this process is repeated infinitely, compute the sum of the areas of the triangles.



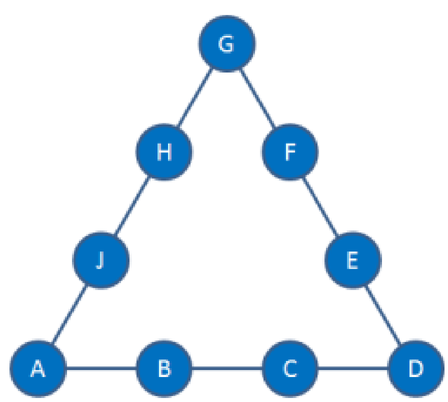
3. The *fractional triangular number* $T(x)$ is defined for all non-negative real numbers x by $T(x) = \sum_{k=0}^{\lfloor x \rfloor} (x-k)$. Compute the value of x such that $T(x) = 2012$.

4. Compute the number of distinct ways to color the nine triangles in the figure below either red, white, or blue such that no two triangles that share a side are the same color.



5. Triangle ABC is equilateral with side length 12. The region S consists of all points in the interior of ABC that are at least $\sqrt{3}$ away from all sides of ABC and less than 2 away from the incenter of ABC. Compute the area of region S.

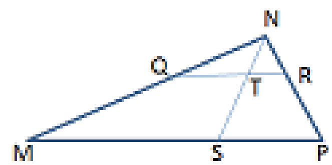
6. In the figure below, distinct non-zero digits are placed in the nine circles such that the sums of the digits along each side of the triangle are equal ($A+B+C+D = D+E+F+G = G+H+J+A$). Compute the smallest possible value of the four-digit number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$.



7. Points A, B, and C lie on a circle. The tangent line to the circle at A and the line $\overleftrightarrow{BC}$ intersect at D, with B between C and D. If $BC = 6$ and $AD = 3\sqrt{3}$, compute BD.

8. If x, y, and z are the angles of a triangle, compute the minimum value of $\sin x \sin y \cos z$.

9. In triangle MNP, Q and R lie on MN and NP, respectively. NS is the angle bisector of $\angle MNP$. If T lies on QR and NS, and $QN : RN : MQ : RP = 2 : 3 : 4 : 5$, compute the ratio NT/NS.



For Question 10, a n-sided die has the integers between 1 and n (inclusive) on its faces. All values on the faces of the die are equally likely to be rolled.

10. Consider the following game. A 20-sided die is rolled, and the player can either receive the number rolled in dollars, or can discard the 20-sided die and roll a 4-sided and an 8-sided die and receive the *product* of the two rolls in dollars. Assuming the player chooses a strategy to keep or pass their first roll that maximizes their expected winnings, compute the expected winnings by the player on a single play of the game.

Theme Solutions

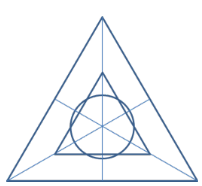
1. $b^2 + (b+d)^2 = (b+3d)^2 \to 2b^2 + 2bd + d^2 = b^2 + 6bd + 9d^2 \to b^2 - 4bd - 8d^2 = 0$. Let $d = 1$ (since the ratio is invariant to scaling both b and d) then $b^2 - 4b - 8 = 0$ has a positive root of $\boxed{2 + 2\sqrt{3}}$.

2. Observe that in the limit, the union of the triangles will be a 30-60-90 triangle with hypotenuse 3, which has area $\boxed{\dfrac{9\sqrt{3}}{8}}$. Alternately, $BC = \sqrt{3}$ (using 30-60-90 triangles) and the altitude of ABC from C has length $\frac{\sqrt{3}}{2}$. The area of ABC is $\frac{3\sqrt{3}}{4}$ and future triangles will be scaled in both base and height by a factor of $\frac{1}{\sqrt{3}}$, so we get an infinite geometric series of areas with common ratio $\frac{1}{3}$, so the sum is $\dfrac{\frac{3\sqrt{3}}{4}}{1 - \frac{1}{3}} = \boxed{\dfrac{9\sqrt{3}}{8}}$.

3. Note that this function matches the triangular numbers on the non-negative integers and is piecewise linear otherwise. 2012 is between the 62^{nd} and 63^{rd} triangular numbers (1953 and 2016). Their difference is 63, so 2012 is 59/63rds of the way from 1953 to 2016, so $T(2012) = \boxed{62\dfrac{59}{63} = \dfrac{3965}{63}}$.

4. The three corners need simply be different than their adjacent triangles, so the answer is eight times the number of ways to color the hexagon such that no adjacent triangles are the same color. Number them 1 to 6 starting from the top and going clockwise. Consider the triangles that share the same color as 1 (e.g., White). Either none of them do (in which case there are 2 possible colorings of the rest: RBRBR and BRBRB), or just 3 or just 5 does (in which case the triangle between the two whites has two possible colorings, and the remainder can be BRB or RBR, or both 3 and 5 (in which case all of the even triangles can be blue or red independent of each other) or just 4 (in which case the two remaining adjacent pairs can each be BR or RB. This makes 22 possible colorings in total if 1 is white, so there are $22 \times 3 \times 8 = \boxed{528}$ possible colorings.

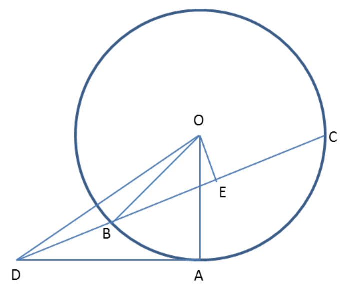
5. The picture below shows the regions of interest. The inside triangle has side length 6. So the region inside both the inside triangle and the circle consists of three equilateral triangles of side length 3 and three 60-degree sectors of a circle of radius 2 (or half a circle). The total area is $\boxed{2\pi + 3\sqrt{3}}$.



6. There are several solutions to this puzzle. Note that the three corner values must sum to a multiple of three, and that the sum along each side will be $15 + \frac{A+D+G}{3}$. However, this is

a necessary, but not sufficient condition (for example, there is no solution with 1, 8, and 9 at the corners). We want $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ as small as possible, so it would be a good idea to try small values for A and B, such as 1 and 2. C and D should be fairly large, as otherwise the side cannot sum to at least 15. Some diligent testing gives a full solution of 1297(35)4(68)1, wrapping from A to J and back to A again. There is no smaller solution, so $\boxed{1297}$ is the answer.

7. Let O be the center of the circle of radius r and drop a perpendicular $\overline{OE}$ to $\overline{CB}$.



Note $CE = EB$ and $ED^2 - EB^2 = CD \times BD$ or $ED^2 = EB^2 + CD \times BD$.

Adding OE^2 to both sides, we get:

$$ED^2 + OE^2 = EB^2 + CD \times BD + OE^2 \rightarrow$$

$$OD^2 = CD \times BD + OB^2 \rightarrow r^2 + AD^2 = CD \times BD + r^2 \rightarrow$$

$$BD \times CD = AD^2 \rightarrow BD \times (BD + 6) = 27 \rightarrow BD = \boxed{3}.$$

8. We want z to be obtuse so that $\cos z < 0$. We claim the minimum value occurs when $x = y$. Assuming the correct value for z, we observe that:

$$(\sin x \cos y - \sin y \cos x)^2 \geq 0$$

$$\sin^2 x \cos^2 y - 2 \sin x \cos y \sin y \cos x + \sin^2 y \cos^2 x \geq 0$$

$$\sin^2 x \cos^2 y + 2 \sin x \cos y \sin y \cos x + \sin^2 y \cos^2 x \geq 4 \sin x \cos y \sin y \cos x$$

$$(\sin x \cos y + \sin y \cos x)^2 \geq \sin 2x \sin 2y$$

$$\sin^2 (x + y) \geq \sin 2x \sin 2y.$$

Let $x = y$ and $z = 180° - 2x$, so we wish to minimize $\sin^2 x \cos(180° - 2x) = -\sin^2 x \cos 2x$, which is the same as maximizing $\sin^2 x \cos 2x = \sin^2 x \left(1 - 2\sin^2 x\right)$. Letting $w = \sin x$, we observe that $w^2 (1 - 2w^2)$ is maximized when $w^2 = 1/4$, so we get $\boxed{-1/8}$ for the minimum value.

9. Let m$\angle MNP = 2\theta$. Then $\dfrac{[QNR]}{[MNP]} = \dfrac{[QNT]+[TNR]}{[MSN]+[SNP]}$, where $[QNR]$ denotes the area of the triangle. Therefore:

$$\frac{\frac{1}{2}QN \times RN \times \sin 2\theta}{\frac{1}{2}MN \times NP \times \sin 2\theta} = \frac{\frac{1}{2}QN \times NT \times \sin\theta + \frac{1}{2}NT \times RN \times \sin\theta}{\frac{1}{2}MN \times NS \times \sin\theta + \frac{1}{2}NS \times NP \times \sin\theta}$$

$$\frac{QN \times RN}{MN \times NP} = \frac{NT}{NS}\frac{QN + RN}{MN + NP}$$

$$\frac{NT}{NS} = \frac{MN+NP}{QN+RN}\frac{QN \times RP}{MN \times NP} = \frac{6+8}{2+3}\frac{2\times 3}{6\times 8} = \frac{14\times 6}{5\times 48} = \boxed{\frac{7}{20}}.$$

10. Consider the following game. A 20-sided die is rolled, and the player can either receive the number rolled in dollars, or can discard the 20-sided die and roll a 4-sided and an 8-sided die and receive the *product* of the two rolls in dollars. Assuming the player chooses a strategy to keep or pass their first roll that maximizes their expected winnings, compute the expected winnings by the player on a single play of the game.

11. The optimal strategy is to keep your first roll if it is greater than or equal to the expected value of the re-roll. The expected value of the re-roll is $2.5 \times 4.5 = 11.25$, so the player should keep a first roll of 12-20 and re-roll 1-11. In the first case, this means that 9/20ths of the time, the player keeps their first roll expects to receive \$16 and 11/20ths of the time the player re-rolls and expects to receive \$11.25. The expected winnings are $0.45 \times 16 + 0.55 \times 11.25 = \boxed{13.3875 = 13\frac{31}{80} = \frac{1071}{80}}$.

Individual Problems

1. Compute the number of positive integers k such that $5^4 \leq k^2 \leq 4^5$.

2. In rectangle $ABCD$, points R and S trisect side $\overline{AB}$ ($AR = RS = SB$). If $CS = 2\sqrt{13}$ and $CR = 10$, compute the area of $ABCD$.

3. A fair 6-sided die is rolled four times. Compute the probability that the numbers rolled were rolled in a strictly increasing order (e.g., 1-2-3-5, not 1-2-2-4 or 1-2-4-3).

4. Compute the side length of the smallest equilateral triangle whose vertices all lie on the $x-$ or $y-$ axes and that contains the point $(20, 12)$ on its boundary.

5. Compute the value of the infinite sum:
$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\frac{1}{4\times 6}+\frac{1}{5\times 7}+\cdots.$$

6. In the convex quadrilateral $ARML$, $AR = AL$, $RM = ML$, and $\mathrm{m}\angle M = 2\mathrm{m}\angle A$. If the area of ARL is 2012 times the area of MRL, compute $\cos\angle A$.

7. Compute the volume of the region $R = \{(x, y, z) : 0 \leq x, y \leq 4, 0 \leq z \leq \lfloor x + y\rfloor\}$, where $\lfloor w\rfloor$ denotes the greatest integer less than or equal to w.

8. Compute the solution (x, y, z) to the following system of equations, where x, y, and z are all positive integers:
$$\begin{aligned} x + xy + xyz &= 564 \\ y + xz + xyz &= 354. \end{aligned}$$

9. Compute the smallest positive integer n such that $77n$ is the product of three *consecutive* integers.

10. Pentagon $ABCDE$ is inscribed in a circle. $AB = 12$, $BC = 32$, $CD = 8$, and diagonal $\overline{BD}$ bisects diagonal $\overline{AC}$. Compute the number of possible integer values of AE.

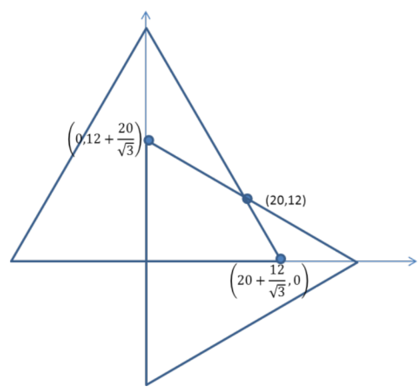
Individual Solutions

1. $5^4 \le k^2 \le 4^5 \to 25^2 \le k^2 \le (2^2)^5 = (2^5)^2 = 32^2$. Hence k can take $32 - 25 + 1 = \boxed{8}$ values.

2. Let $AR = x$ and $AD = y$. Then $x^2 + y^2 = CS^2 = 52$ and $(2x)^2 + y^2 = CR^2 = 100$. Subtracting the two equations, $3x^2 = 48 \to x = 4 \to y = 6 \to$ area $= 3xy = \boxed{72}$.

3. The probability of rolling four different numbers is $\frac{6\times5\times4\times3}{6\times6\times6\times6} = \frac{5}{18}$. The probability those four numbers are in increasing order, given they are different, is $\frac{1}{4!} = \frac{1}{24}$, so the answer is $\boxed{\dfrac{5}{432}}$. Alternately, the set of increasing sequences of four numbers from 1 to 6 is fairly easy to count (15).

4. The triangle is either symmetric around the x or y-axis, meaning that the point on the triangle in the first quadrant should lie on a line segment of slope $-\sqrt{3}$ or $-\frac{1}{\sqrt{3}}$. The line segment of slope $-\sqrt{3}$ through $(20, 12)$ hits the x-axis at $20 + 4\sqrt{3}$. The line segment of slope $-\frac{1}{\sqrt{3}}$ through $(20, 12)$ hits the y-axis at $12 + \frac{20}{3}\sqrt{3}$. Of these two, the latter is smaller, so the length of the side of the triangle is twice this, or $\boxed{24 + \dfrac{40\sqrt{3}}{3}}$.



5. This is the infinite sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1/2}{n} - \frac{1/2}{(n+2)}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \boxed{\frac{3}{4}}$.

6. Let $RL = x$. The area of an isosceles triangle with base x repeated side s and apex angle y is $\dfrac{x^2 \sin y}{2}$, where $\dfrac{x}{\sin y} = \dfrac{s}{\sin\left(90^\circ - \frac{y}{2}\right)} \to s = \dfrac{x\cos\left(\frac{y}{2}\right)}{\sin y} = \dfrac{x}{2\sin\left(\frac{y}{2}\right)}$. Thus the area of the triangle is $\left(\dfrac{x}{2\sin\left(\frac{y}{2}\right)}\right)^2 \cdot \dfrac{\sin y}{2} = \dfrac{x^2\cos\left(\frac{y}{2}\right)}{4\sin\left(\frac{y}{2}\right)} = \dfrac{x^2\cot\left(\frac{y}{2}\right)}{4}$. So we have:

$$\frac{1}{2012} = \frac{\cot\angle A}{\cot\angle A/2} = \frac{\sin\angle A/2}{\cos\angle A/2}\frac{\cos\angle A}{\sin\angle A} = \frac{\cos\angle A}{2\cos^2\angle A/2} = \frac{\cos\angle A}{1+\cos\angle A}.$$

So $\cos\angle A = \boxed{\dfrac{1}{2011}}$.

7. For a fixed integer k between 0 and 7 inclusive, the region where the height of R is equal to k is the region where $k \leq x + y < k + 1$, which is either a triangle (for $k = 0, 7$ or a trapezoid. The areas of the regions from lower left to top right are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$, so the volume is $0 \times \frac{1}{2} + 1 \times \frac{3}{2} + 2 \times \frac{5}{2} + 3 \times \frac{7}{2} + 4 \times \frac{7}{2} + 5 \times \frac{5}{2} + 6 \times \frac{3}{2} + 7 \times \frac{1}{2} = \boxed{56}$

8. We can rewrite the system as:

$$\begin{aligned} x\,(1+y)\,(1+z) - xz &= 564 \\ (xz+1)\,(y+1) - 1 &= 354. \end{aligned}$$

The second equation becomes $(xz + 1)(y + 1) = 355$, which implies that y is either 4 or 70. In the first case, we have $xz = 70$ and $5x(1 + z) = 484$, which has no integer solution. In the second case, $xz = 4$ and $71x(1+z) = 568$ or $(1+z) = 8$, which has $x = 4$, $z = 1$ as a solution. The answer is $\boxed{(4, 70, 1)}$.

9. $77n = 7 \times 11 \times n$, so for the number to be the product of three consecutive integers, you need to look for multiples of 7 and 11 that are no more than two apart. Conveniently, the sequence 20, 21, and 22 fits the bill, so $n = 3 \times 2 \times 20 = \boxed{120}$.

10. Let BD and AC intersect at P. Since $ABCD$ is cyclic, triangles ABP and DCP are similar, as are ADP and BCP. Let $x = AP = CP$, then $BP = \frac{3x}{2}$, and by similar triangles $AD = (\frac{AP}{BP}) \cdot BC = \frac{64}{3}$. Note that $\overline{AD}$ is shorter than $\overline{BC}$, which means that the arc between A and D is less than 180 degrees, in other words, $AE < AD$. Therefore, AE can take on any integer value less than AD, meaning there are $\boxed{21}$ possible values.

Relay Problems

R1-1. The sum of the lengths of the legs of a right triangle is 9. If the length of each leg is doubled, the area of the triangle increases by 24. Compute the length of the hypotenuse of the original triangle.

R1-2. Let $T = TNYWR$. Two fair 6-sided dice are rolled until their sum is greater than or equal to T. Compute the expected number of rolls of the two dice.

R2-1. Compute the two digit number $\underline{A}\,\underline{B}$ such that $3\,(\underline{B}\,\underline{A}) + \underline{A}\,\underline{B} = 300$.

R2-2. Let $T = TNYWR$. Compute the value of a such that the sum of the values x for which $(x - 2a + 3)^{(x+a)} = 1$ is T.

R2-3. Let $T = TNYWR$. Compute the largest value of n such that 5^n divides $1! \times 2! \times \cdots \times T!$.

R3-1. Compute the units digit of $1^{2012} + 3^{2012} + 5^{2012} + 7^{2012} + 9^{2012}$.

R3-2. Let $T = TNYWR$. Compute the number of bases b such that $(122)_b < T^2$.

R3-3. Let $T = TNYWR$. The minimum value of $|x - k| + |x - T|$ is T. Compute the maximum value of k.

R3-4. Let $T = TNYWR$. Compute the probability that the *product* of the rolls of two fair 6-sided dice is greater than or equal to T.

R3-5. Let $T = TNYWR$. $ABCD$ is a square of side length 12. Compute the smallest *integer* radius of a circle with center A whose intersection with $ABCD$ is *at least* T times the area of $ABCD$.

R3-6. Let $T = TNYWR$. The roots of $Tx^2 - 12x + k$ are $\cos\alpha$ and $\sin\alpha$ for an angle α. Compute k.

Relay Solutions

R1-1. Let x and y be the lengths of the legs of the original triangle, then we know that $x + y = 9$ and $\frac{xy}{2} + 24 = \left(\frac{(2x)(2y)}{2}\right) = 2xy \rightarrow \frac{3}{2}xy = 24 \rightarrow xy = 16$. The square of the length of the hypotenuse is $x^2 + y^2 = (x+y)^2 - 2xy = 81 - 32 = 49$, so the original hypotenuse has length $\boxed{7}$.

R1-2. The expected number of tries until the first success for an event with probability p of occurring is $\frac{1}{p}$. The probability of rolling a 7 or higher is $\frac{21}{36} = \frac{7}{12}$, so the answer is $\boxed{\frac{12}{7}}$.

R2-1. $3(10B + A) + (10A + B) = 300 \rightarrow 31B + 11A = 300$. This is satisfied when $A = 4$ and $B = 8$, so the answer is $\boxed{48}$.

R2-2. This equation is satisfied in one of three situations, either the exponent is zero (and the base is non-zero) or the base is 1, or the base is -1 and the exponent is even. The solutions for the three cases are $x = -a$, $x = 2a - 2$, and $x = 2a - 4$ (if a is even), respectively. These sum to $3a - 6$, so the sum of the roots is 48 when $a = \boxed{18}$.

R2-3. Considering the factors of the product, 5! through 9! each contribute one power of 5. 10! through 14! each contribute two powers of 5, and so forth. The total number of powers of 5 in the product are $5 \times 1 + 5 \times 2 + 4 \times 3 = \boxed{27}$.

R3-1. The units digit of 1 or 5 to any power is always 1 and 5, respectively. 9 to an even power always ends in a 1, and 3 and 7 to a power that is a multiple of 4 also end in 1. Thus the units digit of the sum is $1 + 1 + 5 + 1 + 1 = \boxed{9}$.

R3-2. $(122)_b = b^2 + 2b + 2 = (b+1)^2 + 1$. Therefore, if $(122)_b < T^2$, then $3 \leq b < T - 1$. Since $T = 9$, b can take $\boxed{5}$ values (3, 4, 5, 6, and 7).

R3-3. This sum is the sum of the distances from x to k and T. It is minimized when x is between T and k and is equal to $|T - k|$. The maximum value of k for which $|T - k| = T$ is $k = 2T = \boxed{10}$.

R3-4. The solver of this part probably has enough time to simply enumerate all 36 possible cases and wait for an answer. There are 19 products of the numbers between 1 and 6 that are greater than or equal to 10, so the answer is $\boxed{\frac{19}{36}}$.

R3-5. Provided the radius r of the circle is less than or equal to 12, the ratio of the areas of the quarter circle and square is $\dfrac{\frac{\pi r^2}{4}}{144} = \dfrac{\pi}{576}r^2 \geq T$. Since $T = \dfrac{19}{36} \rightarrow \dfrac{\pi}{576}r^2 \geq \dfrac{19}{36} \rightarrow r^2 \geq \dfrac{16 \times 19}{\pi} = \dfrac{304}{\pi}$. This number is just less than 100, but more than 81, so the circle of integer radius $\boxed{10}$ intersects with at least $\dfrac{19}{36}$ of the square.

R3-6. The sum of the roots is $\frac{12}{T} = \sin\alpha + \cos\alpha \rightarrow \frac{144}{T^2} = (\sin\alpha + \cos\alpha)^2 = 1 + 2\sin\alpha\cos\alpha$. The product of the roots is $\frac{k}{T} \rightarrow \frac{144}{T^2} = 1 + \frac{2k}{T} \rightarrow k = \frac{T}{2}\left(\frac{144}{T^2} - 1\right) = \frac{72}{T} - \frac{T}{2} \rightarrow k = \frac{72}{10} - 5 = \boxed{\dfrac{11}{5}}$.

Tiebreaker Problem

1. Point D is located inside equilateral triangle ABC. Perpendiculars of lengths 3 and 6 are dropped from D to sides $\overline{AB}$ and $\overline{AC}$. Compute AD.

Tiebreaker Solution

1. Let $AD = x$, and let the feet of the perpendiculars be E and F. Then $EF^2 = 6^2 + 3^2 - 2 \times 6 \times 3 \times \cos 120^\circ = 63 \to EF = 3\sqrt{7}$. Since $\angle AED$ and $\angle AFD$ are right angles, $AEDF$ is cyclic. Using Ptolemy's Theorem:
$$3\sqrt{7}x = 3\sqrt{x^2-36} + 6\sqrt{x^2-9}$$
$$\sqrt{7}x = \sqrt{x^2-36} + 2\sqrt{x^2-9}$$
$$7x^2 = \left(x^2-36\right) + 4\left(x^2-9\right) + 4\sqrt{x^2-36}\sqrt{x^2-9}$$
$$2x^2 + 72 = 4\sqrt{x^2-36}\sqrt{x^2-9}$$
$$x^2 + 36 = 2\sqrt{x^2-36}\sqrt{x^2-9}$$
$$x^4 + 72x^2 + 36^2 = 4\left(x^2-36\right)\left(x^2-9\right) = 4x^4 - 180x^2 + 36^2$$
$$3x^4 = 252x^2 \to x = \sqrt{84} = \boxed{2\sqrt{21}}.$$

Alternate Solution (by Jeff Soesbe): Let the feet of the perpendiculars be E and F, and let PQ be the segment where P is on $\overline{AB}$, Q is on $\overline{AC}$, and $\overline{PQ}$ passes through D and is parallel to $\overline{BC}$. ABC and APQ are similar, so APQ is also an equilateral triangle, and the sum of the lengths of the two given perpendiculars ($\overline{DE}$ and $\overline{DF}$) is equal to the length of the altitude of APQ (exercise left to the student reader). The altitude of APQ is 9 and the side length of APQ is $6\sqrt{3} = AQ$. DFQ is a 30-60-90 right triangle, so $FQ = 2\sqrt{3}$, meaning $AF = 4\sqrt{3}$. Finally, $AD^2 = 6^2 + \left(4\sqrt{3}\right)^2 = 36 + 48 = 84 \to AD = \sqrt{84} = \boxed{2\sqrt{21}}$.

Answers to ARML Local 2012

Team Round:

1. 268
2. 144
3. $\dfrac{6}{25}$
4. 21
5. $(8, 10, 12, 13, 16)$
6. $36\sqrt{3} + \dfrac{40\pi}{3}$
7. 2880
8. 744
9. $\dfrac{6 + \sqrt{3}}{2}$
10. 7.5

Theme Round:

1. $2 + 2\sqrt{3}$
2. $\dfrac{9\sqrt{3}}{8}$
3. $\dfrac{3965}{63}$
4. 528
5. $2\pi + 3\sqrt{3}$
6. 1297
7. 3
8. $-\dfrac{1}{8}$
9. $\dfrac{7}{20}$
10. $\dfrac{1071}{80}$

Individual Round:

1. 8
2. 72
3. $\dfrac{5}{432}$
4. $24 + \dfrac{40\sqrt{3}}{3}$
5. $\dfrac{3}{4}$
6. $\dfrac{1}{2011}$
7. 56
8. $(4, 70, 1)$
9. 120
10. 21

Relay Round:

Relay 1: 1. 7 2. $\dfrac{12}{7}$

Relay 2: 1. 48 2. 18 3. 27

Relay 3: 1. 9 2. 5 3. 10 4. $\dfrac{19}{36}$ 5. 10 6. $\dfrac{11}{5}$

Tiebreaker:

1. $2\sqrt{21}$

ARML Local 2013

Contents

Team Problems

1. Compute the number of digits in the base-10 representation of $2^3 3^4 4^5 5^6$.

2. Compute the length of the interval of the set of numbers x that satisfy $\lfloor x^2 \rfloor = \lfloor (x+3)^2 \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3. The sequence $x_1, x_2, x_3, \ldots$ is formed by setting $x_1 = 1$, $x_2 = -3$, and for $n > 2$, x_n is obtained by cubing one of the previous terms, chosen uniformly at random. Compute the probability (as a fraction) that $x_1 x_2 x_3 x_4 x_5 > 1000$.

4. Compute the remainder when $8^8 - 4^4$ is divided by $4^4 - 2^2$.

5. A, B, and C are chosen at random (with replacement) from the set of digits $\{0, 1, 2, \ldots, 9\}$. Compute the probability (as a fraction) that the units digit of A^{B^C} is a six. Note that 0^0 is undefined.

6. Compute the smallest integer b with exactly 4 positive integer divisors such that, when expressed in base b, the number $b!$ ends in exactly 10 zeroes.

7. Three balls of radius 1 are sitting on the surface of a table and are mutually tangent to each other. A solid circular cone of height 2 has its base on the surface of the table, and the lateral surface of the cone is tangent to each of the three spheres. Compute the radius of the base of the cone.

8. Compute the smallest positive angle x in degrees such that $\cos^2 x = \dfrac{1 - \sin 46^\circ}{2\sin^2 112^\circ - \sin 46^\circ + 1}$.

9. For non-zero complex numbers a and b, the chain of equalities $\dfrac{a}{b+1} = \dfrac{b-7a}{a+5} = \dfrac{3b+a}{4b+5}$ is satisfied. Compute the value of $a + 6b$.

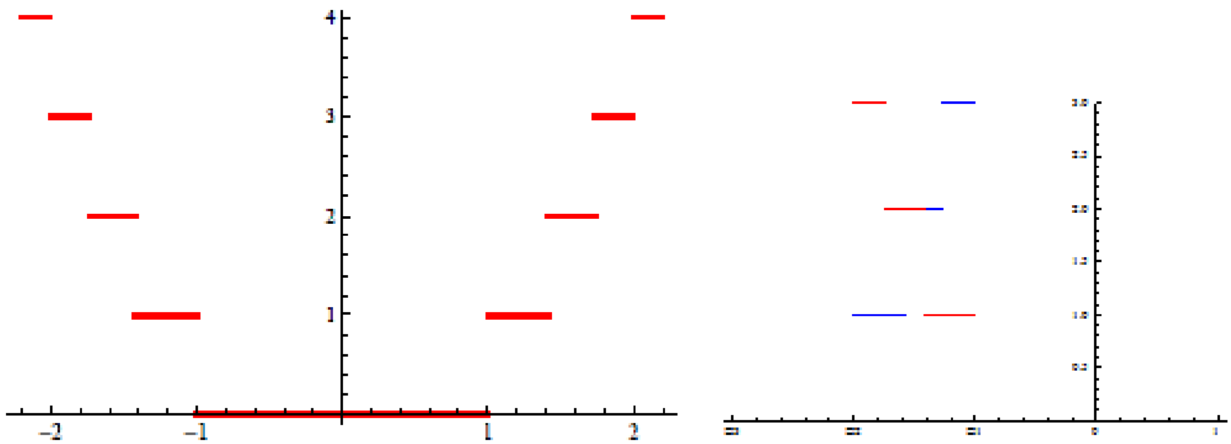
10. Let S be the set $\{1, 2, 3, 4\}$. Compute the number of functions $f : S \to S$ such that

$$f(1) + f(f(1)) + f(f(f(1))) + f(f(f(f(1)))) = 13.$$

Team Solutions

1. $2^3 3^4 4^5 5^6 = 10^6 \times 2^7 \times 3^4 = 10^6 \times 128 \times 81 \approx 10^6 \times 128 \times 80 = 10^7 \times 1024 = 1.024 \times 10^{10}$ which has $\boxed{11}$ digits.

2. Look at the graph of $y = \lfloor x^2 \rfloor$ and find plateaus of the graphs that are separated by a distance of 3, as $y = \left\lfloor (x+3)^2 \right\rfloor$ is simply the same graph left-shifted by three units. The largest distance between two points on the level $y = 0$ is 2, on the level $y = 1$ is $2\sqrt{2}$, and on the level $y = 2$ is $2\sqrt{3}$. Points in different plateaus on higher levels are separated by a distance greater than three, so all solutions occur on the level $y = 2$. The left interval on the level $y = 2$ of the graph of $y = \lfloor x^2 \rfloor$ is $(-\sqrt{3}, -\sqrt{2}]$, and the right interval on the level $y = 2$ on the graph of $y = \left\lfloor (x+3)^2 \right\rfloor$ is $[\sqrt{2} - 3, \sqrt{3} - 3)$. The intersection between the two intervals is $[\sqrt{2} - 3, -\sqrt{2}]$, so the length of the interval is $\boxed{3 - 2\sqrt{2}}$.



3. For the product to be positive, there must be two or four negative terms in the sequence. Having only two negative terms implies that -3 and -27 are the only terms in the sequence not equal to one, and thus the product is less than 1000. Therefore, the last four terms must all be not equal to one. In that case, the product is at least $-3 \times -27 \times -27 \times -27 > 1000$, so it suffices for all of the terms to be negative. This occurs with probability $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \boxed{\dfrac{1}{4}}$.

4. $4^4 - 2^2 = 2^8 - 2^2 = 2^2(2^6 - 1)$ and

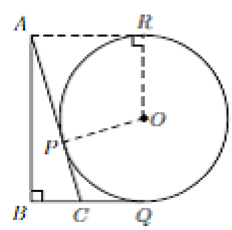
$$\begin{aligned}
8^8 - 4^4 &= 2^{24} - 2^8 = 2^2(2^{22} - 2^6) \\
&= 2^2\left(2^{22} - 2^{16} + 2^{16} - 2^{10} + 2^{10} - 2^4\right) + 2^2\left(2^4 - 2^6\right) \\
&= 2^2\left(2^6 - 1\right)\left(2^{16} + 2^{10} + 2^4\right) + \left(2^6 - 2^8\right) \\
&= 2^2\left(2^6 - 1\right)\left(2^{16} + 2^{10} + 2^4\right) + \left(-2^8 + 2^6 - 2^2 + 2^2\right) \\
&= 2^2\left(2^6 - 1\right)\left(2^{16} + 2^{10} + 2^4\right) - \left(2^8 - 2^2\right) + \left(2^6 - 2^2\right) \\
&= 2^2\left(2^6 - 1\right)\left(2^{16} + 2^{10} + 2^4\right) - 2^2\left(2^6 - 1\right) + \left(2^6 - 2^2\right).
\end{aligned}$$

The first two terms are multiples of $4^4 - 2^2$, so we are only interested in the last term, which is $2^6 - 2^2 = 64 - 4 = \boxed{60}$, which is less than $4^4 - 2^2$.

5. We know that A must be even and non-zero. If A is 6, then $B^C > 0$ so there are $9 \times 10 = 90$ choices for B and C. If A is 4, then B^C must be even and greater than zero, so there are $4 \times 9 = 36$ choices for B and C (as B must be non-zero and even and C must be non-zero). If A is 2 or 8, then B^C must be a multiple of four and greater than zero, so there are $2 \times 9 + 2 \times 8 = 34$ choices for B and C (conditioning on whether B is 0 or 2 mod 4: if B is a multiple of 4 then C must be non-zero, if B is even but not a multiple of four, then C cannot be 0 or 1). The total is 194, so the probability (as a fraction) is $\frac{194}{1000} = \boxed{\frac{97}{500} = 0.194}$.

6. If $b = p \times q$, with p and q primes and $p < q$, then $b!$ is divisible by q^p (as there are exactly p multiples of q between 1 and pq) but not divisible by q^{p+1}, so $(b!)_b$ ends in p zeroes, which won't work, since 10 is not prime. Therefore, $b = p^3$ for a prime p. Then $b!$ is divisible by p^{p^2+p+1} (as there are p^2 multiples of p between 1 and p^3, p multiples of p^2, and 1 multiple of p^3) but not divisible by p^{p^2+p+2}, so $(b!)_b$ ends in $\left\lfloor \frac{p^2+p+1}{3} \right\rfloor$ zeroes. Testing values of p quickly gets $p = 5$, so $b = 5^3 = \boxed{125}$.

7. Let the apex of the cone be A, the center of its base B, and let P be a point of tangency between the side of the cone and one of the spheres. In the vertical cross-section below we label O, Q, and R as the center, bottom, and top of that sphere, and C the point on the base of the cone such that P is on $\overline{AC}$.



If we projected the centers of all three spheres onto the table's surface, B would be the center of an equilateral triangle of side length 2. So $BQ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Note that $AP = AR = BQ$ and $BC + CP = BQ$, so we know that if the radius of the base of the cone is r, then $AP = AR = BC + CQ = r + PC = r + AC - AP \to AC = 2 \times AP - r$:

$$2^2 + r^2 = AC^2 = (2 \times AP - r)^2 = \left(\frac{4\sqrt{3}}{3} - r\right)^2 = r^2 - \frac{8\sqrt{3}}{3}r + \frac{16}{3}$$
$$\frac{8\sqrt{3}}{3}r = \frac{4}{3} \to r = \boxed{\frac{\sqrt{3}}{6}}.$$

8. Invert and subtract 1 from both sides, then take the square root to get

$$|\tan x| = \sqrt{\frac{2\cos^2 22^\circ}{1 - \sin 46^\circ}} = \sqrt{\frac{1 + (2\cos^2 22^\circ - 1)}{1 - \cos 44^\circ}} = \sqrt{\frac{1 + \cos 44^\circ}{1 - \cos 44^\circ}} = \cot 22^\circ.$$

And the smallest positive angle x in degrees that satisfies this is $x = \boxed{68^\circ}$.

9. Using the property $\frac{u}{v}+\frac{x}{y}=\frac{u+x}{v+y}$ if $\frac{u}{v}=\frac{x}{y}$ (unless v, y, or $v+y=0$):

$$\frac{a}{b+1}=3\left(\frac{b-7a}{3b+a}\right)+22\left(\frac{a}{b+1}\right)-\frac{3b+a}{4b+5}=\frac{0}{32+3a+18b}$$

unless $32+3a+18b=0$. If $32+3a+18b\neq 0$, then $a=0$, a contradiction to the conditions in the problem. Therefore $32+3a+18b=0$, so $a+6b=\boxed{-\frac{32}{3}}$.

10. Consider the orbit of 1 under repeated applications of f.

If $f(1)=1$, then the orbit is $1\to 1\to 1\to 1\to 1$, but $1+1+1+1<13$.

If $f(1)=2$, then the next three values must sum to 11, so must be $1\to 2\to 3\to 4\to 4$ or $1\to 2\to 4\to 3\to 4$ ($1\to 2\to 4\to 4\to 3$ is impossible, since 4 maps to two different numbers).

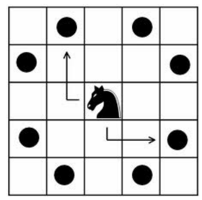
If $f(1)=3$, then the next three values must sum to 10, so must be some arrangement of $4,4,2$ or $4,3,3$, leading to two solutions $1\to 3\to 2\to 4\to 4$ and $1\to 3\to 4\to 2\to 4$. The other possible arrangements lead to contradictions.

If $f(1)=4$, then the next three values must sum to 9, so must be some arrangement of $4,4,1$ or $4,3,2$ or $3,3,3$, leading to three solutions: $1\to 4\to 3\to 3\to 3$, $1\to 4\to 2\to 3\to 4$, and $1\to 4\to 3\to 2\to 4$. The other possible arrangements lead to contradictions. Note that for the first orbit, there are four possible functions with this orbit, since $f(2)$ can be anything in S. The answer is $\boxed{10}$.

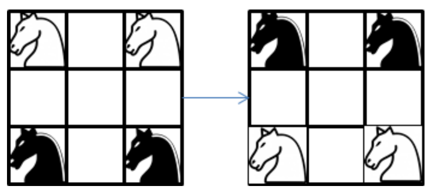
Theme Problems

Theme Round: Ain't It Funny How the Knight Moves?[1]

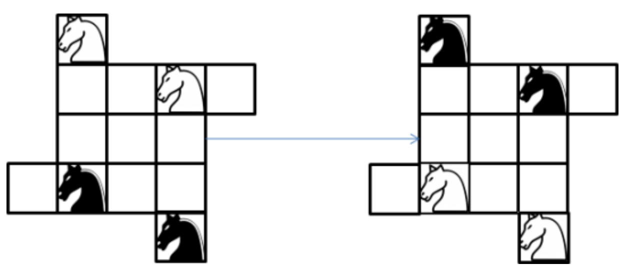
In chess, a knight can move either two squares horizontally and one square vertically, or two squares vertically and one square horizontally. The graphic below shows the eight possible locations to which the knight in the center of the 5×5 board can move. Unlike all other standard chess pieces, the knight can 'jump over' all other pieces (of either color) to its destination square.



1. Compute the minimum number of moves to exchange the positions of the two white and black knights as shown in the graphic below. Two knights may not occupy the same square at the same time. Alternating black and white knight moves is not required.

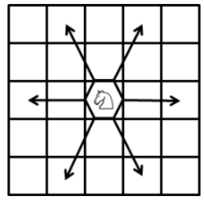


2. Compute the minimum number of moves to exchange the positions of the two white and black knights as shown in the graphic below. Two knights may not occupy the same square at the same time. Alternating black and white knight moves is not required.

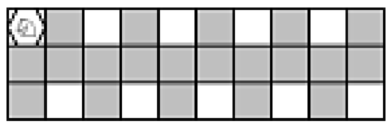


[1]Apologies to Bob Seger…to everyone, actually. That pun was horrible.

A *hexaknight* can move either two squares horizontally, or two squares vertically and one square horizontally. The graphic below shows the six possible locations to which the hexaknight in the center of the board can move.

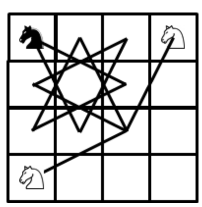


3. In the 3×10 board below, two of the eight non-corner white squares are colored gray. If all eight squares are equally likely to be colored gray, compute the probability (as a fraction) that there exists a path for the hexaknight in the upper left corner to reach the lower right corner of the board while traveling only on white squares.



4. Same as Question 3, but with *three* of the eight non-corner white squares colored gray.

5. Consider the path formed by joining in order the centers of the squares visited by a knight. The path is *simple* if none of the edges cross in their interiors and is *closed* if the knight returns to the square where it began. In the board below, the white knight's path is simple but not closed, while the black knight's path is closed but not simple. If the centers of adjacent squares are one unit apart, compute the maximum area enclosed by a knight's simple closed path on an 8×8 board.

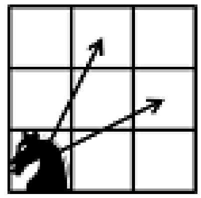


6. On a board, a square is considered *attacked* if it contains a knight or can be reached by a knight in one move (on the 5×5 board showing the possible moves of a knight, nine squares are attacked). Two knights are randomly placed on distinct squares of a 3×3 board, with all squares equally likely. Compute the expected number of squares that are attacked.

7. Compute the minimum number of knights to attack every square of a 4×6 board.

8. This question is an estimation problem. If the answer given is within 10% of the correct answer, your team will receive credit. A knight is on a square on an infinite chess board. Compute the number of distinct squares where the knight can end up after exactly 10 moves.

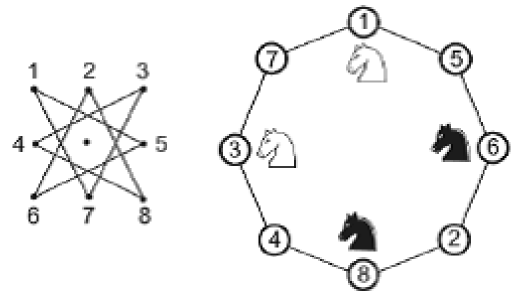
9. For integers m and n with $0 \leq m \leq n$, a (m,n)-*restricted* knight can only move either up m squares and right n squares or up n squares and right m squares. The possible moves of a $(1,2)$-restricted knight are shown below. Compute the number of distinct paths a $(1,2)$-restricted knight can take from the lower left corner to the upper right corner of a 10×10 board.



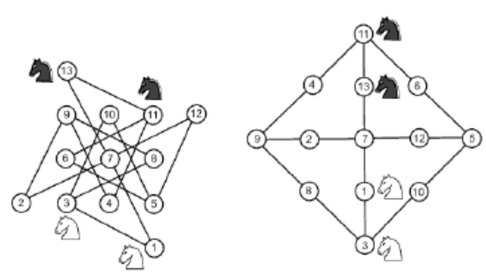
10. Compute the number of ordered pairs (m,n) with $0 \leq m \leq n$ such that there exist at least 20 distinct paths for a (m,n)-restricted knight from the lower left corner to the upper right corner of a 17×17 board.

Theme Solutions

1. Consider the graph with each vertex corresponding to a square on the border of the board (the center square cannot be visited), and two vertices are joined by an edge if a knight can move from one square to the other in a single move. The graph can be simplified to the one below. The fewest moves required is $\boxed{16}$, as no knight can pass another on this graph, so each knight must move four squares so that both knights of one color can reach the initial location of the knights of an opposite color.

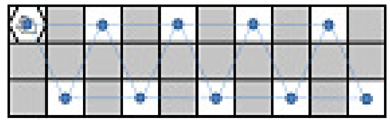


2. The solution to this question is similar to question 1, just with a slightly more complex graph. One possible solution is moving the black knight at 13 to 2 (2 moves), then the black knight from 11 to 8 (3 moves), then the white knights from 1 to 11 (3 moves) and 3 to 13 (3 moves), then the black knight from 8 to 3 (1 move) and the black knight from 2 to 1 (2 moves), for a total of 14. It remains to show that 14 moves is minimal. The minimum number of moves to interchange the knights *allowing* multiple knights to occupy the same square is 12, but there is no way to avoid two knights occupying square 1, 7, or 13 with only 12 moves. In addition, note that if the grid were colored alternating black and white, there would be one knight of each color on a grid square of each color. Accordingly, any solution which interchanges the knights would have an even number of moves (since knights change square color each move). Therefore, the minimum number of moves is $\boxed{14}$.



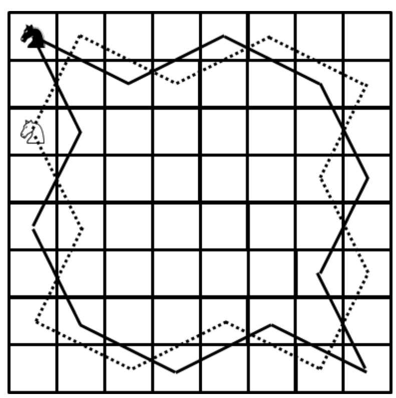
3. Consider the graph formed by joining two white squares if a hexaknight can get from one to the other in one move. Removing any single non-corner vertex from this graph will not disconnect the rest of the graph (that is, there will still be a path from the top left corner to the lower right). Removing two adjacent non-corner vertices on different rows will, however, and that is the only way to break all paths from the upper left corner to the lower right

corner. There are seven such pairs of vertices, and there are $\binom{8}{2} = 28$ possible pairs of vertices to remove, so the answer is $\frac{28-7}{28} = \boxed{\frac{3}{4}}$.

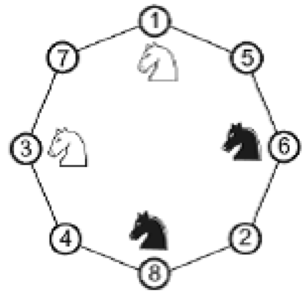


4. There are $\binom{8}{3} = 56$ possible triplets to remove. The same condition holds in that all paths from the two corners will be broken if and only if two adjacent non-corner vertices on different rows are removed. For each of the seven pairs of adjacent non-corner vertices on different rows, there are 6 other vertices that can be chosen to complete the triplet. However, each of the six triangles in the graph is double counted, so we remove those duplicates from the count leaving 36, so $\frac{20}{56} = \boxed{\frac{5}{14}}$ of the triplets can be removed and still have a path between the corners.

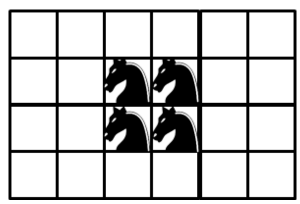
5. One would suspect that the solution uses a path that remains as close to the border of the board as possible. Two such paths are shown in the grid below, with the one by the white knight enclosing a greater area. Note that the area enclosed by the white knight's path is contained in a 7×7 square minus 12 1×2 right triangles, for a total area of $\boxed{37}$. Note that the area contained by the black knight's path is 35.



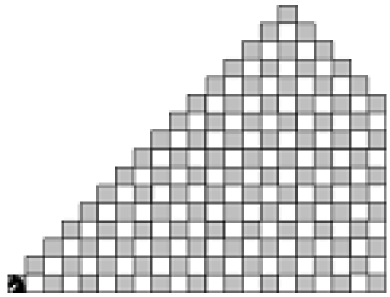
6. Note that a knight attacks either 1 or 3 squares depending on whether it is in the center or on the boundary, respectively. If one knight is in the center, then 4 squares will be attacked in total. If both are on the boundary, recalling the octagonal graph from problem 1, we can assume by symmetry that one of the knights is on square 1. If the other knight is on square 5 or 7, 4 squares are attacked in total. If the other knight is on square 3 or 6, then five squares are attacked in total. Otherwise (on 2, 4, or 8), six squares are attacked in total. Thus, if both are on the boundary, then on average $\frac{2 \times 4 + 2 \times 5 + 3 \times 6}{7} = \frac{36}{7}$ squares are attacked. The probability that both knights are on the boundary is $\frac{\binom{8}{2}}{\binom{9}{2}} = \frac{28}{36} = \frac{7}{9}$, so the expected number of squares attacked is $\frac{7}{9} \times \frac{36}{7} + \frac{2}{9} \times 4 = 4 + \frac{8}{9} = \boxed{4\frac{8}{9} \text{ or } \frac{44}{9}}$.



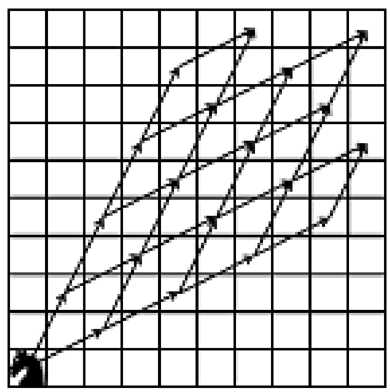
7. The arrangement below uses four knights, it is easy to show that three knights would not be able to attack every square since it is impossible for any individual knight to attack more than seven squares. Thus, the answer is $\boxed{4}$.



8. Even though the knight is on an infinite board, it will remain on the 41×41 board centered on the starting square in ten moves. In addition, if adjacent squares colors alternate, the color the knight begins on will be the same as the color of the square where it finishes, which knocks out almost half of the squares on the 41×41 board. If we give each square a coordinate with the knight starting on $(0, 0)$, the grey squares in the graphic below shows all squares (i, j) with $i \geq j$ and $j \geq 0$ that can be reached in exactly ten moves (there is vertical, horizontal, and diagonal symmetry to fill out the rest of the squares on the 41×41 board). It is relatively straightforward to show that the squares on the right boundary are reachable in ten moves, it is a bit more difficult to show (an exercise) that all of the other squares are also reachable. There are $1 + 2 + 3 + 4 + 5 + 6 + 6 + 7 + 7 + 8 + 8 + 9 + 9 + 10 + 10 = 105$ grey squares in the graphic (not counting the starting square for the knight). Creating the full octagon would have 840 squares, but we double count the 15 squares on the diagonal four times as well as the 10 squares on the same row as the knight starts, reducing the total to 740, adding in the starting square to get 741. Thus any answer between 666.9 and 815.1 is considered correct. One way to estimate this value is to note that the centers of all of the squares it can reach are contained in a circle of radius $10\sqrt{5}$, giving an area of 500π, but only half of the squares can be the final spot, so an estimate of $250\pi \approx 785$ would be an overestimate but within the allowable error bounds.



9. The picture below shows all possible paths for the $(1, 2)$-restricted knight on the 10×10 board. It should be straightforward to note the number of paths to the squares on the diagonal k moves away is simply the k^{th} row of Pascal's Triangle, so the answer is $\binom{6}{3} = \boxed{20}$.



10. Building on the previous solution, if $0 \le m < n$, we observe that it is necessary and sufficient for there to exist a path for a (m, n)-restricted knight from the lower left corner to the upper right corner of a $k \times k$ board provided that $m + n$ divides $k - 1$. The number of paths will be $\binom{\frac{2k-2}{m+n}}{\frac{k-1}{m+n}}$, as there will be $\dfrac{k-1}{m+n}$ up m right n moves and $\dfrac{k-1}{m+n}$ up n right m moves, which can occur in any order. From the answer to the last question, we know that $\dbinom{6}{3} = 20$ and we also know $\dbinom{2n}{n} \ge \dbinom{6}{3} = 20$ for $n \ge 3$. So $\dfrac{k-1}{m+n} \ge 3$. If $0 < m = n$, there would be a single path between the two corners if m divides $k - 1$. If $0 = m = n$, there are no solutions. Therefore, we count all pairs of integers m and n with $0 \le m < n$, with $(m + n) \mid 16$ and $\dfrac{16}{m+n} \ge 3$. This means that $m + n$ can equal 1, 2, or 4. Therefore, there are $\boxed{4}$ pairs: $(0, 1), (0, 2), (0, 4)$, and $(1, 3)$.

Individual Problems

1. Primes p, q, and r sum to 50. Compute the largest possible value of p.

2. On the planet Octa, each day has eight hours, and each hour has M minutes, where M is a positive integer multiple of eight. On the clocks on Octa, the hour hand goes around the clock once a day, and the minute hand goes around the clock once an hour. At 8:00, the hour and minute hand on a clock are both facing the same direction. The hour and minute hand face the same direction for the first time after 8:00 an integral number of minutes later. Compute the smallest possible value for M.

3. Let the sequence $a_1, a_2, \ldots, a_{20}$ be defined by $a_1 = 72$ and $a_n = \phi(a_{n-1})$ for $2 \leq n \leq 20$, where $\phi(n)$ is the number of positive integral divisors of n. Compute the sum of the twenty elements in the sequence.

4. The cubic equation $ax^3 + bx^2 + cx + d = 0$ has non-zero integer coefficients and three distinct integer solutions. Compute the smallest possible value of $|a| + |b| + |c| + |d|$.

5. When today's date (i.e., April 20, 2013) is written in the format YYYYMMDD, it reads 20130420 (including the trailing zero in the month). Compute the earliest future date which contains 8 distinct digits in its YYYYMMDD representation.

6. Points D and E are on side $\overline{BC}$ of triangle ABC, such that ABC, ABD, and ACE are all similar to each other. If $\cos \angle DAE = \frac{7}{9}$, compute $\cos \angle BAC$.

7. For a number $n > 1$, let $f(n)$ denote the largest number of iterations of $\log_2 n$ under which n stays strictly greater than 1. For example, $f(\sqrt{3}) = 0$ and $f(2) = 0$ because $\log_2 \sqrt{3} < 1$ and $\log_2 2 = 1$. And $f(25) = 3$ because $1 < \log_2 \log_2 \log_2 25 < 2$. Compute
$$f(2) + f(3) + \cdots + f(2013).$$

8. Let $a_1, \ldots, a_{10}$ and $b_1, \ldots, b_{10}$ be two increasing arithmetic sequences of positive integers with exactly four terms in common, the largest of which is 2013. Compute the largest possible value of b_7.

9. A rectangular box measures $6 \times 8 \times 10$. A ball of radius 2 rests in one corner of the box (that is, it is tangent to three walls). Compute the minimum distance from the ball's surface to the opposite corner of the box.

10. Compute all values for $\lfloor x \rfloor + \lfloor y \rfloor$, where x and y are positive real numbers for which $\left\lfloor x^{\lfloor y \rfloor} \right\rfloor = 20$ and $\left\lfloor y^{\lfloor x \rfloor} \right\rfloor = 13$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Individual Solutions

1. One of the primes must be 2, so assume $p + q = 48$. Testing primes less than 48, we quickly get that $\boxed{43}$ is the largest possible value for p with $q = 5$.

2. An hour later, the hour hand has moved 1/8 around the clock and the minute hand has returned to its position at 8:00. If it takes an additional k minutes for the minute hand to catch the hour hand, then $\frac{k}{M} = \frac{1}{8} + \frac{k}{8M} \to k\left(\frac{1}{M} - \frac{1}{8M}\right) = \frac{1}{8} \to k\left(\frac{7}{8M}\right) = \frac{1}{8} \to k = \frac{M}{7}$. Since k is an integer, M must be a multiple of seven as well as a multiple of eight, the smallest of which is $M = \boxed{56}$.

   Alternate Solution: The hour and minute hand intersect again seven times during the day. By symmetry, these intersections occur at equally spaced times through the day, so the next crossing occurs in 8/7 of an hour. The smallest M such that $8M/7$ is an integer is $M = \boxed{56}$.

3. In general, if $p_1^{e_1} \times \cdots \times p_k^{e_k}$ is the prime factorization of n, then $\phi(n) = (e_1 + 1) \times \cdots \times (e_k + 1)$. $\phi(72) = \phi(2^3 \times 3^2) = 4 \times 3 = 12$. Repeated applications of the ϕ-function gives the sequence $72, 12, 6, 4, 3, 2, 2, 2, \ldots$ so the sum of the first twenty terms of the sequence is $72 + 12 + 6 + 4 + 3 + 15 \times 2 = \boxed{127}$.

4. The minimum value is $\boxed{6}$, satisfied by $f(x) = (x + 1)(x - 1)(x - 2) = x^3 - 2x^2 - x + 2$. We need to show that the sum of the absolute value of the coefficients cannot be smaller than 6 with a different cubic polynomial. Since $f(x)$ is required to have non-zero coefficients, the sum is at least 4. Since the solutions are distinct, then $|d| \geq 2$. It remains to show that no polynomial of the form $x^3 \pm x^2 \pm x \pm 2$ has distinct integer valued zeroes. Note that the zeroes would have to be ± 1 and one of $\{-2, 2\}$, but $x^3 \pm x^2 \pm x \pm 2$ is odd for $x = \pm 1$, so they cannot be zeroes.

5. If the year is prior to 2100, then neither the day nor the month can contain a zero. This leaves no possibilities for the month, which must contain a 1 but now has no choice for the second digit. If the year is in the 2100s, then the month must start with a 0 and the day must start with a 3, but in this case there is no choice for the second digit of the day. If the year is in the 2300s, then the month can start with a 0 and the day can start with a 1. The priority should be to minimize the year, then the month, then the date, which results in the date $\boxed{23450617}$.

6. Points D and E only exist if A is the largest angle of the triangle. If triangle ABC is acute then $\angle DAE = A - (A - B) - (A - C) = B + C - A = 180° - 2A$. If triangle ABC is obtuse then $\angle DAE = A - B - C = 2A - 180°$. Since $\cos DAE = \frac{7}{9}$, the acute case is ruled out, as that would imply $A < 60°$, which would mean that A is not the largest angle in the triangle, so A is obtuse and:

$$\cos(2A - 180°) = \frac{7}{9}$$
$$-\cos(2A) = \frac{7}{9}$$
$$1 - 2\cos^2 A = \frac{7}{9}$$
$$\cos^2 A = \frac{1}{9}.$$

Since A is obtuse, we conclude that $\cos A = \boxed{-\frac{1}{3}}$.

7. Consider the range of integers for which $f(n) = 0, 1, 2, 3$, they are 2, 3 to $2^2 = 4$, 5 to $(2^2)^2 = 16$, and 17 to $2^{2^{2^2}} = 65536$. So $f(2) + f(3) + \cdots + f(2013) = 0 \times 1 + 1 \times 2 + 2 \times 12 + 3 \times 1997 = \boxed{6017}$.

8. Let the common differences of the sequences be d_a and d_b, respectively. And let $a_M = b_N = 2013$. If $N = 4$, then the common values must be b_1, b_2, b_3, b_4 and a_1, a_4, a_7, a_{10}, as any other subsequence of a matching b_1, b_2, b_3, b_4 will result in more than four terms in common. For example, if the terms in a in common were a_1, a_3, a_5, a_7, then $b_5 = a_9$. Thus $3d_a = d_b$ and $b_7 = 2013 + 3d_b = 2013 + 9d_a$. Since $a_1 > 0$, the largest value for d_a is $\left\lfloor \frac{2012}{9} \right\rfloor = 223$, giving $b_7 = 4020$. This corresponds to the pair of sequences $a_1 = 6, d_a = 223$ and $b_1 = 6, d_b = 669$.

   If $N = 5$, then $b_7 = 2013 + 2d_b$. Since $b_1 > 0$, that forces $d_b \le \left\lfloor \frac{2012}{4} \right\rfloor = 503$ and $b_7 \le 3019$.

   If $N = 6$, then $b_7 = 2013 + d_b$ and $d_b \le \left\lfloor \frac{2012}{5} \right\rfloor = 402$, and so $b_7 \le 2415$.

   If $N \ge 7$, then $b_7 \le b_N = 2013$. The answer is $\boxed{4020}$.

9. Place one corner of the box at the origin and the corner with the ball at $(6, 8, 10)$. The center of the ball is at $(4, 6, 8)$, which is a distance $\sqrt{4^2 + 6^2 + 8^2} = \sqrt{116} = 2\sqrt{29}$ from the origin. Therefore, a point on the ball is $\boxed{2\sqrt{29} - 2 = 2\left(\sqrt{29} - 1\right)}$ from the origin.

10. If $\lfloor x \rfloor = 1$, then we need $13 \le y < 14$ and $20 \le x^{13} < 21$. Since such (x, y) exist ($(x, y) = (\sqrt[13]{20}, 13.5)$, for example), we have $(\lfloor x \rfloor, \lfloor y \rfloor) = (1, 13)$ as a possible pair.

    If $\lfloor x \rfloor = 2$, then we need $13 \le y^2 < 14 \to \sqrt{13} \le y < \sqrt{14}$, there are solutions of this form as well, $(x, y) = (\sqrt[3]{20}, \sqrt{13.5})$, for example. This gives another solution pair $(\lfloor x \rfloor, \lfloor y \rfloor) = (2, 3)$.

    If $\lfloor x \rfloor = 3$, we need $13 \le y^3 < 14 \to \lfloor y \rfloor = 2$, but then $x^2 < 16$, so there are no solutions.

    If $\lfloor x \rfloor \ge 4$, we need $13 \le y^{\lfloor x \rfloor} < 14 \to \lfloor y \rfloor = 1$. Thus, the only remaining solution is for $\lfloor x \rfloor = 20$, with an example solution of $(x, y) = (20, \sqrt[20]{13.5})$, so $(\lfloor x \rfloor, \lfloor y \rfloor) = (20, 1)$ is the final pair. The possible sums are $\boxed{5, 14, \text{ and } 21}$.

Relay Problems

R1-1. Compute the smallest positive integer whose representations in bases 2, 3, 5, and 7 all have different numbers of digits.

R1-2. Let $T = TNYWR$. Compute the length of the shortest path along the surface of a cube of volume T from one corner to the corner furthest away from it.

R2-1. Compute the number of rearrangements of the integers 1, 2, 3, and 4 such that some set of consecutive digits in the rearrangement sums to 5.

R2-2. Let $T = TNYWR$. If x and y are positive integers, compute the smallest value for x such that $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = T$.

R2-3. Let $T = TNYWR$. Compute the smallest positive integer n such that the sum of the 3 largest divisors of n is greater than T.

R3-1. A six sided die is rolled three times. Compute the probability (as a fraction) that the three numbers rolled are in strictly increasing order.

R3-2. Let $T = TNYWR$. Let $S = 6^3 \cdot T$. Compute the probability (as a fraction) that the number thrown on a six sided die is a factor of the number thrown on an S sided die.

R3-3. Let $T = TNYWR$. In right triangle ABC (with right angle at C), $\cos A = T$ and the perimeter of the triangle is $\dfrac{7+\sqrt{21}}{4}$. Compute the length of side $\overline{AC}$.

R3-4. Let $T = TNYWR$. Two weighted coins each come up heads with probability T. Compute the expected number of tosses of both coins until they both show tails simultaneously.

R3-5. Let $T = TNYWR$. $x_1, \ldots, x_{10}$ is a sequence of positive integers where $|x_i - x_{i+1}| \leq T$ for $1 \leq i \leq 9$. If $x_1 = 2$ and $x_{10} = 12$, compute the largest integer that can appear in the sequence.

R3-6. Let $T = TNYWR$. Compute the sum of all values k for which the sum of the x- and y-intercepts of $y = |x - 3k| - k$ is T.

Relay Solutions

R1-1. Consider the number of digits in the base 7 representation of the answer N. If it is a single digit, then $N < 7$, but no base 3 representation of a number less than 7 has three or more digits. If the base 7 representation of N has 2 digits, then we test whether the number of digits in the base 2, 3, 5, and 7 representations can be 5, 4, 3, and 2, respectively. Looking at the base 2 and 3 representations of N, we get that $27 \leq N \leq 31$. Trying $\boxed{27}$ is sufficient, as $27_{10} = 36_7 = 202_5 = 1000_3 = 11011_2$.

R1-2. $T = 27$, so the side length of the cube is 3. Envision unfolding the cube, then opposing corners fall on the opposite corners of a rectangle formed by two sides of the cube. The length of this path will be $\sqrt{5}$ times the side length of the cube, which in this case is $\boxed{3\sqrt{5}}$.

R2-1. Some consecutive digits in the rearrangement will sum to 5 provided the digits 1 and 4 or 2 and 3 are adjacent to each other. Letting a box around the pair of numbers denote both orderings of the digits, the rearrangements that work are $\boxed{14}\boxed{23}$, $2\boxed{14}3$, $3\boxed{14}2$, $\boxed{23}\boxed{14}$, $1\boxed{23}4$, $4\boxed{23}1$, so there are $\boxed{16}$ in total.

R2-2. $T = 16$. Solving for x in terms of y gives us $\sqrt{x} + \sqrt{y} = T\left(\sqrt{x} - \sqrt{y}\right) \rightarrow \sqrt{x} = \frac{T+1}{T-1}\sqrt{y} \rightarrow x = \left(\frac{T+1}{T-1}\right)^2 y \rightarrow x = \left(\frac{17}{15}\right)^2 y$. As 15 and 17 are relatively prime, the solution to this problem that minimizes x is when $y = 15^2 \rightarrow x = 17^2 = \boxed{289}$.

R2-3. $T = 289$. The largest divisor of n is itself, the second largest divisor is at most $n/2$, the third largest divisor is at most $n/3$, so the sum is at most $n + \frac{n}{2} + \frac{n}{3} = \frac{11n}{6}$. The sum must be greater than T, which means that n must be greater than $\frac{6T}{11} = \frac{6\times 289}{11}$, which is greater than 157. Testing values of n greater than or equal to 158, the first number whose three largest divisors sum to greater than 289 is $\boxed{162}$.

R3-1. The probability of getting three different values is $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} = \frac{20}{36} = \frac{5}{9}$. Given the event of rolling three different values, all $3! = 6$ orderings are equally likely, so the probability of getting a strictly increasing sequence is $\boxed{\dfrac{5}{54}}$.

R3-2. $T = \frac{5}{54} \rightarrow S = 20$. Another way to ask the question is if the number on the S sided die is a multiple of the number on the 6 sided die, so the answer is $\dfrac{S + \lfloor S/2 \rfloor + \lfloor S/3 \rfloor + \lfloor S/4 \rfloor + \lfloor S/5 \rfloor + \lfloor S/6 \rfloor}{6S}$.
With $S = 20$, this becomes $\dfrac{20 + 10 + 6 + 5 + 4 + 3}{120} = \dfrac{48}{120} = \boxed{\dfrac{2}{5}}$.

R3-3. $T = \frac{2}{5}$. The hypotenuse has length $\frac{AC}{T}$ and the other side has length $AC\sqrt{\frac{1}{T^2} - 1}$. Therefore the perimeter $\frac{7+\sqrt{21}}{4} = AC\left(1 + \frac{1}{T} + \sqrt{\frac{1}{T^2} - 1}\right) = AC\left(1 + \frac{5}{2} + \frac{\sqrt{21}}{2}\right) = AC\left(\frac{7+\sqrt{21}}{2}\right) \to AC = \boxed{\frac{1}{2}}$.

R3-4. $T = \frac{1}{2}$. The probability that two weighted coins will come up tails is $(1-T)^2$. Accordingly, the expected number of tosses is $\frac{1}{(1-T)^2}$. With $T = \frac{1}{2}$, the answer is $\boxed{4}$.

R3-5. $T = 4$. Consider the sequence of values $f_1, \ldots, f_{10}$ that are generated by moving forward from x_1 and increasing each term by T and the sequence $b_1, \ldots, b_{10}$ formed by moving backward from x_{10} and increasing each term by T. Note that $x_i \le \min\{f_i, b_i\}$ for $1 \le i \le 10$. With $T = 4$, the sequences are $f = 2, 6, 10, 14, 18, 22, 26, 30, 34, 38$ and $b = 48, 44, 40, 36, 32, 28, 24, 20, 16, 12$. The minimum of these two sequences is also the maximal sequence for the x values: $x = 2, 6, 10, 14, 18, 22, 24, 20, 16, 12$, so $\boxed{24}$ is the answer.

R3-6. $T = 24$. If $k > 0$, then there are x-intercepts at $2k$ and $4k$ and a y-intercept at $2k$. If $k < 0$, then there is a y-intercept at $-4k$. If $k = 0$, the only solution is at the origin. Thus there is a positive solution for $k = 3$ and a negative solution for $k = -6$, so the sum is $\boxed{-3}$.

Tiebreaker Problem

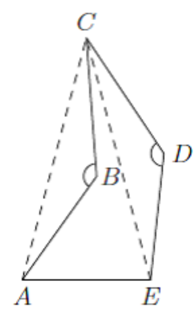
1. Equilateral pentagon $ABCDE$ has sides of length 2. Given that

$$\cos B = \cos D = -\frac{3}{4} \quad \text{and} \quad \tan(A + C) + \tan E = 0,$$

compute the area of $ABCDE$.

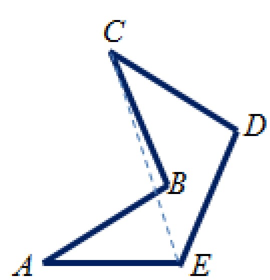
Tiebreaker Solution

1. The cosine equality implies that angles B and D are both 2^{nd} or 3^{rd} quadrant angles. From the value of the cosine we can see that in any case both angles are greater than $135° \to A+C+E < 270°$. By the tangent equality we get that one of the angles $A+C$ or E is in the 2^{nd} quadrant and the other is in the 1^{st} or 3^{rd} quadrant. If it is in the first quadrant we get that $A+C+E = 180°$, if it is in the third quadrant we get that $A+C+E = 360°$. From what we observed regarding $B+D$, since $A+C+E < 270°$, we conclude that $A+C+E = 180°$. This implies $B+D = 360°$, meaning one of the two angles is obtuse and the other is convex.



Note that by dissection we see that the area of the pentagon $ABCDE$ is equal to the area of triangle ACE. By the Law of Cosines we get that $AC = CE = \sqrt{14}$, so the height of ACE is $\sqrt{(\sqrt{14})^2 - 1^2}$ and the area of the triangle ACE is $\frac{1}{2} \times 2 \times \sqrt{13} = \boxed{\sqrt{13}}$.

Note: It has been pointed out that another possible shape for the pentagon with angle D obtuse is below, with B either on or to the right of $\overline{CE}$. For the particular angle chosen in this problem, the graphic in the problem is correct, but it remains an exercise for the reader to show that the graphic as given is correct (that is, B is to the left of $\overline{CE}$) and also that for the pentagon below, the area of the pentagon is still equal to the area of ACE.



Answers to ARML Local 2013

Team Round:

1. 11	2. $3-2\sqrt{2}$	3. $\dfrac{1}{4}$	4. 60	5. $\dfrac{97}{500}$
6. 125	7. $\dfrac{\sqrt{3}}{6}$	8. 68°	9. $-\dfrac{32}{3}$	10. 10

Theme Round:

1. 16	2. 14	3. $\dfrac{3}{4}$	4. $\dfrac{5}{14}$	5. 37
6. $\dfrac{44}{9}$	7. 4	8. 741[1]	9. 20	10. 4

Individual Round:

1. 43	2. 56	3. 127	4. 6	5. 23450617
6. $-\dfrac{1}{3}$	7. 6017	8. 4020	9. $2\sqrt{29}-2$	10. 5, 14, and 21

Relay Round:

Relay 1:	1. 27	2. $3\sqrt{5}$				
Relay 2:	1. 16	2. 289	3. 162			
Relay 3:	1. $\dfrac{5}{54}$	2. $\dfrac{2}{5}$	3. $\dfrac{1}{2}$	4. 4	5. 24	6. -3

Tiebreaker:

1. $\sqrt{13}$

[1] Any number between 666.9 and 815.1 inclusive was considered correct.

ARML Local 2014

Contents

Team Problems

Note: In 2014, the Theme Round was removed and the Team Round was expanded to 15 questions and 45 minutes.

1. There are two times between noon and 1 pm where the hour and minute hands of a clock are perpendicular. Compute the number of minutes between these two times.

2. The three distinct integer roots of $x^3 + qx^2 - 2qx - 8 = 0$ form an arithmetic progression. Compute q.

3. Suppose that n leaves a remainder of 24 when divided by 77. If n leaves a remainder of A when divided by 7 and a remainder of B when divided by 11, compute $A + B$.

4. Compute the length of the interval of values x for which $\dfrac{1}{x+\sqrt{x}} + \dfrac{1}{x-\sqrt{x}} \geq 1$.

5. Compute the number of positive integer factors of $N = 1{,}004{,}006{,}004{,}001$.

6. For a positive integer n, let $p(n)$ be the product of the digits of n. Compute the number of positive integers less than 1000 for which $p(p(n)) = 6$.

7. The continued fraction $1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{\ddots}}}} = \frac{A+\sqrt{B}}{C}$ for integers A, B, and C. Compute the minimum value of $A + B + C$.

8. Compute the number of ways to color the cells of a 3×3 grid red, green, or blue such that each color appears in at least two cells and no cells that share an edge have the same color. Note that reflections and rotations of a grid are considered distinct.

9. A quadrilateral $MATH$ is inscribed in a circle of radius 10. $MA = AT = 12$ and $MH = TH = 16$. Compute the radius of the circle inscribed within $MATH$.

10. Let the legs of a right triangle have lengths x and y. If $x + y = 98$ and $x^3 + y^3 = 517832$, compute the area of the triangle.

11. The expression $n^2 + n - 9$ is a multiple of 101 for two positive integers less than 100. One of these integers is 10. Compute the other one.

12. The graph of the line $y = kx$ intersects the graph of $y = |x - 4|$ at A and B, where A is to the left of B. Let O be the origin. If $OA = AB$, compute k.

13. Two circles pass through the points $(9, 0)$ and $(8, 7)$ and are tangent to the y-axis. Compute the sum of the radii of these two circles.

14. If $2\,(\underline{G}\,\underline{E}\,\underline{R}\,\underline{M}\,\underline{A}\,\underline{N}) = 5\,(\underline{M}\,\underline{A}\,\underline{N}\,\underline{G}\,\underline{E}\,\underline{R})$, and every letter corresponds to a different non-zero digit, compute the value of $\underline{E}\,\underline{N}\,\underline{G}\,\underline{R}\,\underline{A}\,\underline{M}$.

15. An *interior diagonal* of a polyhedron P is a line segment whose endpoints are vertices of P and which lies entirely in the interior of P (that is, does not intersect any faces of P), except for its endpoints. Find the maximum possible number of interior diagonals if P has 12 vertices.

Team Solutions

1. The minute hand moves at a rate of 360 degrees per hour (or 6 degrees per minute), the hour hand moves at a rate of 30 degrees per hour (or 0.5 degrees per minute). Therefore, the angle between the hour and the minute hand increases by 5.5 degrees per minute. The first time the hands are perpendicular, the angle between them is 90 degrees. The second time is when the angle is 270 degrees, which occurs 180/5.5 or $\boxed{360/11}$ minutes later.

2. The product of the roots is 8, and the possible sets of distinct integer roots are $\{\pm1, \pm2, \pm4\}$, $\{\pm2, \pm2, \pm2\}$, and $\{\pm1, \pm1, \pm8\}$ where either zero or two of the roots are negative. Of all of the cases listed here, only one has distinct roots in arithmetic progression with two roots negative, $\{-4, -1, 2\}$. The corresponding polynomial is $(x-2)(x+1)(x+4) = x^3+3x^2-6x-8$, and the answer is $q = \boxed{3}$.

3. $n = 77k+24$ for some integer value of k. When n is divided by 7, the $77k$ term divides evenly, so the remainder is the residue of 24 modulo 7, or 3. Similarly when n is divided by 11, the remainder is 2. The answer is $3 + 2 = \boxed{5}$.

4. $\dfrac{1}{x+\sqrt{x}} + \dfrac{1}{x-\sqrt{x}} = \dfrac{2x}{x(x-1)}$. The domain of the function is all positive x except 1. For x between 0 and 1, the denominator is negative, so the function is negative. For x greater than 1, the function is strictly decreasing. It equals 1 when $\dfrac{2x}{x(x-1)} = 1 \to 2x = x^2 - x \to x^2 - 3x = 0 \to x = 3$, so the interval is $(1, 3)$, which has length $\boxed{2}$.

5. Considering that the non-zero digits of N look like binomial coefficients, N is the binomial expansion of $(1000 + 1)^4$. Since $1001 = 7 \times 11 \times 13$, $1{,}004{,}006{,}004{,}001 = 7^4 \times 11^4 \times 13^4$. The prime factorization of any factor of N will have 0 to 4 powers of each of the primes 7, 11, and 13, so the total number of factors is $5^3 = \boxed{125}$.

6. In order for $p(p(n)) = 6$, $p(n)$ must be 6, 16, or 32 (no other product of up to three digits k has the property that $p(k) = 6$). Treat the cases individually.

   $p(n) = 6$: If n has one digit, then $n = 6$. If n has two digits, then n is either 16, 23, 32, or 61. If n has three digits, the digits of n must be some permutation of 116 or 123. There are three of the former (ignoring duplicates) and six of the latter, so there are 14 numbers less than 1000 for which $p(n) = 6$.

   $p(n) = 16$: All digits will be powers of two. There are three two digit numbers: 28, 44, and 82, and $\binom{6}{4} - 3 = 12$ three digit numbers (think of the digits as distinct boxes and the four powers of two as identical balls to be distributed in three buckets, but we need to throw out the three cases where all four balls go into one bucket) so there are 15 numbers less than 1000 for which $p(n) = 16$.

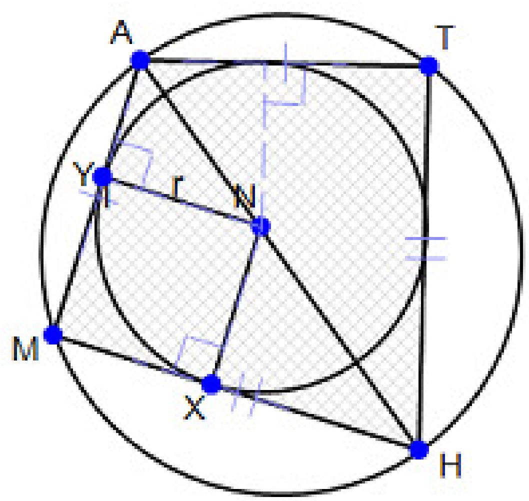
   $p(n) = 32$: All digits will be powers of two. There are two two-digit numbers: 48, and 84, and $\binom{7}{5} - 6 - 3 = 12$ three digit numbers (again, think of the digits as distinct boxes and the five powers of two as identical balls to be distributed, but we have to remove the six cases where one box gets 4 balls and the three cases where one box gets 5 balls, so there are 14 numbers less than 1000 for which $p(n) = 32$.

The total is $14 + 15 + 14 = \boxed{43}$.

7. $x = 1 + \frac{1}{3+\frac{1}{1+\frac{1}{3+\frac{1}{\ddots}}}} \to x = 1 + \frac{1}{3+\frac{1}{x}} \to x - 1 = \frac{1}{3+\frac{1}{x}} \to 3x + 1 - 3 - \frac{1}{x} = 1 \to 3x^2 - 3x - 1 = 0 \to x = \frac{3\pm\sqrt{21}}{6}$. The positive root is the value, so $A + B + C = 3 + 21 + 6 = \boxed{30}$.

8. Assume the center is blue. Then 1, 2, 3, or 4 of the corners must be blue. If 1 corner is blue, there are 4 possible corners to choose from and 2 ways to finish the grid with alternating red and green squares. If 2 corners are blue and the 2 corners are opposite, there are two ways to choose opposing corners and 2 ways to color each of the remaining paths of 3 squares with alternating colors. If 2 corners are blue and the corners are consecutive, there are 4 ways to choose consecutive corners, 2 ways to color the square between the two blue corners, and 2 ways to color the remaining squares. If three of the corners are blue, there are 4 ways to choose three blue corners, 2 choices for the color of each of the squares between 2 blue corners, and 2 ways to color the path of length 3. However, we must discard two of those cases that only contain 1 red or green square in the corner. If 4 corners are blue, there are $C(4, 2) = 6$ ways to pick the two green (or red) cells. Totaling these cases up we get $(4 \times 2) + (2 \times 2 \times 2) + (4 \times 2 \times 2) + (4 \times (2 \times 2 \times 2 - 2)) + 6 = 62$. Since there are three colors to choose for the center square, the answer is $\boxed{186}$.

9. Since AH is a diameter of length 20, we know that HAM is a right triangle with right angle at M. Let the center of the inscribed circle be N, and let radii of the inscribed circle intersect $\overline{MA}$ at Y and $\overline{MH}$ at X. Then $MYNX$ is a square of side length r. Further, triangles AYN and AMH are similar; thus $\frac{YN}{MH} = \frac{AY}{AM} \to \frac{r}{16} = \frac{12-r}{12} \to 12r = 192 - 16r \to r = \frac{192}{28} = \boxed{\frac{48}{7}}$.



10. We can factor $x^3+y^3 = (x+y)(x^2-xy+y^2)$, so $517832 = 98(x^2-xy+y^2) \to x^2-xy+y^2 = 5284$. In addition, $(x+y)^2 = x^2+2xy+y^2 = 98^2 = 9604$. Subtracting the first equation from the second, we get that $3xy = 4320$, so the area of the triangle is $\frac{xy}{2} = \frac{4320}{6} = \boxed{720}$.

11. Let $n = 10+t$, then the expression is $(10+t)^2+(10+t)-9 = t^2+21t+101$. For this value to be a multiple of 101, it means that $t^2+21t = t(t+21)$ must be a multiple of 101. Since 101 is prime, one of the two factors must be a multiple of 101. Since t can be at most 89, $t+21 = 101 \to t = 80 \to n = \boxed{90}$.

12. A and B must be on different branches of the absolute value graph, and the x-coordinates of O, A, and B must be evenly spaced. Let the points be $O(0,0)$, $A(t, 4-t)$, and $B(2t, 2t-4)$. Equating OA and AB, we are looking for the solution to $\sqrt{t^2+(4-t)^2} = \sqrt{t^2+(3t-8)^2}$. Squaring both sides and subtracting t^2 from each side yields $(4-t)^2 = (3t-8)^2$, yielding two solutions, $t = 2, 3$. If $t = 2$, the points are $(0,0), (2,2)$, and $(4,0)$, which are not collinear. If $t = 3$, the points are $(0,0), (3,1)$, and $(6,2)$, which are collinear, so $k = \boxed{1/3}$.

13. Assume the circle has center (x, y) and radius r. The following three equations must be satisfied:

$$\begin{aligned}(x-9)^2+y^2 &= r^2\\ (x-8)^2+(y-7)^2 &= r^2\\ x &= r.\end{aligned}$$

Subtracting the first equation from the second, we get $2x-17-14y+49 = 0 \to 14y-2x-32 = 0 \to x = 7y-16$. Squaring the third equation and subtracting it from the second, we get $-16x+64+(y-7)^2 = 0$. Substituting $x = 7y-16$ gives $-16(7y-16)+64+(y-7)^2 = 0 \to y^2-126y+369 = 0$. Therefore, y is either 3 or 123, with corresponding values of x (and radii) of 5 and 845. The sum of the radii is $\boxed{850}$.

14. Let $x = \underline{G}\,\underline{E}\,\underline{R}$, $y = \underline{M}\,\underline{A}\,\underline{N}$, then $2(1000x+y) = 5(1000y+x) \to 1995x = 4998y \to \frac{x}{y} = \frac{4998}{1995} = \frac{238}{95}$. Since both x and y must be three digit integers with all non-zero digits, we multiply the numerator and denominator by 3 to get $x = 714$ and $y = 285$, rearranging the digits to get $\underline{E}\,\underline{N}\,\underline{G}\,\underline{R}\,\underline{A}\,\underline{M} = \boxed{157482}$.

15. It is fair to assume that each face of a polyhedron with the maximal number of interior diagonals is a triangle (otherwise some diagonals are on the faces of the polyhedron). If a triangular faced polyhedron P has F faces, E edges, and V vertices, then it has $\binom{V}{2} - E$ interior diagonals. For a polyhedron with all triangular faces, $3F = 2E$. Using Euler's formula for polyhedra ($V+F = E+2$), we get $V + \frac{2E}{3} = E+2 \to E = 3V-6$, so the maximal number of interior diagonals is $\binom{V}{2} - 3(V-2) = \frac{(V-3)(V-4)}{2}$, which for $V = 12$ is $\boxed{36}$. A sample polyhedron that has 36 interior diagonals is two convex pyramids that share a common decagon base.

Individual Problems

1. Compute the area of the region defined by $2\,|x-y|+|y|\leq 2$.

2. Compute the sum of all integers of the form $\underline{2}\,\underline{0}\,\underline{1}\,\underline{4}\,\underline{A}\,\underline{B}\,\underline{5}$ that are multiples of 225.

3. Compute the sum of all solutions to $\log_4 x+\log_{x^2}\frac{1}{8}=1$.

4. In triangle COW, $CO=OW=2014$, and $m\angle O=2\theta$. In triangle PIG, $PI=IG=2014$, and $m\angle I=\theta$. If COW and PIG have the same area, compute θ in degrees.

5. For a positive integer n, let $s(n)$ be the sum of the digits of n. Compute the number of positive integers less than 1000 for which the digits are non-decreasing from left to right and $s(s(s(n)))=3$.

6. Compute the integer value of a such that $2-i$ is a solution to $x^3+ax=-20$.

7. Compute the whole number N such that $N^6=6{,}321{,}363{,}049$.

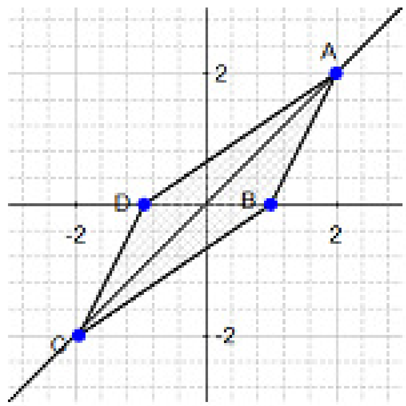
8. Given a circle O with center E and diameter $\overline{AB}$, C is on O and D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. If $AE=13$ and $BC=10$, compute ED.

9. Jef is playing a game with three rounds. In Round 1, he flips 4 coins. In Round 2 and Round 3 he can re-flip some or all (or none) of the coins and wins the game if all of the coins are heads or all of the coins are tails at the end of any round. Assuming Jef uses an optimal strategy, compute the probability he wins the game.

10. In rectangle $MATH$, $MA=3$ and $AT=1$. Equilateral triangles $\triangle THE$ and $\triangle HUM$ are constructed in the same plane as the rectangle. Compute the ratio between the largest and smallest possible area of $\triangle UAE$.

Individual Solutions

1. Consider splitting the plane into four regions, depending on whether $x-y$ and y are positive or negative. If both are positive, the boundary of the region is $2(x-y)+y=2 \to 2x-y=2$. If $x-y$ is negative and y is positive, the boundary of the region is $2(y-x)+y=2 \to 3y-2x=2$. If $x-y$ is positive and y is negative, the boundary of the region is $2(x-y)-y=2 \to 2x-3y=2$. If both are negative, then the boundary of the region is $2(y-x)-y=2 \to y-2x=2$. Putting these together gives the region shown below, two triangles $\triangle DBA$ and $\triangle DBC$ with base and height of 2, giving a total area of $\boxed{4}$.



2. To be a multiple of 225, the number must be both a multiple of 25 (which means B is either 2 or 7) and a multiple of 9 ($2+0+1+4+A+B+5=12+A+B$ must be a multiple of 9, giving A is 4 or 8, respectively). The numbers are 2014425 and 2014875, giving a sum of $\boxed{4029300}$.

3. Using the fact that $\log_a b = \dfrac{\log_c b}{\log_c a}$, we get $\log_4 x + \log_{x^2} \dfrac{1}{8} = \dfrac{\log_2 x}{\log_2 4} + \dfrac{\log_2(1/8)}{\log_2 x^2} = \dfrac{\log_2 x}{2} - \dfrac{3}{2\log_2 x}$. Letting $u = \log_2 x$, $\dfrac{u}{2} - \dfrac{3}{2u} = 1 \to u^2 - 3 = 2u \to u^2 - 2u - 3 = 0 \to (u-3)(u+1) = 0 \to u = 3, -1 \to x = 8, 1/2$. The sum of the solutions is $\boxed{17/2}$.

4. Since the areas are equal, we have $\frac{1}{2} \times 2014 \times 2014 \times \sin(2\theta) = \frac{1}{2} \times 2014 \times 2014 \times \sin(\theta) \to \sin(2\theta) = \sin(\theta)$. Using the double-angle formula,

$$2\sin(\theta)\cos(\theta) = \sin(\theta) \to \cos(\theta) = 1/2 \to \theta = \boxed{60^\circ}.$$

5. $s(s(s(n))) = 3 \to s(n) = 3$, 12, or 21. Consider the cases separately.

   $s(n) = 3$: There are three solutions: 3, 12, and 111.

   $s(n) = 12$: There are 4 two-digit solutions: 39, 48, 57, 66. For the three-digit solutions, condition off the hundreds digit h: find the smallest solution starting with each h and each new solution is 9 more until the units digit is less than the tens digit: 129, 138, 147, 156, 228, 237, 246, 255, 336, 345, and 444 are the solutions.

$s(n) = 21$: There are no two-digit solutions. For the three-digit solutions, condition off the hundreds digit h: find the smallest solution starting with each h and each new solution is 9 more until the units digit is less than the tens digit: 399, 489, 579, 588, 669, 678, and 777 are the solutions.

There are $\boxed{25}$ solutions in total.

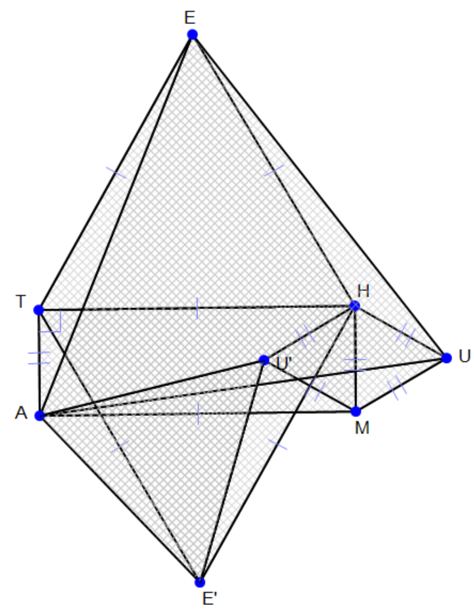
6. If $2 - i$ is a root of $x^3 + ax + 20$, so is $2 + i$. Since $(2 + i)(2 - i) = 5$ and the product of the roots is -20, the third root is -4. The linear coefficient is the sum of the pairwise products of the roots, which is $(-4(2 + i)) + (-4(2 - i)) + ((2 + i)(2 - i)) = -8 + 4i - 8 - 4i + 5 = \boxed{-11}$.

7. The only units digits whose sixth powers end in 9 are 3 and 7. Since $40^6 = 4{,}096{,}000{,}000$ and $50^6 = 15{,}625{,}000{,}000$, $40 < N < 50$, so $N = 43$ or $N = 47$. Since N^6 is closer to 40^6 than 50^6, N must be $\boxed{43}$.

8. Since $\angle ACB$ is inscribed on a semicircle, it is a right angle, and so $\triangle ABC$ is right with hypotenuse 26 and $BC = 10$, so $AC = 24$. By the Angle Bisector Theorem, $\dfrac{AD}{DB} = \dfrac{AC}{CB}$, so $\dfrac{13 + ED}{13 - ED} = \dfrac{24}{10}$ can be solved to obtain $ED = \boxed{\dfrac{91}{17}}$.

9. The probability of getting 0 to 4 heads on the first flip are 1/16, 4/16, 6/16, 4/16, and 1/16, respectively. If 0 or 4 heads are flipped, Jef wins. If 1 or 3 heads are flipped, the one different coin is flipped with the goal of matching the other three. In this case, Jef will succeed with probability 3/4. If 2 heads are flipped, either pair should be reflipped with the goal of matching the other pair. Each coin will match the other pair with probability 3/4, so Jef wins in this case with probability 9/16. The total probability of winning is $\dfrac{1}{16}(1) + \dfrac{4}{16}\left(\dfrac{3}{4}\right) + \dfrac{6}{16}\left(\dfrac{9}{16}\right) + \dfrac{4}{16}\left(\dfrac{3}{4}\right) + \dfrac{1}{16}(1) = \dfrac{182}{256} = \boxed{\dfrac{91}{128}}$.

10. The largest and smallest triangles UAE and $U'AE'$ are shown in the diagram. Note that the sides of UAE are all the longest side of a triangle whose other sides have length 1 and 3 and included angle 150 degrees, so $\triangle UAE$ is equilateral with area $\frac{(AE)^2\sqrt{3}}{4}$. Using the Law of Cosines: $AE^2 = 3^2 + 1^2 - 2 \times 3 \times 1 \times \cos(150°) = 10 + 3\sqrt{3}$.



Note that the sides of $U'AE'$ are all the third side of a triangle whose other sides have length 1 and 3 and included angle 30 degrees, so $\triangle U'AE'$ is equilateral with area $\frac{(AE')^2\sqrt{3}}{4}$. Using the Law of Cosines: $AE^2 = 3^2 + 1^2 - 2 \times 3 \times 1 \times \cos(30°) = 10 - 3\sqrt{3}$.

The ratio of the areas is $\frac{10 + 3\sqrt{3}}{10 - 3\sqrt{3}} = \boxed{\frac{127 + 60\sqrt{3}}{73}}$.

Relay Problems

R1-1. Compute the value of k such that May k^{th} and June $(4k)^{\text{th}}$ fall on the same day of the week. (Note: May has 31 days and June has 30 days.)

R1-2. Let $T = TNYWR$. A set of k numbers has the sum S. Each number in the set is decreased by T, then multiplied by T, then increased by T. The sum of the new set of numbers is $mS + nk$. Compute $m + n$.

R2-1. Compute the sum of the solutions (in radians) of $2(\cos x - \sin x)(\sin x + \cos x) + 1 = 0$ for $0 \leq x \leq 2\pi$.

R2-2. Let $T = TNYWR$. Two concentric circles have radii that differ by $\frac{1}{2}$ and the region inside the outer circle but outside the inner circle has area T. Compute the radius of the inner circle.

R2-3. Let $T = TNYWR$. $RYAN$ is a rectangle with $\dfrac{RY}{YA} = 4T$. If the perimeter of $RYAN$ is also $4T$, compute the area of $RYAN$.

R3-1. In the multiplication problem $\underline{A} \times \underline{B}\,\underline{C} = \underline{B}\,\underline{B}\,\underline{B}$, each letter represents a different digit. Compute the value of $A + B + C$.

R3-2. Let $T = TNYWR$. The sequence of numbers $a_1, a_2, \ldots, a_k$ is defined by $a_i = 2 \times a_{i-1} + b$. If $a_2 = 5$ and $a_5 = T$, compute b.

R3-3. Let $T = TNYWR$. The graphs of $y = 2x^2 + Tx + 1$ and $y = 19 - x^2$ intersect at two points. Compute the slope of the line through these two points.

R3-4. Let $T = TNYWR$. The numbers $1 + T, 1 + T^2, 1 + T^3, 1 + T^4, 1 + T^5$, and $1 + T^6$ are on the faces of a fair six sided die. Compute the expected value of one roll of this die.

R3-5. Let $T = TNYWR$. A fair six sided die is rolled, getting a number x. Then x fair six sided dice are rolled. Compute the expected sum of the x die rolls, given that $x > T$.

R3-6. Let $T = TNYWR$, and let $S = 2\lceil T/2 \rceil$. Compute the smallest positive integer x that satisfies the following $\dfrac{S}{2} - 1$ conditions: x has a remainder of 2 when divided by 4, x has a remainder of 4 when divided by 6, $\ldots$, x has a remainder of $S - 2$ when divided by S.

Relay Solutions

R1-1. The number of days between May k^{th} and June $(4k)^{\text{th}}$ is $4k + (31 - k) = 31 + 3k$, which must be divisible by 7 for the dates to fall on the same day of the week, which occurs when $k = \boxed{6}$ $(31 + 18 = 49)$.

R1-2. For simplicity, assume all of the numbers in the set are S/k. The transformations described produce new numbers equal to $T + T(S/k - T) = TS/k + (T - T^2)$ whose sum is $TS + k(T - T^2)$. So $m = T$ and $n = T - T^2$ and the sum is $2T - T^2$. Since $T = 6$, the answer is $\boxed{-24}$.

R2-1. $2(\cos x - \sin x)(\sin x + \cos x) + 1 = 2\left(\cos^2 x - \sin^2 x\right) + 1 = 2\left(\cos^2 x - (1 - \cos^2 x)\right) + 1 = 2\left(2\cos^2 x - 1\right) + 1 = 4\cos^2 x - 3$ which equals 0 when $\cos x = \pm\dfrac{\sqrt{3}}{2}$. This occurs when $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$, the sum is $\boxed{4\pi}$.

R2-2. Let R be the radius of the inner circle. The difference in areas is $\pi\left(R + \frac{1}{2}\right)^2 - \pi R^2 = \pi\left(R + \frac{1}{4}\right)$. Since the area of the region is 4π, $R = \boxed{15/4}$.

R2-3. Let $s = YA$. Then the side lengths are s and $4Ts$, so the perimeter is $s(2 + 2(4T)) = 4T \to s = \dfrac{4T}{(2 + 2(4T))}$. The area of the rectangle is $4Ts^2 = \dfrac{(4T)^3}{(2 + 2(4T))^2}$. Since $4T = 15$, the area is $\dfrac{15^3}{32^2} = \boxed{\dfrac{3375}{1024}}$.

R3-1. $\underline{B}\,\underline{B}\,\underline{B}$ is a multiple of 111, so $\underline{B}\,\underline{C}$ is a multiple of 37 (either 37 or 74) and A is a multiple of 3. B cannot be 7, since $777 > 74 \times 9$, so B is 3, and A is 9, giving $9 \times 37 = 333$, and the sum of the digits is $9 + 3 + 7 = \boxed{19}$.

R3-2. $a_3 = 2 \times 5 + b = 10 + b$, $a_4 = 2 \times (10 + b) + b = 20 + 3b$, $T = a_5 = 2 \times (20 + 3b) + b = 40 + 7b$. Since $T = 19$, $b = \dfrac{T - 40}{7} = \dfrac{19 - 40}{7} = \boxed{-3}$.

R3-3. $2x^2 + Tx + 1 = 19 - x^2 \to 3x^2 + Tx - 18 = 0$. The solutions are $x = \dfrac{-T \pm \sqrt{T^2 + 216}}{6}$. Since $T = -3$, the roots are $x = 3$ and $x = -2$, and the corresponding y-values are 10 and 15, so the slope is $\boxed{-1}$.

R3-4. The expected value of one roll of this die is the average of the six faces which is

$$\frac{6+T+T^2+T^3+T^4+T^5+T^6}{6} = \frac{5+1+T+T^2+T^3+T^4+T^5+T^6}{6} = \frac{5+\frac{1-T^7}{1-T}}{6}.$$

With $T=-1$, this expected value is $\boxed{1}$.

R3-5. If x dice are rolled, the expected sum is $3.5x$. The possible expected values are 3.5, 7, 10.5, 14, 17.5, and 21. Since $x>1$, we only consider the last five cases, all of which are equally likely, so the expected value is the average of the last 5 numbers, which is also the middle number, $\boxed{14}$.

R3-6. Since the remainder is always 2 fewer than the dividend, the answer is 2 fewer than the least common multiple of the dividends. $\operatorname{lcm}(4,6,8,10,12,14)=840$, so the answer is $\boxed{838}$.

Tiebreaker Problem

1. If x and y are strings of letters, x is an *ordered substring* of y if and only if the letters in x appear in y in the same order. For example, AR, AM, AL, RM, RL, and ML are all of the ordered substrings of length 2 of ARML, but RA is not, as order is not preserved. Compute the number of distinct ordered substrings of length 4 of the string WORLDCUPBRAZIL.

Tiebreaker Solution

1. There are $\binom{14}{4} = 1001$ ordered substrings of length 4 in total ignoring duplicates. There are two Rs and two Ls in the string which will cause duplicates (we will call them L1 and L2 and R1 and R2). We will consider cases where duplicates occur, with x, y, and z denoting any other letter besides R and L.

   WORL: This is duplicated twice: R1L1, R1L2, and R2L2. [2 duplicates]

   Rxyz: Duplicates occur when xyz is an ordered substring of AZIL [4 duplicates]

   xRyz: Duplicates can occur when x = W or O and yz is an ordered substring of AZIL. [12 duplicates]

   WORx: Duplicates can occur when x = A, Z, or I [3 duplicates]

   The total is $1001 - 12 - 4 - 3 - 2 = \boxed{980}$.

Answers to ARML Local 2014

Team Round:

1. $\dfrac{360}{11}$	2. 3	3. 5	4. 2	5. 125
6. 43	7. 30	8. 186	9. $\dfrac{48}{7}$	10. 720
11. 90	12. $\dfrac{1}{3}$	13. 850	14. 157482	15. 36

Individual Round:

1. 4	2. 4029300	3. $\dfrac{17}{2}$	4. 60°	5. 25
6. -11	7. 43	8. $\dfrac{91}{17}$	9. $\dfrac{91}{128}$	10. $\dfrac{127+60\sqrt{3}}{73}$

Relay Round:

Relay 1:	1. 6	2. -24				
Relay 2:	1. 4π	2. $\dfrac{15}{4}$	3. $\dfrac{3375}{1024}$			
Relay 3:	1. 19	2. -3	3. -1	4. 1	5. 14	6. 838

Tiebreaker:

1. 980

Part III

ARML Power Contests

Basimal Fractions

The Background

Any rational number $\frac{p}{q}$ between 0 and 1 can be represented using negative powers of b: $\frac{p}{q} = c_1 b^{-1} + c_2 b^{-2} + c_3 b^{-3} + \ldots + c_n b^{-n} + \ldots$ where $0 \leq c_i \leq b$. This is the meaning behind the representation $\frac{p}{q} = 0.c_1 c_2 c_3 \ldots c_n \ldots$. If the base were TEN, we would call this representation a decimal. So for any base b, we will call this representation a **base *b* basimal**. Like a decimal, a basimal can be terminating (0.25), totally repeating ($0.\overline{25}$), or partially repeating ($0.2\overline{5}$). In base TEN, these basimals represent the rational numbers $\frac{1}{4}, \frac{25}{99}$, and $\frac{23}{90}$. In base SIX , these basimals represent the rational numbers $\frac{17}{36}, \frac{17}{35}$, and $\frac{1}{2}$. Converting a basimal to a fraction is relatively easy, but converting a fraction to a basimal requires a few steps. The following algorithm is quite helpful:

> To compute the **base** b basimal representation for the rational number $\frac{p}{q}$:
> Let d_0 be the decimal representation for $\frac{p}{q}$. Let $i = 0$.
> **Repeat** the following four steps **until** $d_i = 0$ or $d_i = d_k$ for some k, where $0 \leq k < i$.
>
> Increment i.
> Let $a_i = d_{i-1} \cdot b$.
> Let $c_i = \lfloor a_i \rfloor$, the integer part of a_i.
> Let $d_i = a_i - \lfloor a_i \rfloor$, the fractional part of a_i.
> If $d_i = 0$, the desired basimal is $0.c_1 c_2 c_3 \ldots c_i$ or, if $d_i = d_k$, it is $0.c_1 c_2 \ldots \overline{c_{k+1} \ldots c_i}$.
>
> For example, compute the base THREE and base FOUR basimal representations for $\frac{3}{8}$.
>
> $$\begin{array}{llll} & & & d_0 = 0.375 \\ \text{Base THREE}: & a_1 = 1.125 & c_1 = 1 & d_1 = 0.125 \\ & a_2 = 0.375 & c_2 = 0 & d_2 = 0.375 \end{array}$$
>
> Since $d_2 = d_0$, exit loop and $\frac{3}{8} = \left(0.\overline{10}\right)_{\text{THREE}}$.
>
> $$\begin{array}{llll} & & & d_0 = 0.375 \\ \text{Base FOUR}: & a_1 = 1.5 & c_1 = 1 & d_1 = 0.5 \\ & a_2 = 2.0 & c_2 = 2 & d_2 = 0 \end{array}$$
>
> Since $d_2 = 0$, exist loop and $\frac{3}{8} = (0.12)_{\text{FOUR}}$.

While Part A of this contest deals with converting a rational number to a basimal and back again, Part B deals with the function, $f_b(\frac{p}{q}) = b \cdot x - \lfloor b \cdot x \rfloor$, where x is the base b basimal representation of $\frac{p}{q}$.

As examples, $f_4(\frac{27}{64}) = f_4(0.123_{\text{FOUR}}) = (0.23_{\text{FOUR}}) = \frac{11}{16}$ and $f_2(\frac{1}{3}) = f_2(0.\overline{01}_{\text{TWO}}) = (0.\overline{10}_{\text{TWO}}) = \frac{2}{3}$. Notice that multiplying a base b basimal by b shifts the basimal point one digit to the right. Subtracting the floor of this number removes the integer part and the function then returns the rational number (between 0 and 1) represented by this shifted basimal.

The input and output of this function are always rational numbers between 0 and 1 written in the form $\frac{p}{q}$.

By convention, ${f_b}^3(\frac{p}{q}) = f_b\left(f_b\left(f_b(\frac{p}{q})\right)\right)$.

For example, ${f_3}^2(\frac{4}{5}) = f_3\left(f_3(\frac{4}{5})\right) = f_3\left(f_3(0.\overline{2101}_{\text{THREE}})\right) = f_3(0.\overline{1012}_{\text{THREE}}) = 0.\overline{0121}_{\text{THREE}} = \frac{1}{5}$.

The following page contains a chart of conversions to different bases, which may be helpful.

Chart of Conversions to Different Bases

$\frac{p}{q}$	TWO	THREE	FOUR	FIVE	TEN
$\frac{1}{2}$	0.1	$0.\overline{1}$	0.2	$0.\overline{2}$	0.5
$\frac{1}{3}$	$0.\overline{01}$	0.1	$0.\overline{1}$	$0.\overline{13}$	$0.\overline{3}$
$\frac{2}{3}$	$0.\overline{10}$	0.2	$0.\overline{2}$	$0.\overline{31}$	$0.\overline{6}$
$\frac{1}{4}$	0.01	$0.\overline{02}$	0.1	$0.\overline{1}$	0.25
$\frac{3}{4}$	0.11	$0.\overline{20}$	0.3	$0.\overline{3}$	0.75
$\frac{1}{5}$	$0.\overline{0011}$	$0.\overline{0121}$	$0.\overline{03}$	0.1	0.2
$\frac{2}{5}$	$0.\overline{0110}$	$0.\overline{1012}$	$0.\overline{12}$	0.2	0.4
$\frac{3}{5}$	$0.\overline{1001}$	$0.\overline{1210}$	$0.\overline{21}$	0.3	0.6
$\frac{4}{5}$	$0.\overline{1100}$	$0.\overline{2101}$	$0.\overline{30}$	0.4	0.8
$\frac{1}{6}$	$0.0\overline{01}$	$0.0\overline{11}$	$0.0\overline{2}$	$0.\overline{04}$	$0.1\overline{6}$
$\frac{5}{6}$	$0.1\overline{10}$	$0.2\overline{11}$	$0.3\overline{1}$	$0.\overline{40}$	$0.8\overline{3}$
$\frac{1}{7}$	$0.\overline{001}$	$0.\overline{010212}$	$0.\overline{021}$	$0.\overline{032412}$	$0.\overline{142857}$
$\frac{2}{7}$	$0.\overline{010}$	$0.\overline{021201}$	$0.\overline{102}$	$0.\overline{120324}$	$0.\overline{285714}$
$\frac{3}{7}$	$0.\overline{011}$	$0.\overline{102120}$	$0.\overline{123}$	$0.\overline{203241}$	$0.\overline{428571}$
$\frac{4}{7}$	$0.\overline{100}$	$0.\overline{120102}$	$0.\overline{210}$	$0.\overline{241203}$	$0.\overline{571428}$
$\frac{5}{7}$	$0.\overline{101}$	$0.\overline{201021}$	$0.\overline{231}$	$0.\overline{324120}$	$0.\overline{714285}$
$\frac{6}{7}$	$0.\overline{119}$	$0.\overline{212010}$	$0.\overline{312}$	$0.\overline{412032}$	$0.\overline{857142}$
$\frac{1}{8}$	0.001	$0.\overline{01}$	0.02	$0.\overline{03}$	0.125
$\frac{3}{8}$	0.011	$0.\overline{10}$	0.12	$0.\overline{14}$	0.375
$\frac{5}{8}$	0.101	$0.\overline{12}$	0.22	$0.\overline{30}$	0.625
$\frac{7}{8}$	0.111	$0.\overline{21}$	0.32	$0.\overline{41}$	0.875
$\frac{1}{9}$	$0.\overline{000111}$	0.01	$0.\overline{013}$	$0.\overline{023421}$	$0.\overline{1}$
$\frac{2}{9}$	$0.\overline{001110}$	0.02	$0.\overline{032}$	$0.\overline{102342}$	$0.\overline{4}$
$\frac{5}{9}$	$0.\overline{111000}$	0.12	$0.\overline{203}$	$0.\overline{234210}$	$0.\overline{5}$
$\frac{7}{9}$	$0.\overline{110001}$	0.21	$0.\overline{301}$	$0.\overline{342102}$	$0.\overline{7}$
$\frac{8}{9}$	$0.\overline{100011}$	0.22	$0.\overline{320}$	$0.\overline{421023}$	$0.\overline{8}$

The Problems

Part A – Basic Basimal Conversions – SHOW WORK FOR EACH PROBLEM

1. a. Compute the base THREE basimal representation for $\frac{5}{12}$.

   b. Compute the base EIGHT basimal representation for $\frac{13}{16}$.

   c. Compute the rational number, $\frac{p}{q}$, such that $\frac{p}{q} = (0.132)_{\text{FOUR}}$.

   d. Compute the rational number, $\frac{p}{q}$, such that $\frac{p}{q} = (0.3\overline{12})_{\text{FIVE}}$.

   e. The rational number $\frac{5}{24}$ can be written as $(0.113)_b$ for some base b. Compute the value of b.

   f. The rational number $\frac{37}{60}$ can be written as $(0.2\overline{13})_b$ for some base b. Compute the value of b.

2. a. Letting $N =$ the length of the repetend of a base TWO totally repeating basimal and $M =$ the position of the first 1 in the repetend, the following table is produced. What would be the entries in column six?

M/N	1	2	3	4
1	$(0.\overline{1})_{\text{TWO}} = \frac{1}{1}$	$(0.\overline{10})_{\text{TWO}} = \frac{2}{3}$	$(0.\overline{100})_{\text{TWO}} = \frac{4}{7}$	$(0.\overline{1000})_{\text{TWO}} = \frac{8}{15}$
2	–	$(0.\overline{01})_{\text{TWO}} = \frac{1}{3}$	$(0.\overline{010})_{\text{TWO}} = \frac{2}{7}$	$(0.\overline{0100})_{\text{TWO}} = \frac{4}{15}$
3	–	–	$(0.\overline{001})_{\text{TWO}} = \frac{1}{7}$	$(0.\overline{0010})_{\text{TWO}} = \frac{2}{15}$
4	–	–	–	$(0.\overline{0001})_{\text{TWO}} = \frac{1}{15}$

   b. Use the entries in column six to determine the rational number $\frac{p}{q}$ represented by $(0.\overline{010110})_{\text{TWO}}$.

   c. If the rational number $\frac{p}{q}$ is in row M and column N in the table above, express $\frac{p}{q}$ in terms of M and N.

   d. If you made a table similar to the table in problem 2a) but for base THREE , what would be the entries in column five?

   e. Use the entries in column five of the base THREE table to determine the rational number $\frac{p}{q}$ represented by $(0.\overline{12010})_{\text{THREE}}$.

3. Determine under what conditions $\frac{p}{q}$ will be a terminating, totally repeating, or partially repeating basimal in base b.

4. In base b, the rational number X is $(0.\overline{32})_b$ and the rational number Y is $(0.\overline{23})_b$. In base a, $X = (0.\overline{31})_a$ and $Y = (0.\overline{13})_a$. If $a < 10$ and $b < 10$, compute a and b.

Part B – The Shifting Function

5. On three separate axes, graph the function $f_b(x)$ for $b = 2, 3,$ and 4.

6. Compute $f_2\left(f_3(\frac{4}{7})\right)$ and $f_3\left(f_2(\frac{4}{7})\right)$.

7. If $\frac{a}{b} \leq x < \frac{a+1}{b}$, determine a linear function rule for $f_b(x)$ where $0 \leq x < 1$. (N.B. This function may be useful in solving questions 8 – 13.)

8. Compute the value of $f_{12}(\frac{13}{15})$.

9. Find the sum of the solutions to $f_{10}(x) = x$.

10. Find the sum of the solutions to ${f_4}^3(x) = \frac{5}{9}$.

11. Find all x such that $x < {f_2}^2(x) < f(x)$.

12. a. For fixed $y \in [0, 1)$, in terms of b and n, how many solutions does the equation, ${f_b}^n(x) = y$, have?
    b. For fixed $y \in [0, 1)$, in terms of $b, n,$ and y, what is the sum of the solutions to the equation ${f_b}^n(x) = y$? Hint: For various values of n, check out the solutions to ${f_2}^n(x) = y$ and generalize. (You do not have to prove your generalization.)

13. If $x = \frac{1}{b+1}$, show that $f_b(x), {f_b}^2(x), {f_b}^3(x), {f_b}^4(x), \ldots$ is periodic. Hint: Show $f_b(\frac{1}{b+1}) = \frac{b}{b+1}$.

The Solutions

1. a. $x = \frac{5}{12} = (0.41\overline{6})_{\text{TEN}} = (0.1\overline{02})_{\text{THREE}}$.

$$\begin{array}{lll} & & d_0 = 0.41\overline{6} \\ a_1 = 1.25 & c_1 = 1 & d_1 = 0.25 \\ a_2 = 0.75 & c_2 = 0 & d_2 = 0.75 \\ a_3 = 2.25 & c_3 = 2 & d_3 = 0.25 \Rightarrow \text{STOP} \end{array}$$

b. $x = \frac{13}{16} = (0.8125)_{\text{TEN}} = (0.64)_{\text{EIGHT}}$.

$$\begin{array}{lll} & & d_0 = 0.8125 \\ a_1 = 6.5 & c_1 = 6 & d_1 = 0.5 \\ a_2 = 4.0 & c_2 = 4 & d_2 = 0.0 \Rightarrow \text{STOP} \end{array}$$

c. $(0.132)_{\text{FOUR}} = \frac{1}{4} + 3(\frac{1}{16}) + 2(\frac{1}{64}) = \frac{15}{32}$.

d. $(0.3\overline{12})_{\text{FIVE}} = \frac{3}{5} + \frac{1}{25} + \frac{2}{125} + \frac{1}{625} + \frac{2}{3125} + \ldots = \frac{3}{5} + \frac{1}{25} + \frac{1}{625} + \ldots + \frac{2}{125} + \frac{1}{3125} + \ldots = \frac{3}{5} + \frac{1}{24} + \frac{2}{120} = \frac{79}{120}$.

e.

$$\begin{aligned} (0.113)_b = \tfrac{5}{24} &= \tfrac{1}{b} + \tfrac{1}{b^2} + \tfrac{1}{b^3} \\ 5b^3 &= 24b^2 + 24b + 72 \\ 5b^3 - 24b^2 - 24b - 72 &= 0 \end{aligned}$$

6 is the only integer solution to this equation:

$$\begin{array}{r|rrrr} & 5 & -24 & -24 & -72 \\ 6 & & 30 & 36 & 72 \\ \hline & 5 & 6 & 12 & |\underline{0} \end{array}$$

f.

$$\begin{aligned} (0.2\overline{13})_b = \tfrac{37}{60} &= \tfrac{2}{b} + \tfrac{1}{b^2} + \tfrac{3}{b^3} + \tfrac{1}{b^4} + \tfrac{3}{b^5} + \tfrac{1}{b^6} + \tfrac{3}{b^7} + \ldots \text{ (i)} \\ \tfrac{37b^2}{60} &= 2b + 1 + \tfrac{3}{b} + \tfrac{1}{b^2} + \tfrac{3}{b^3} + \tfrac{1}{b^4} + \tfrac{3}{b^5} + \ldots \text{ (ii)} \end{aligned}$$

Subtracting (i) from (ii): $\frac{37b^2}{60} - \frac{37}{60} = 2b + 1 + \frac{3}{b} - \frac{2}{b}$.

Multiplying by $60b$ and rearranging: $37b^3 - 120b^2 - 97b - 60 = 0$.

4 is the only integer solution to this equation:

$$\begin{array}{r|rrrr} & 37 & -120 & -97 & -60 \\ 4 & & 148 & 112 & 60 \\ \hline & 37 & 28 & 15 & |\underline{0} \end{array}$$

2. a.

$(0.\overline{100000})_{\text{TWO}} = \frac{32}{63}$	$(0.\overline{001000})_{\text{TWO}} = \frac{8}{63}$	$(0.\overline{000010})_{\text{TWO}} = \frac{2}{63}$
$(0.\overline{010000})_{\text{TWO}} = \frac{16}{63}$	$(0.\overline{000100})_{\text{TWO}} = \frac{4}{63}$	$(0.\overline{000001})_{\text{TWO}} = \frac{1}{63}$

b.

$$\begin{aligned}(0.\overline{010110})_{\text{TWO}} &= (0.\overline{010000})_{\text{TWO}} + (0.\overline{000100})_{\text{TWO}} + (0.\overline{000010})_{\text{TWO}} \\ &= \tfrac{16}{63} + \tfrac{4}{63} + \tfrac{2}{63} = \tfrac{22}{63}\end{aligned}$$

c. $\dfrac{p}{q} = \dfrac{2^{N-M}}{2^N - 1}$.

d.

$(0.\overline{10000})_{\text{THREE}} = \frac{81}{242}$	$(0.\overline{00100})_{\text{THREE}} = \frac{9}{242}$	$(0.\overline{00001})_{\text{THREE}} = \frac{1}{242}$
$(0.\overline{01000})_{\text{THREE}} = \frac{27}{242}$	$(0.\overline{00010})_{\text{THREE}} = \frac{3}{242}$	

e.

$$\begin{aligned}(0.\overline{12010})_{\text{THREE}} &= (0.\overline{10000})_{\text{THREE}} + (0.\overline{01000})_{\text{THREE}} + (0.\overline{00010})_{\text{THREE}} \\ &= \tfrac{81}{242} + \tfrac{54}{242} + \tfrac{3}{242} = \tfrac{69}{121}\end{aligned}$$

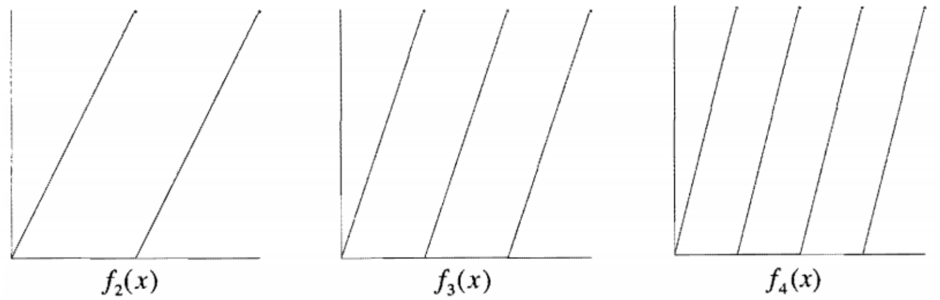
3. Let $B = \{\text{prime factors of } b\}$ and $Q = \{\text{prime factors of } q\}$.
   - $\frac{p}{q}$ is totally repeating if $B \cap Q = \varnothing$, i.e., if b and q are relatively prime.
   - $\frac{p}{q}$ is terminating if $Q \subset B$.
   - $\frac{p}{q}$ is partially repeating if $B \cap Q \neq \varnothing$ and $Q \not\subset B$.

4. $X = \dfrac{3b+2}{b^2-1} = \dfrac{3a+1}{a^2-1}$ and $Y = \dfrac{2b+3}{b^2-1} = \dfrac{1a+3}{a^2-1}$.

   Adding these together: $\dfrac{5b+5}{b^2-1} = \dfrac{4a+4}{a^2-1}$.

   Simplifying yields $5a - 4b = 1$, with solutions $(1,1), (5,6), (9,11), \ldots$. Only $(5,6)$ satisfies the constraints of the problem. However, this solution does not work when they are plugged into the equations above. Therefore, there are no solutions to this problem.

5.



6. We have $f_2\left(f_3(\frac{4}{7})\right) = f_2\left(f_3(0.\overline{120102}_{\text{THREE}})\right) = f_2(0.\overline{201021}_{\text{THREE}}) = f_2(\frac{5}{7}) = f_2(0.\overline{101}_{\text{TWO}}) = 0.\overline{011}_{\text{TWO}} = \frac{3}{7}$.
   We have $f_3\left(f_2(\frac{4}{7})\right) = f_3\left(f_2(0.\overline{100}_{\text{TWO}})\right) = f_3(0.\overline{001}_{\text{TWO}}) = f_3(\frac{1}{7}) = f_3(0.\overline{010212}_{\text{THREE}}) = 0.\overline{102120}_{\text{THREE}} = \frac{3}{7}$.

7. Looking at the graphs in #5, each segment has endpoints $(\frac{a}{b}, 0)$ and $(\frac{a+1}{b}, 1)$. The equation of the line through these two points is $f_b(x) = bx - a$.

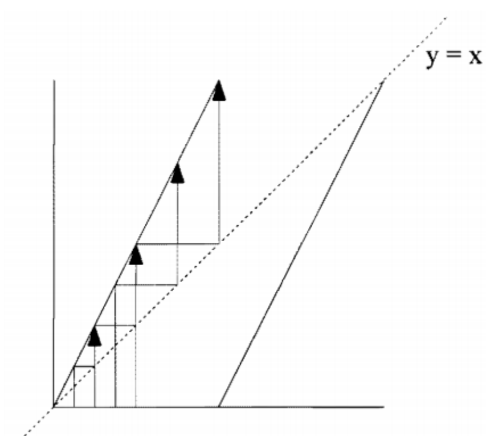
8. Using the formula in #7, $\frac{a}{12} < \frac{13}{15} < \frac{a+1}{12} \to 9.4 < a < 10.4 \to a = 10$ and $f_{12}(\frac{13}{15}) = 12(\frac{13}{15}) - a = \frac{52}{5} - 10 = \frac{2}{5}$.

9. $10x - a = x \to 10x - x = a \to x = \frac{a}{9}$, where $0 \le a < 9$. So the sum of all the solutions is $\frac{0}{9} + \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{4}{9} + \frac{5}{9} + \frac{6}{9} + \frac{7}{9} + \frac{8}{9} = 4$.

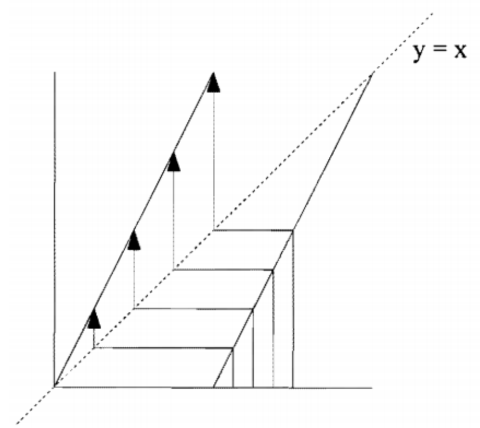
10. ${f_4}^3(x) = \frac{5}{9} \to 4x - a = \frac{5}{9}$, where $a \in \{0,1,2,3\}$. Therefore, $x = \frac{9a+5}{36}$ and so $x = \{\frac{5}{36}, \frac{14}{36}, \frac{23}{36}, \frac{32}{36}\} = \frac{9k+5}{36}$, where $k = \{0,1,2,3\}$. ${f_4}^2(x)\frac{9k+5}{36}$ where $k = \{0,1,2,3\} \to 4x - a = \frac{9k+5}{36}$, where $a \in \{0,1,2,3\}$ and $k = \{0,1,2,3\}$. Therefore, $x = \{\frac{5}{144}, \frac{14}{144}, \frac{23}{144}, \dots, \frac{140}{144}\} = \frac{9m+5}{144}$ where $m \in \{0,1,2,\dots,15\}$.
    So $f_4(x) = \frac{9m+5}{144}$ where $m = \{0,1,2,\dots,15\} \to 4x - a = \frac{9m+5}{144}$, where $k = \{0,1,2,\dots,15\}$ and for every k, $a \in \{0,1,2,3\}$. There are 64 answers! $x = \{\frac{5}{576}, \frac{14}{576}, \frac{23}{576}, \dots, \frac{572}{576}\} = \frac{9n+5}{576}$ where $n \in \{0,1,2,\dots,63\}$. The set of solutions forms an arithmetic progression whose sum is $\frac{577}{18}$.

11. Let's solve this problem using graphical iteration. There are four cases:

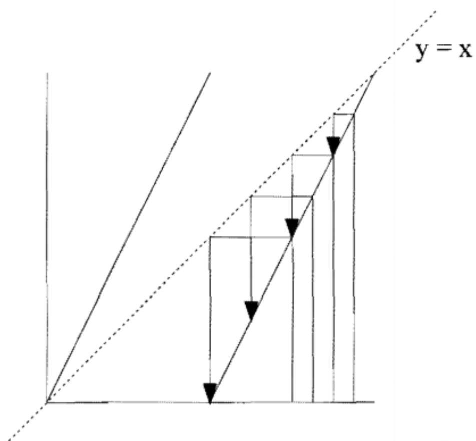
    i) if $0 \le x < \frac{1}{4}$, then ${f_2}^2(x) > f_2(x)$.



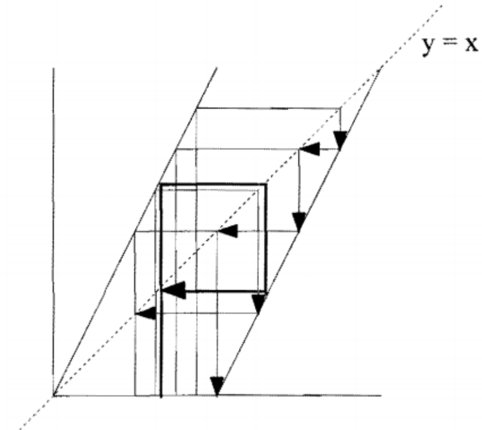
ii) if $\frac{1}{2} < x < \frac{3}{4}$, then ${f_2}^2(x) > f_2(x)$.



iii) if $\frac{3}{4} < x < 1$, then $x > {f_2}^2(x)$.



iv) if $\frac{1}{4} < x < \frac{1}{2}$, then ${f_2}^2(x) < f_2(x)$ and sometimes $x \leq {f_2}^2(x)$.



As you can see from this last graphical iteration, there is one value $x = c$ where ${f_2}^2(x) = x$ and for any value $c < {f_2}^2(x) < x$, ${f_2}^2(x) > x$. So let's find this value. When $\frac{1}{4} < x < \frac{1}{2}$ then $f_2(x) = 2x$ and so $\frac{1}{2} \leq f_2(x) < 1$. Therefore, ${f_2}^2(x) = f_2(f_2(x)) = 2(2x) - 1 = 4x - 1$. Since ${f_2}^2(x) = x$, $4x - 1 = x \to 3x = 1 \to x = \frac{1}{3}$. Therefore, the solution to this problem is $\frac{1}{3} < x < \frac{1}{2}$.

12. a. b^n

    b. $y + \frac{b^n - 1}{2}$. Observe some cases and generalize. If $f_3(x) = y$, then $3x - a = y$ where $a \in \{0, 1, 2\}$ and $x = \frac{y+a}{3} = \{\frac{y}{3}, \frac{y+1}{3}, \frac{y+2}{3}\}$. The sum of these values is $y + 1$. If ${f_3}^2(x) = y$, then $f_3(x) = \{\frac{y}{3}, \frac{y+1}{3}, \frac{y+2}{3}\}$ and $3x - a = \frac{y}{3}$ or $\frac{y+1}{3}$ or $\frac{y+2}{3}$ where $a \in \{0, 1, 2\}$. Therefore, $x = \frac{y+3a}{9}$ or $\frac{y+1+3a}{9}$ or $\frac{y+2+3a}{3} = \{\frac{y}{9}, \frac{y+1}{9}, \frac{y+2}{9}, \ldots, \frac{y+8}{9}\}$. The sum of these values is $y + 4$.

13. Lemma 1: $\left(\frac{1}{b+1}\right)_b = 0.\overline{0X}$, where $X = b - 1$.

    $(\frac{1}{b+1}) \cdot b = \frac{b}{b+1}$, which is < 1. So $c_1 = 0$.
    $(\frac{b}{b+1}) \cdot b = \frac{b^2}{b+1} = \frac{b^2-1+1}{b+1} = \frac{b^2-1}{b+1} + \frac{1}{b+1} = b - 1 + \frac{1}{b+1}$. So $c_2 = b - 1 = X$.
    Since the remainder repeats, the process stops and $0X$ is the repetend.

Lemma 2: $\left(\frac{b}{b+1}\right)_b = 0.\overline{X0}$, where $X = b-1$.

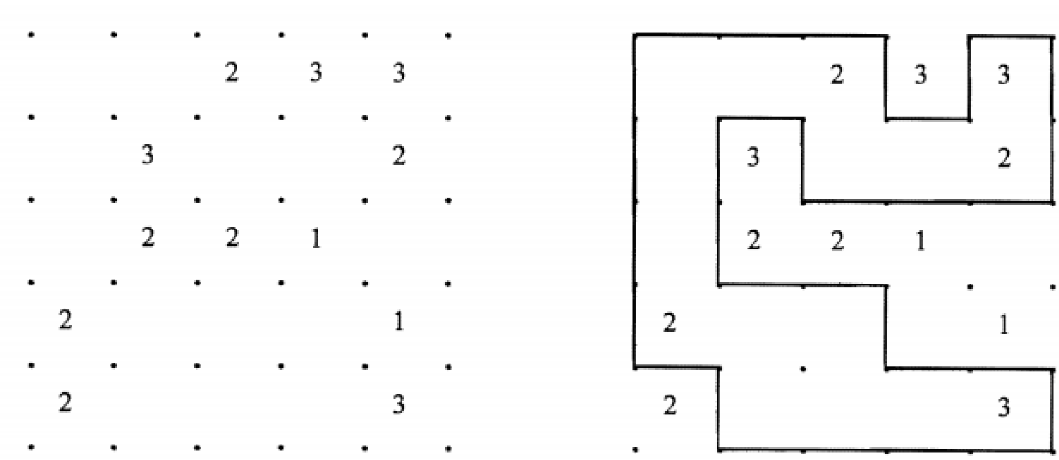
$(\frac{b}{b+1}) \cdot b = \frac{b^2}{b+1} = b - 1 + \frac{1}{b+1}$. So $c_1 = b - 1 = X$.
$(\frac{1}{b+1}) \cdot b = \frac{b}{b+1}$, which is < 1. So $c_2 = 0$.
Since the remainder repeats, the process stops and $X0$ is the repetend. So the process repeats with a period of 2.

Slitherlinks

The Background

While at the Mathematical Association of America's MathFest 2009 last summer, I was introduced to a logic puzzle called **Slitherlinks**. The simplicity of the puzzle was fascinating, and soon solving them became quite addictive. The puzzle was invented in Japan in 1989 by Nikoli Puzzles, the same company that created Sudoku, Kakuro, and Hitori. Since then the puzzle has been refined, algorithms for designing and solving them have been created, and it is now a popular app on iPhones.

A Slitherlink puzzle consists of a square array of dots, like the lattice points in the first quadrant of a coordinate system. The four points: (m, n), $(m + 1, n)$, $(m, n + 1)$, and $(m + 1, n + 1)$, make up the corners of a cell. Inside some of the cells is a numeral. The goal of the puzzle is to form a simple, closed curve (a loop with no crossings, i.e., every lattice point has degree zero or two) by connecting the dots of a cell with horizontal or vertical segments, and the number of connected segments around a cell must equal the number inside the cell. If a cell does not contain a number then the number of connected segments around the cell is unknown. Since the solution is a simple, closed curve, the Jordan Curve Theorem from topology states it must have an inside and an outside, and a line segment connecting a point on the inside to a point on the outside must cross the curve an odd number of times. Like Sudoku there is only one solution to each puzzle, which can be arrived at by pure logic, and like Sudoku, a random guess, can often have disastrous results. The best beginning strategy is to place segments in the grid where you know two dots must be connected and place a × between two dots that you know cannot be connected. Enjoy!



A 5 by 5 Slitherlink Puzzle and Solution

The Problems

1. Solve these 5 by 5 Slitherlink puzzles. (When submitting your solutions, you may submit one copy of **Special Answer Sheet #1** with all three puzzles solved on it or three copies of **Special Answer Sheet #1**, each with one puzzle solved on it.)

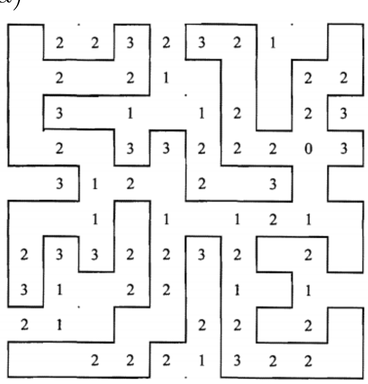
a)

b)

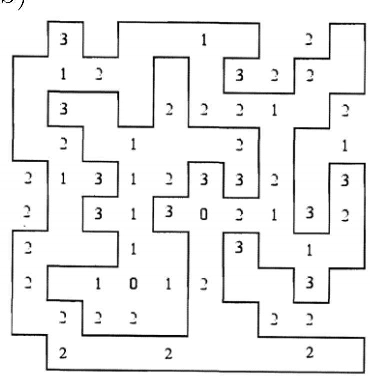
c)

2. Each of the following puzzles has one error. a) Identify or describe the error. b) Correct the error by erasing and/or drawing segments, not by changing the numbers. (Hint: More than 90% of each of the puzzles is correct, so isolate your corrections near the error you found.) On **Special Answer Sheet #2** you only have to draw the part of the Slitherlink loop that you changed. Again you may submit the three solutions on one answer sheet or use three separate sheets for your solutions.

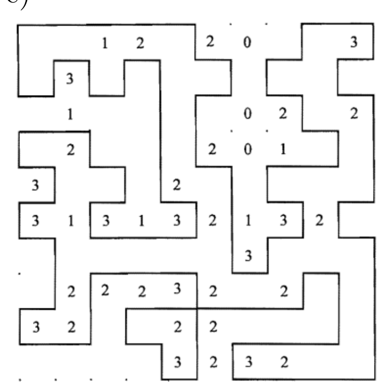
a)



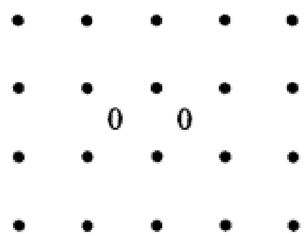
b)

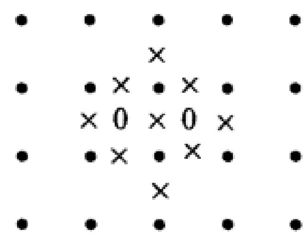


c)



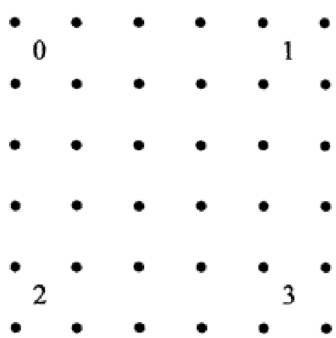
3. Sometimes a given solution allows one to know for sure that a segment can be drawn between two dots or an × should be placed between two dots. These are called "basic theorems" and can be proven with logical reasoning. Each of the following problems is such a hypothetical situation that is part of a larger Slitherlink puzzle and you need to supply the conclusion, i.e., which segments and ×'s can be drawn. To receive full credit, <u>all</u> guaranteed segments and all guaranteed ×'s must be in your conclusion. For example, given the image below and to the left, you can conclude the image below and to the right.



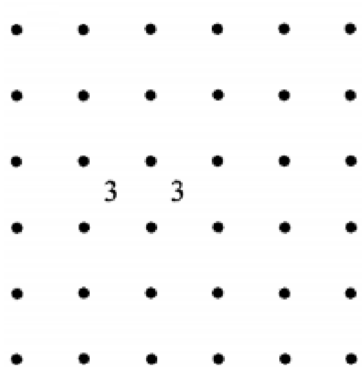


Please submit only one copy of **Special Answer Sheet #3** with all seven conclusions on it.

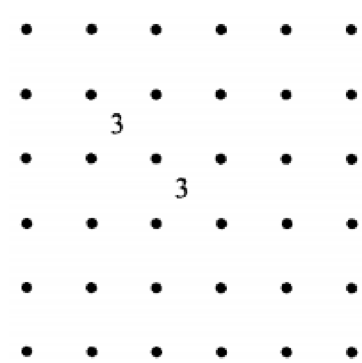
a) Numbers in corners:



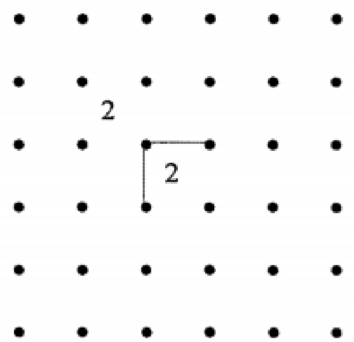
b) i. Adjacent threes:



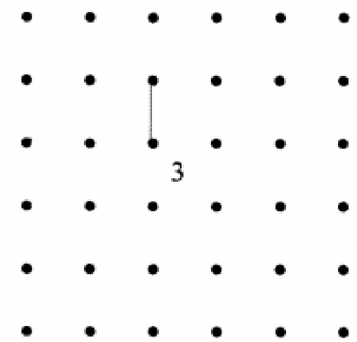
ii. Diagonal threes:



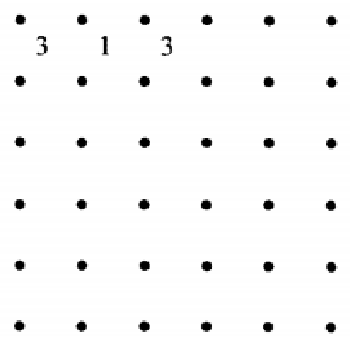
c) Diagonal twos with two segments drawn:



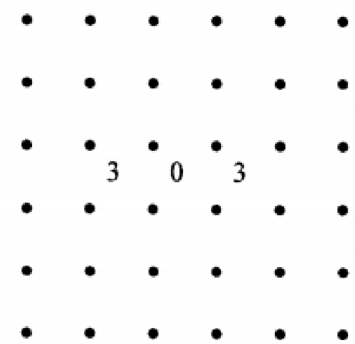
d) A three with a segment drawn:



e) i. Three-one-three in a corner:

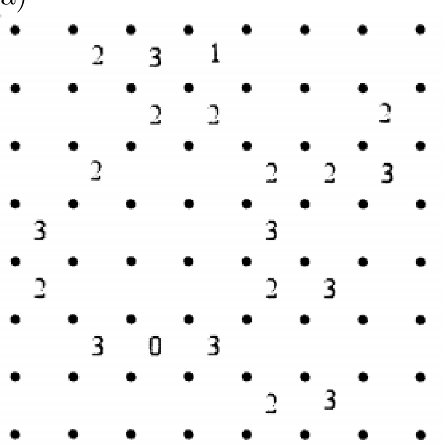


ii. Three-zero-three in a row:

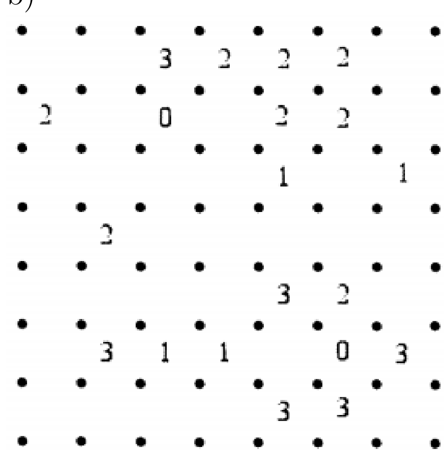


4. These puzzles are harder than those in problem #1. You will need to use the theorems from the last problem, lots of $\times$'s, and as in chess, think a few moves ahead. Draw your final loop on the answer sheet that will be turned in. Partial credit will be calculated by adding correct segments minus incorrect segments. So even if you do not complete the loop, record the segments you know for sure. Submit four copies of **Special Answer Sheet #4**, each with one solution.

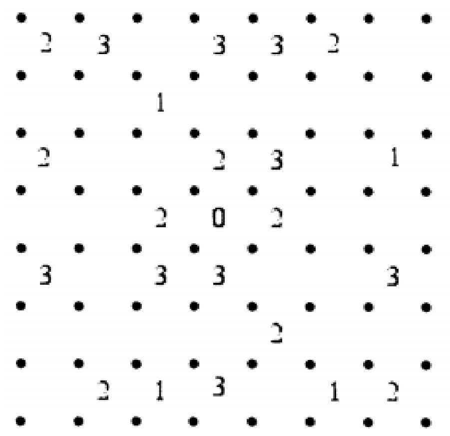
a)



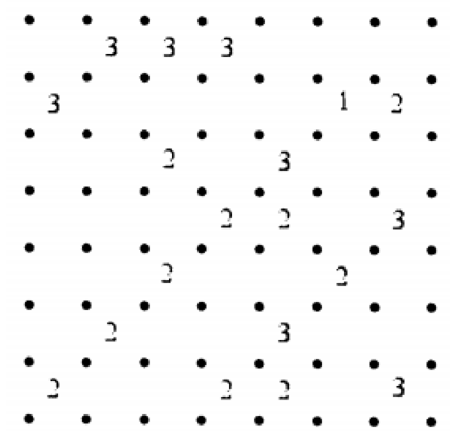
b)



c)



d)



5. Once a Slitherlink puzzle is solved you could place the correct numbers in all the cells of the grid. (This is how the first Slitherlink puzzle appeared!) Prove that the sum of all the numbers inside the Slitherlink loop must be even.

   You can search the web to find more Slitherlink puzzles of various sizes and difficulty. Some are interactive; some are not limited to square grids. (See Simon Tatham's Portable Puzzle Collection.) Enjoy!

ARML Power Contest
2009–2010
Special Answer Sheet #1

a)

2	3			2
2		2		1
3	2		3	
2				
			3	3

b)

		2	2	
3		3		3
	3			
1			2	
			1	

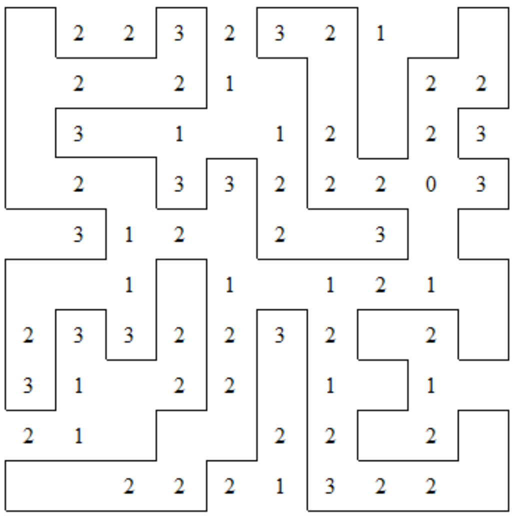
c)

	2			
	3			2
3				3
2	3		3	2
		1		2

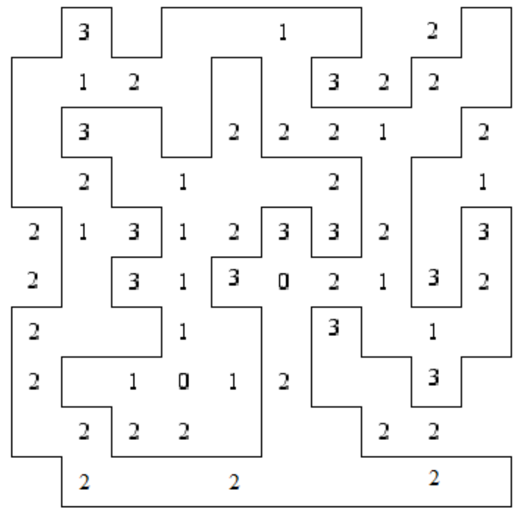
ARML Power Contest
2009–2010
Special Answer Sheet #2

Circle the number which is in a cell with an incorrect number of edges or describe the error in each Slitherlink.

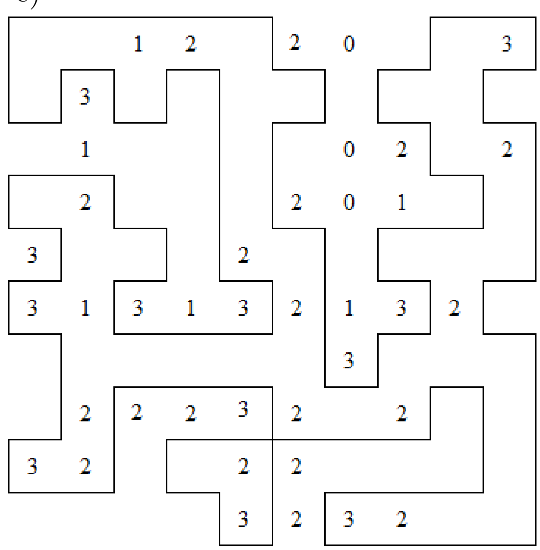
a)



b)

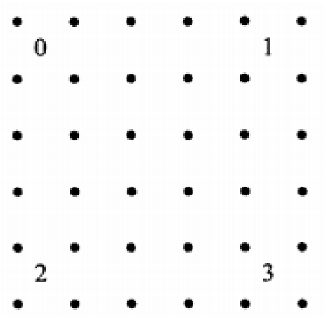


c)

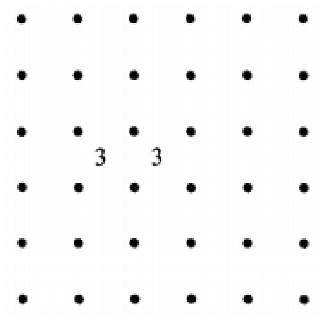


ARML Power Contest
2009–2010
Special Answer Sheet #3

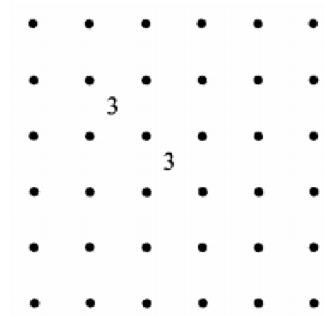
a)



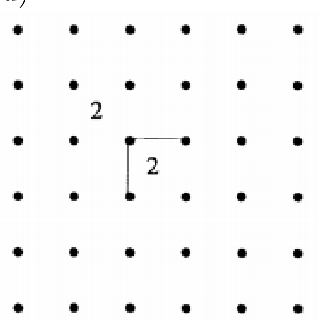
b)



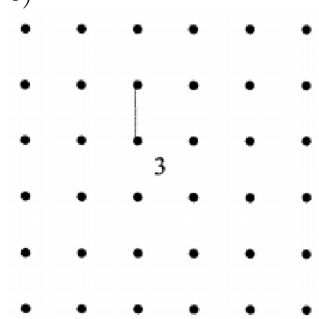
c)



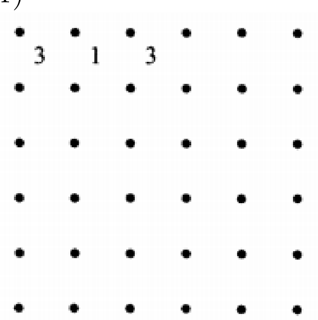
d)



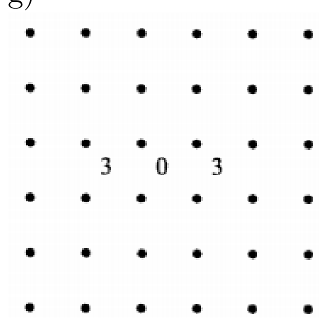
e)



f)

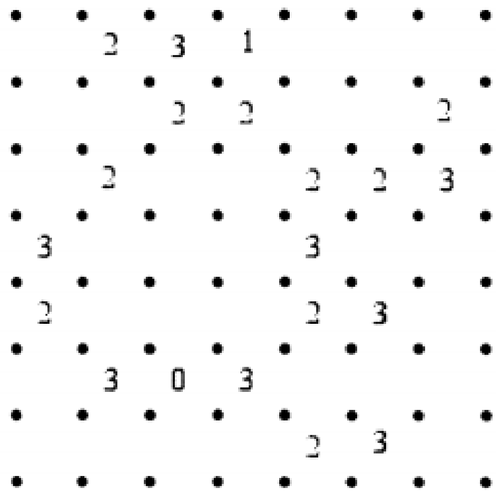


g)

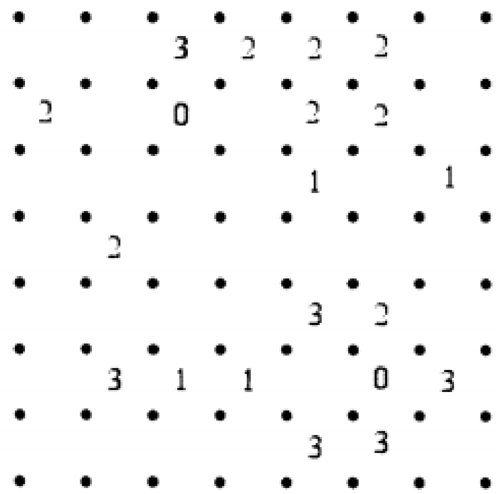


ARML Power Contest
2009–2010
Special Answer Sheet #4

a)



b)



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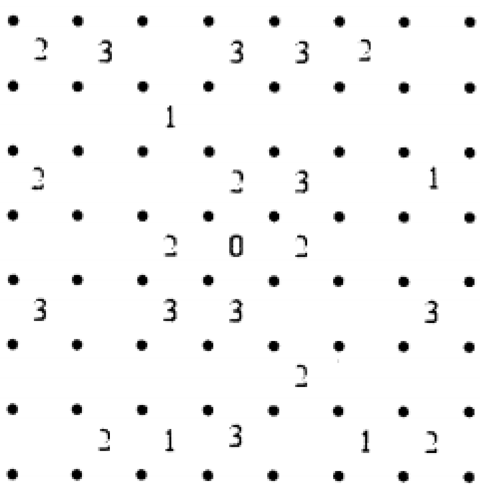
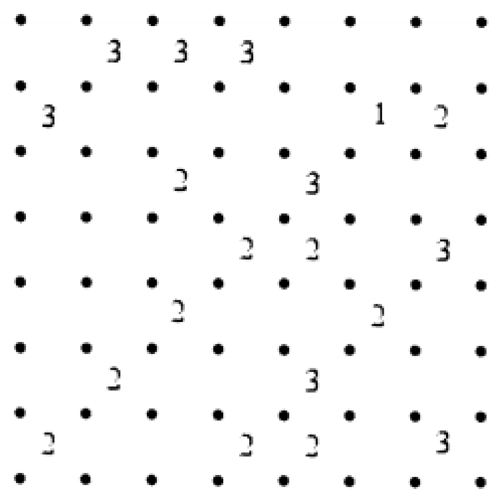
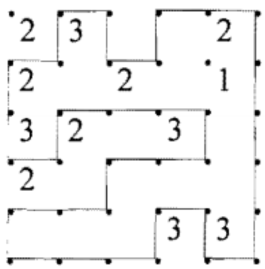


d)



The Solutions

1. a)



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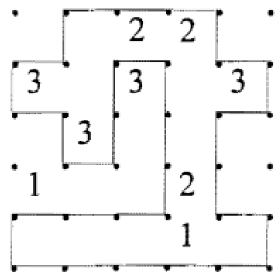

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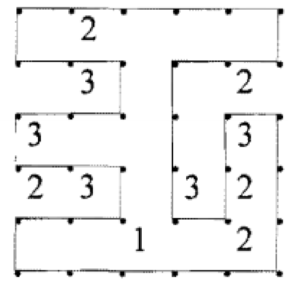
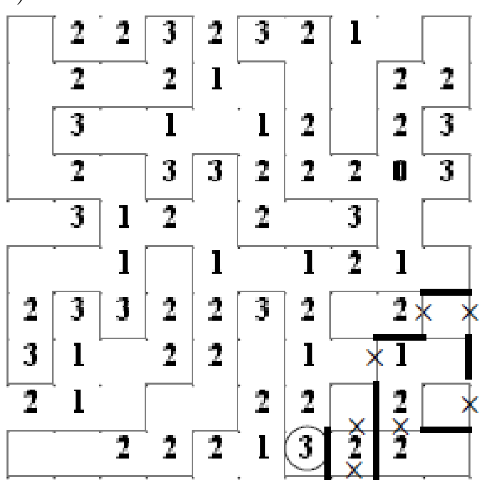


2. a)



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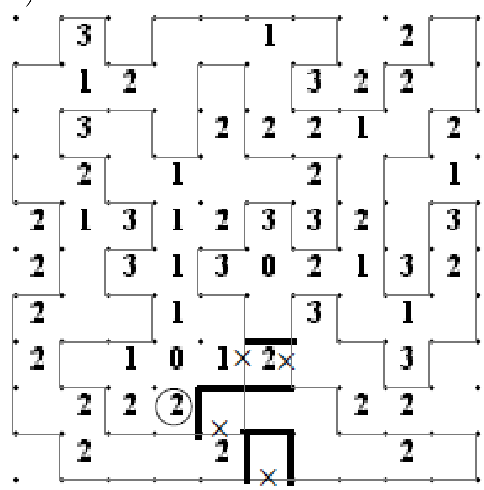

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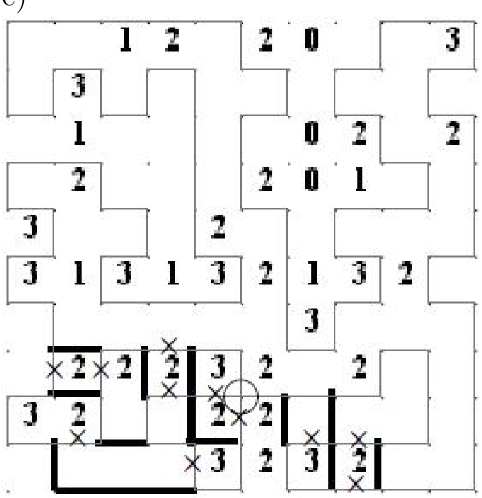
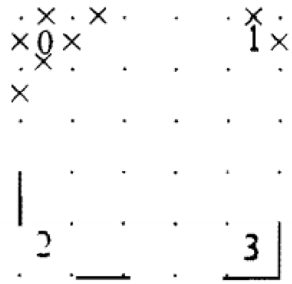
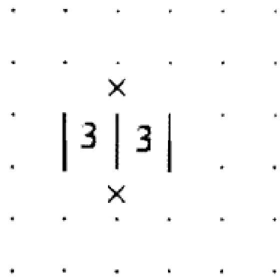


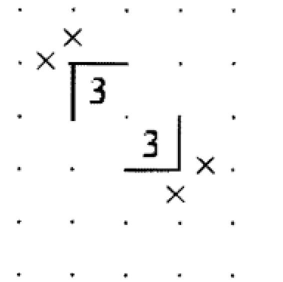
3. a)



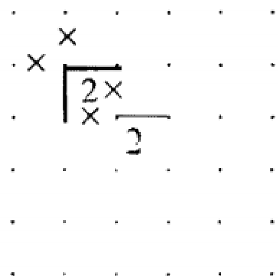
b) i.



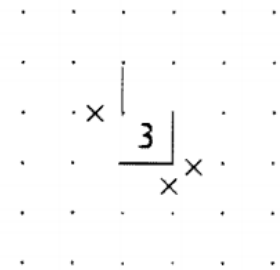
b) ii.



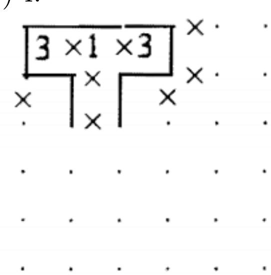
c)



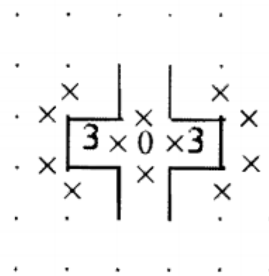
d)



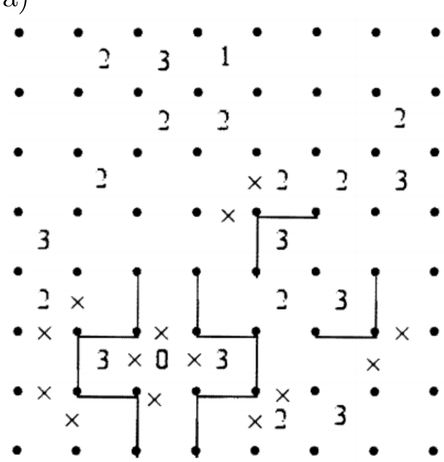
e) i.

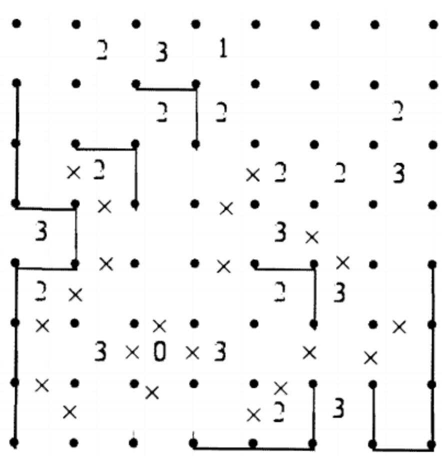


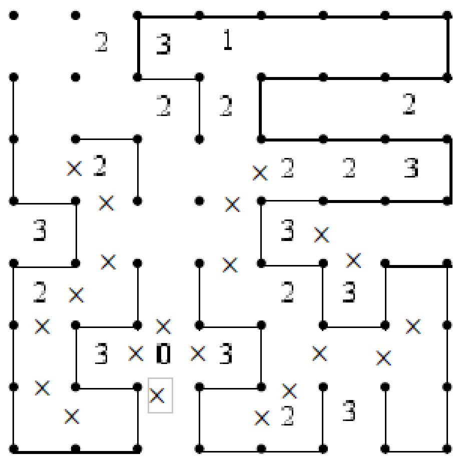
e) ii.

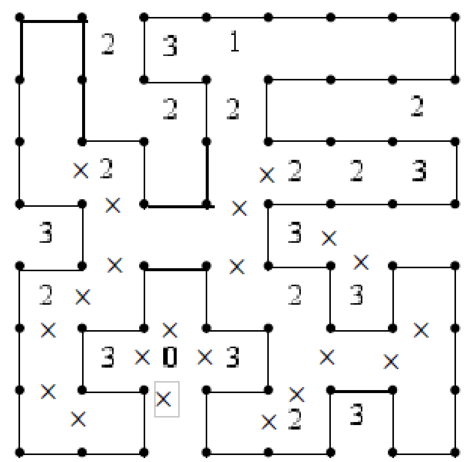


4. a)









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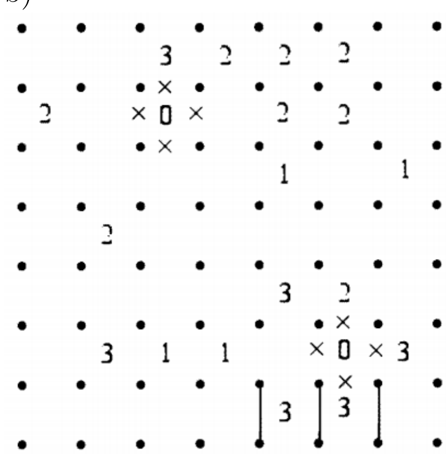

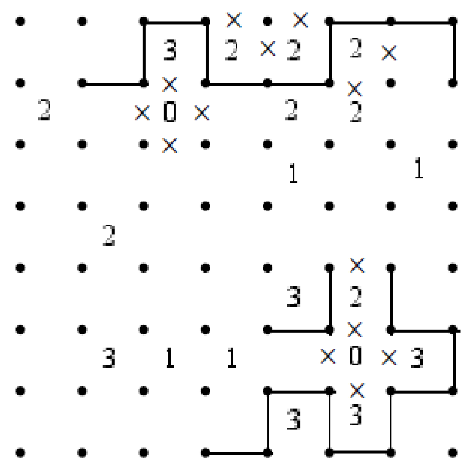

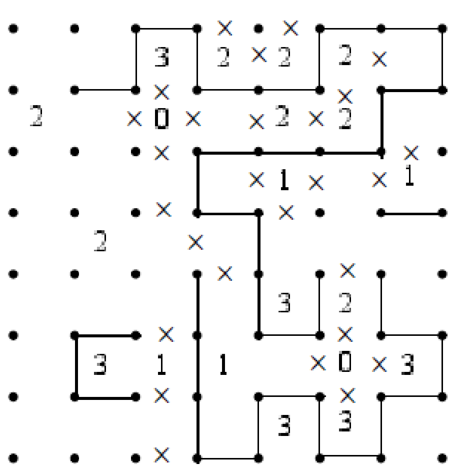

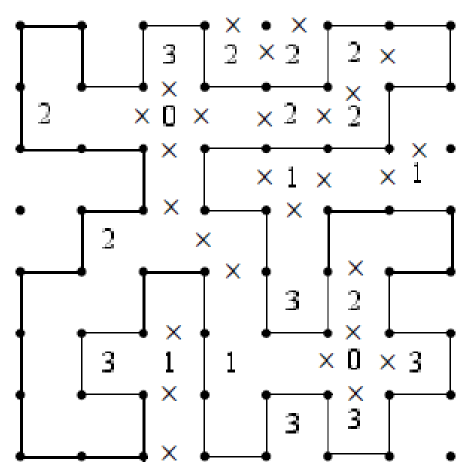

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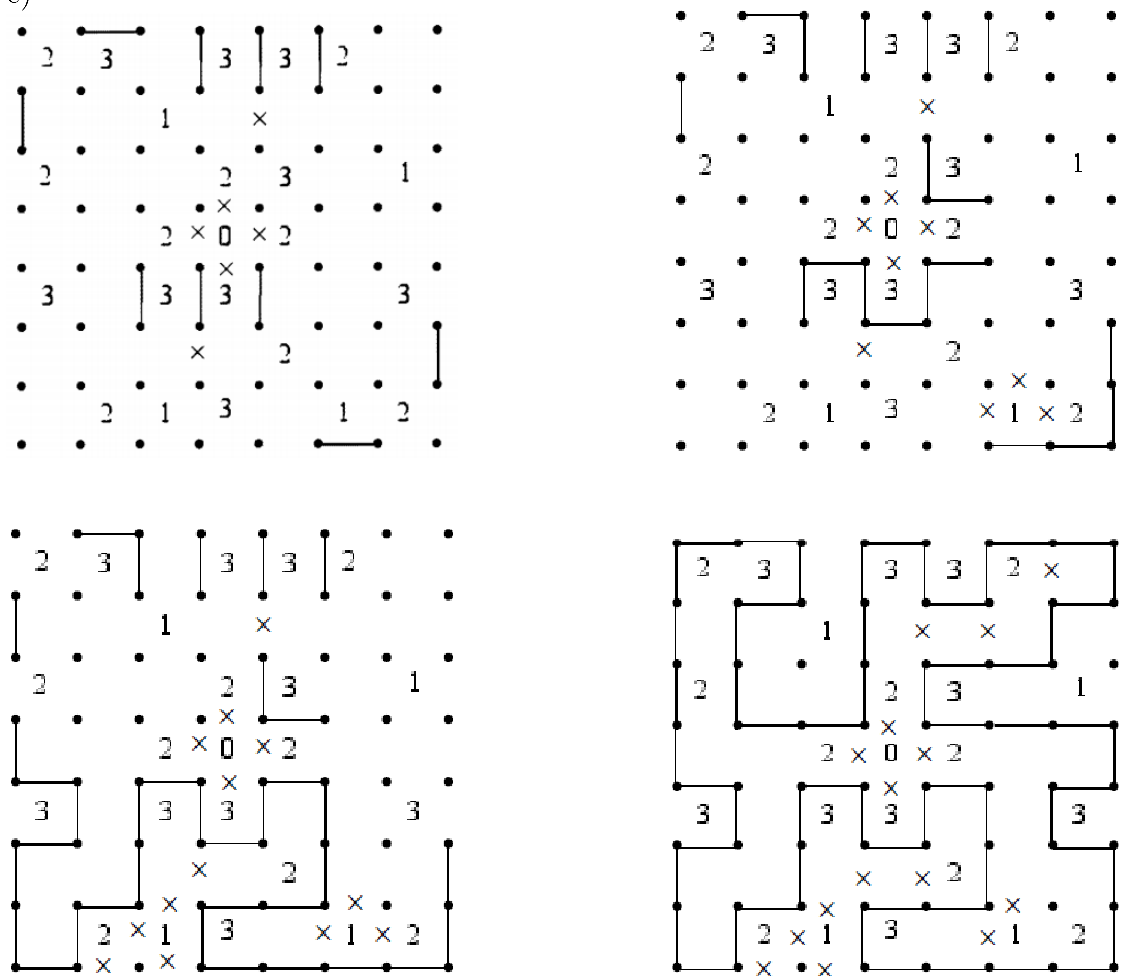

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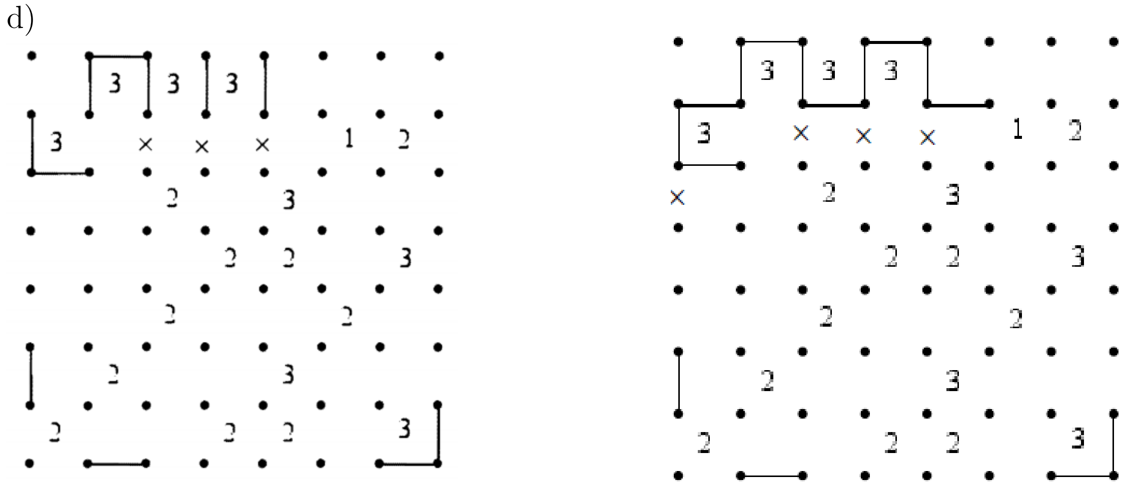
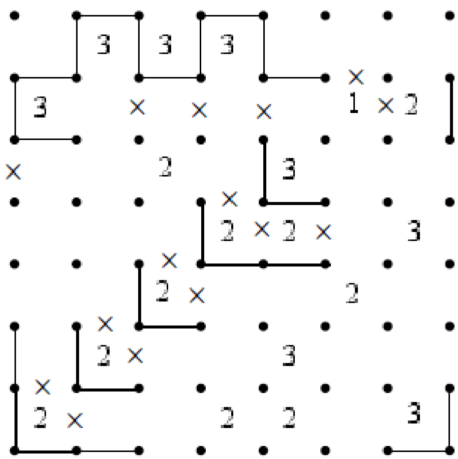
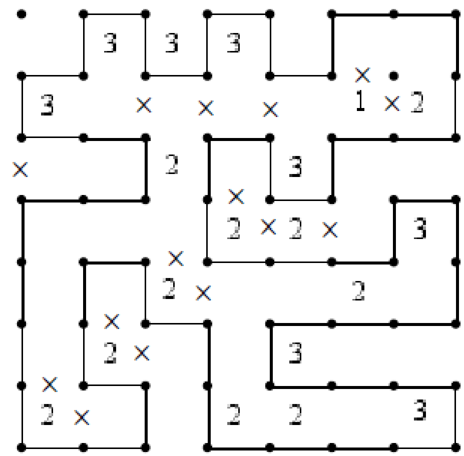
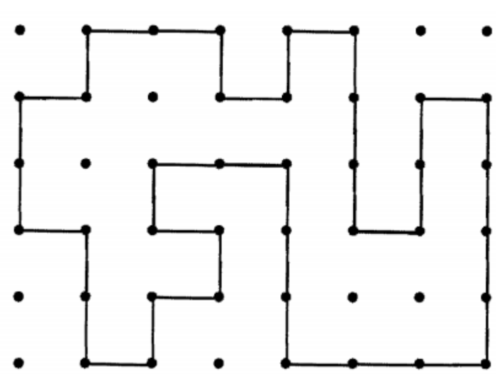
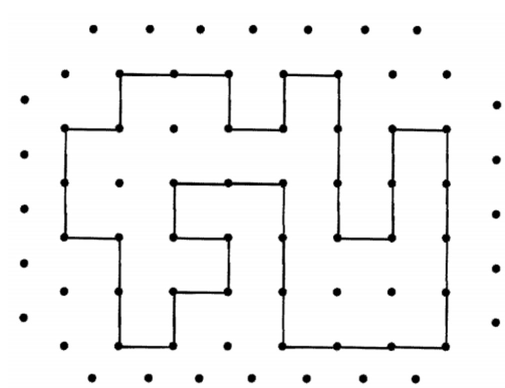
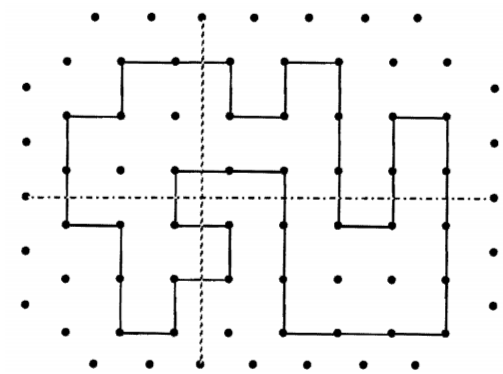


5. Since the sum of all the numbers inside the Slitherlink represents the total of all the horizontal and vertical segments that make up the Slitherlink or the perimeter of the Slitherlink, we need to prove that the perimeter of a Slitherlink is always even. Given a rectangular array of points on which some Slitherlink is drawn. Above and below each column of square cells add a point. To the left and right of each row of square cells add a point. (See diagram below.) These points are all clearly outside the simple closed curve that makes up the Slitherlink. The Jordan Curve Theorem states that if you connect with a segment two points on the outside of any simple closed curve, this segment will cross the curve an even number of times. Therefore, connecting a point above a column of cells with a point below the column of cells will be even and will count the number of horizontal segments from that column that are part of the Slitherlink. Doing this for each of the columns will always produce an even number and the sum of all these numbers will be the total number of horizontal segments that make up the Slitherlink. Connecting the points to the left and right of a row of cells with a segment will count the vertical segments that are in the Slitherlink from squares in that row. Again by the Jordan Curve Theorem this number will always be even. The sum of all the numbers determined by doing this for each of the rows in the array, will be the total number of vertical segments that make up the Slitherlink. Since the number of horizontal and vertical segments is even the sum of the numbers inside any Slitherlink must always be even.

This problem can also be proven using Pick's Theorem or by using a parity argument. In a parity argument assume that the array of points on which a Slitherlink is drawn is associated with the first quadrant of the rectangular (x, y) coordinate system. Each point of the Slitherlink is assigned a number represented by the sum of its coordinates. A segment of a Slitherlink will always connect an "even" and an "odd" point. Since a Slitherlink is closed, it returns to its starting number, indicating an even number of segments.



There are many questions about Slitherlink puzzles that remain unsolved. For example, what is the greatest number of numbers that can be removed from a Slitherlink and still guarantee a solution?

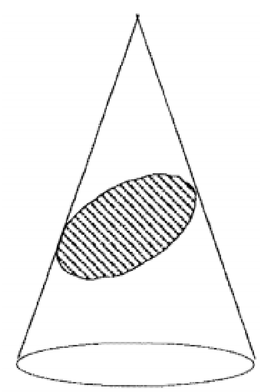
Drawing Ellipses

The Background

The study of conic sections, curves formed by the intersection of a plane with a cone, began about 200 B.C. by Apollonius of Perga, the last of the great mathematicians of the Golden Age of Greek mathematics. In his eight-volume treatise called *Conics*, he gave names to the various curves (parabola, hyperbola, and ellipse) and developed most of the geometric properties of the curves. But conics remained pure mathematics for almost 2000 years until 1609 when Johannes Kepler revolutionized astronomy by declaring that the planets revolved around the sun in elliptic paths with the sun as a focus. As an ellipse is a circle stretched horizontally and/or vertically, ellipses share some similar formulas with circles. For example, the area of a circle is $\pi \cdot r \cdot r$ while the area of an ellipse is $\pi \cdot a \cdot b$. However, while the circumference of a circle is $2 \cdot \pi \cdot r = \pi(r + r)$, the circumference of an ellipse is not $\pi(a + b)$ and cannot even be expressed in a closed form. I find it amazing that both a cone and a cylinder cut diagonally by a plane produce an ellipse, and that if you wrap a strip of paper around a cylinder, slice the cylinder diagonally, and the unwrap the paper, the edge of the paper formed by the circumference of the ellipse is a sine wave! There is much to explore in this old topic of mathematics!

Definitions for an ellipse:

1. The curve formed when a plane intersects a right circular cone at an angle to its axis of symmetry less than 90° but greater than the slant angle of the cone.
2. The set of all points in a plane the sum of whose distances from two given points (called the foci) is always constant.
3. The set of all points whose ratio of its distance from a fixed point (called the focus) to its distance from a fixed line (called the directrix) is constant and between 0 and 1.
4. The graph formed by the general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where $4AC - B^2 > 0$.
5. The curve formed when a cylinder is intersected by a plane not parallel to the axis of symmetry of the cylinder.



Other vocabulary:

A segment connecting two points of an ellipse is called a chord. A chord through the center of an ellipse is called a diameter. The longest diameter of an ellipse is called the major axis. The foci will always be on the major axis. The shortest diameter of an ellipse is called the minor axis. It will always be perpendicular to the major axis. The focal length is the distance between the two foci. The eccentricity of an ellipse, a number between 0 and 1, refers to the "roundness" of the ellipse. The closer the eccentricity is to 0, the closer the ellipse is to a circle.

Equations of an ellipse:

1. General equation: $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ where $4AC-B^2>0$. If the ellipse is centered at the origin, then $D=E=0$. If the major and minor axes of the ellipse are parallel with the coordinate axes, then $B=0$.

2. Standard form: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The length of the major axis is $2a$ and the length of the minor axis is $2b$. The focal length is $2c$, where $c^2=|a^2-b^2|$. The eccentricity e is $\dfrac{c}{a}$. If the ellipse is translated so that $x=x'+h$ and $y=y'+k$, where (x',y') is the point on the translated coordinate axes, it is now centered at (h,k) and the equation becomes $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$. If the ellipse is rotated an angle of θ, then $x=x'\cos\theta-y'\sin\theta$ and $y=x'\sin\theta-y'\cos\theta$ and the general equation must now be used to describe the curve.

3. Parametric equations: $\begin{cases} x=a\cos(\theta) \\ y=b\sin(\theta) \end{cases}$

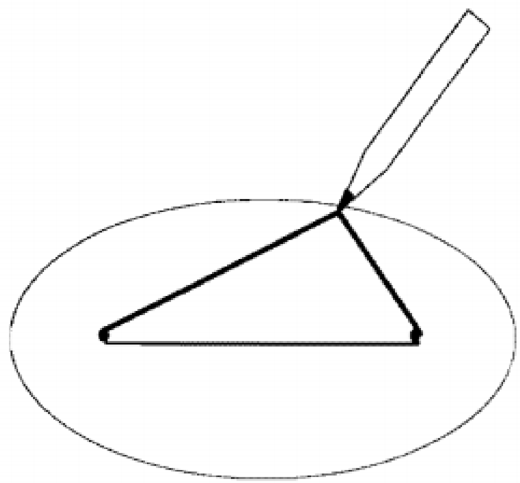
4. Polar equation: $r=\dfrac{ep}{1-e\cos\theta}$ with directrix $x=\pm p$ or $r=\dfrac{ep}{1-e\sin\theta}$ with directrix $y=\pm p$. One focus is at the pole and $0<e<1$ is the eccentricity of the ellipse.

Through various algebraic techniques these equations can be shown to be equivalent.

The problems in this contest deal with methods that are used by artists, engineers, draftsmen, astronomers, and mathematicians for drawing ellipses. In several problems you are given the method of construction and must prove that the curve formed is an ellipse. This can be done by showing that points on the curve satisfy one of the above equations or that the curve has one of the properties that defines an ellipse. Most of the problems are independent of one another and therefore can be solved in any order.

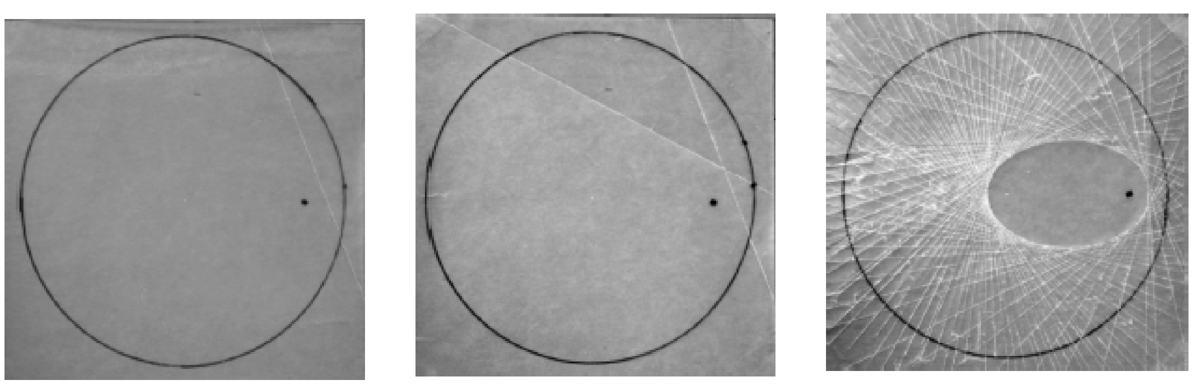
The Problems

1. An ellipse can easily be drawn using two push-pins and a loop of string as shown at the right. By keeping the triangular loop of the string taut as you move the pencil, a smooth curve can be drawn.



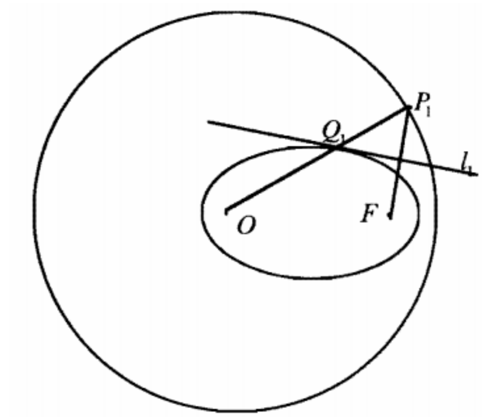
a) How do you know the curve formed is an ellipse?

b) If you want to draw an ellipse with a major axis of length 12 and a minor axis of length 8, how far apart should the push-pins be placed and what should be the length (or circumference) of the loop of string?



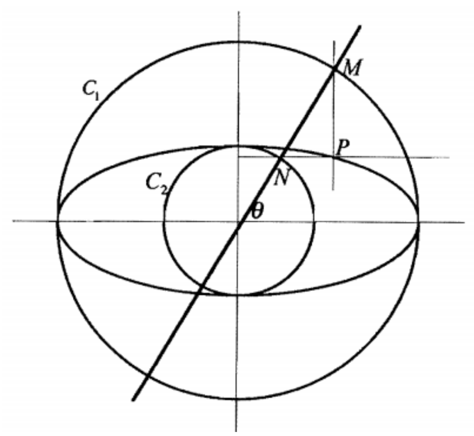
2. An ellipse can be made by drawing a large circle on a piece of wax paper and marking a point F somewhere inside the circle but not at the center. Fold the paper so that some point P on the circle coincides with F and crease the paper on this fold. Unfold the paper and repeat the process using another point P on the circle. Continue repeating the process.

a) Using the diagram at the right, prove that the points Q determined by this technique trace out an ellipse.



b) If circle O, centered at the origin, has a radius of 10 units and F is 8 units from the center, determine an equation for the ellipse formed by this technique.

3. On a set of coordinate axes, draw a circle C_1 centered at the origin and having a radius equal to a, and a second circle C_2 centered at the origin and having a radius equal to b, where $a > b$. Draw a line through the origin (at an angle of θ), intersecting C_1 and C_2 at points M and N, respectively. Draw a line parallel to the vertical axis through M and another line parallel to the horizontal axis through N. Label the intersection of these two lines P. Repeat the process above for different values of θ, determining a locus of points P. Connect all the points P to form an ellipse.



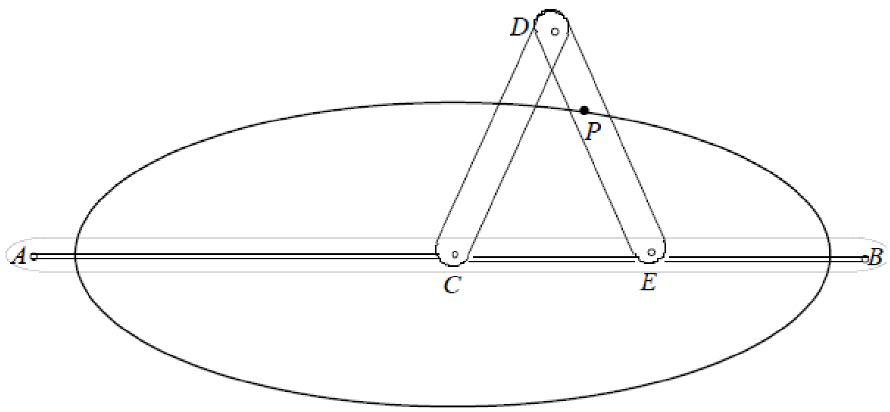
a) If $P = (x, y)$, show that it lies on the curve defined by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

b) If the area between the two circles (called an annulus) is equal to the area of the ellipse, what is the ratio of $\dfrac{a}{b}$?

4. The Dutch mathematician Franz Van Schooten came up with the following simple device for drawing ellipses (see the diagram on the following page). Points A and B are stationary, with $AB > 4 \cdot CD$. $CD = DE$ and points C and D are hinges with C anchored at the midpoint between A and B. At point P is a pen which traces out a curve as E moves from A to B.

a) Let $CD = a$ and $DP = b$. If C is the origin and if this curve is an ellipse, what would be its equation?

b) Let $P = (x, y)$ and $E = (t, 0)$. Prove $P(x, y)$ satisfies the equation above for any value of t in the interval $[-2a, 2a]$.

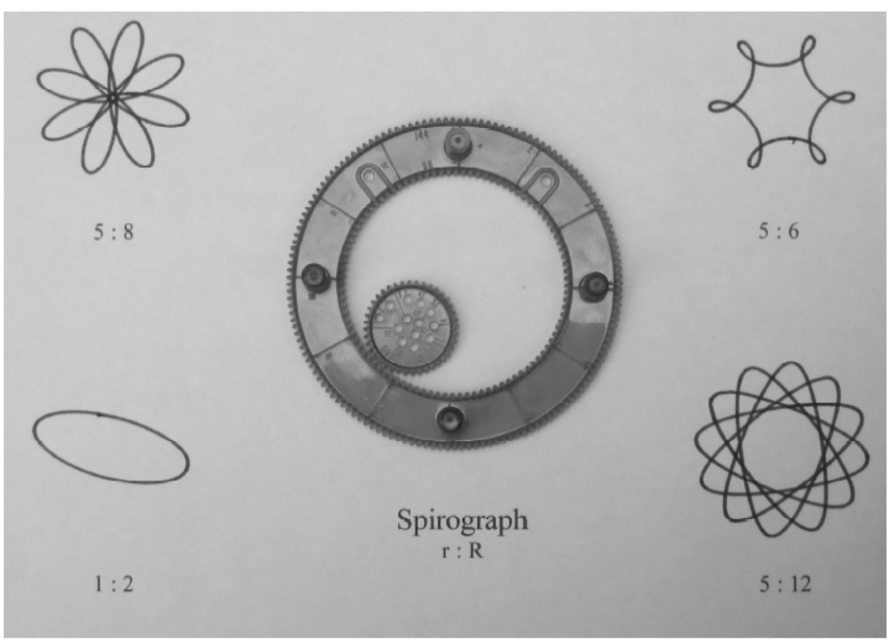


5. Using a Spirograph, a circular disk can be rolled inside of a fixed circular ring without slipping, producing a figure called a *trochoid*. A point P on the disk marks the position of a pen which traces out a curve as the disk is rolled. If the inner radius of the fixed circular ring is R, the radius of the circular disk is r, and d is the distance from P to the center of the disk, the coordinates of P are (x, y), where

$$x = (R-r)\sin\left(\frac{r}{R}\theta\right) + d\sin\left(\left(\frac{r-R}{R}\right)\theta\right) \text{ and } y = (R-r)\cos\left(\frac{r}{R}\theta\right) + d\cos\left(\left(\frac{r-R}{R}\right)\theta\right).[1]$$

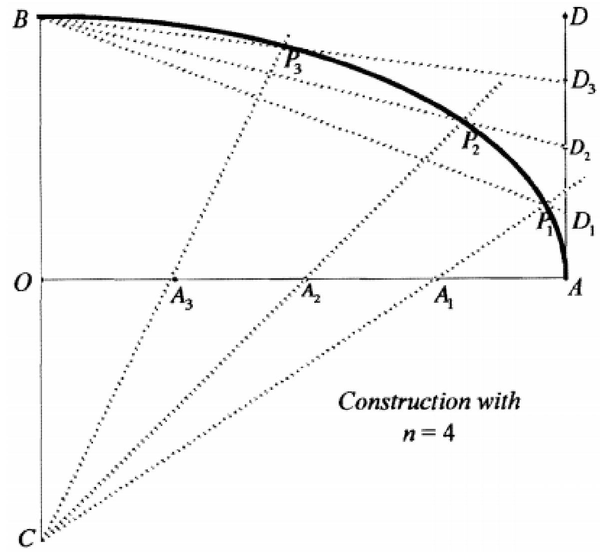
The figure below shows four trochoids and the radio $r : R$ that produced them.

If $R = 2r$, prove the trochoid is an ellipse.

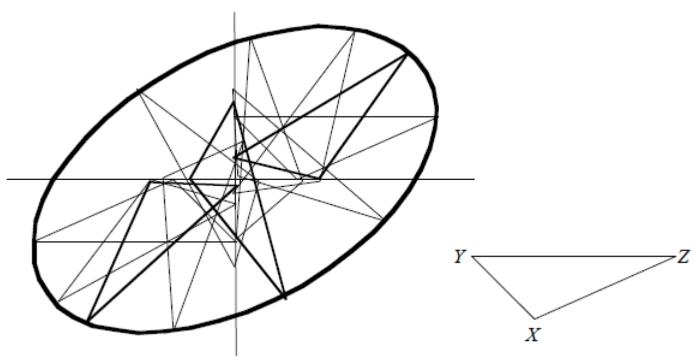


[1] See `http://web.archive.org/web/20090929103228/http://the-robinson-family.org/nigel/spiro.htm` for a nice derivation of these formulas.

6. Let O be the origin, $A = (a, 0)$, $B = (0, b)$, $C = (0, -b)$, and $D = (a, b)$. Divide segments $\overline{OA}$ and $\overline{AD}$ into n congruent segments using $n - 1$ points. On $\overline{AD}$, starting at A, label these points $D_1, D_2, D_3, \ldots, D_k, \ldots, D_{n-1}$. On $\overline{OA}$, starting at A, label the points $A_1, A_2, A_3, \ldots, A_k, \ldots, A_{n-1}$. Let (x, y) be coordinates of point P_k, the intersection of lines $\overleftrightarrow{CA_k}$ and $\overleftrightarrow{BD_k}$. Prove P_k is on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Hint: One method might be to find the equations of lines $\overleftrightarrow{CA_k}$ and $\overleftrightarrow{BD_k}$ by first expressing the coordinates of D_k and A_k in terms of a, b, n, and k.)



7. Take any triangle XYZ and position it on a coordinate axes so that X is on the x-axis and Y is on the y-axis. Mark the location of point Z. Reposition the triangle so that again X is on the x-axis and Y is on the y-axis and plot point Z. Repeat several times. The locus of all such points Z will be an ellipse, whose axes are not parallel with the coordinate axes.



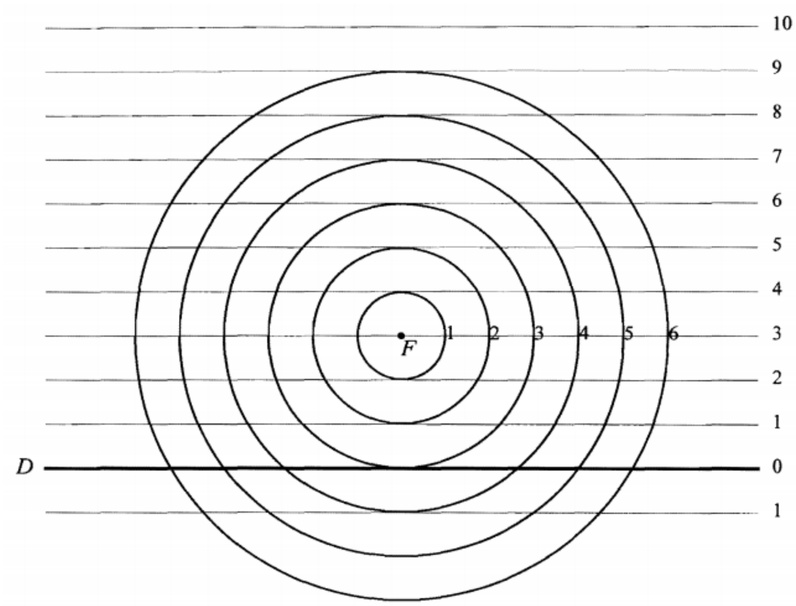
a) If XYZ is an equilateral triangle with sides of length 2, what is the equation of the ellipse formed? (Hint: Determine the exact coordinates of some points on the ellipse and determine the general equation.)

b) If XYZ is a $30-60-90$ triangle with angle $Z = 30°$ and hypotenuse XZ of length 2, what is the equation of the ellipse formed?

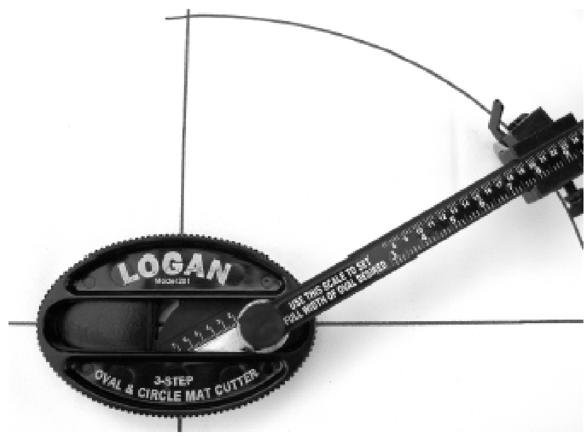
8. Given a line D (called the directrix) and a point F (called the focus). Let P be any point (x, y). If PF is the distance from the point to the focus and PD is the distance from the point to the directrix, then an ellipse is the set of all points P where the ratio of PF to PD is a constant less than 1. The ratio PF/PD is called the *eccentricity* of the ellipse.

a) Use the graph below to **accurately** plot at least 10 points of the ellipse with an eccentricity of $\frac{2}{3}$. (Hint: $\frac{2}{3} = \frac{4}{6} = \frac{3}{4.5} = \ldots$) Use the **Special Answer Sheet** to record your answers to 8a and 8b.

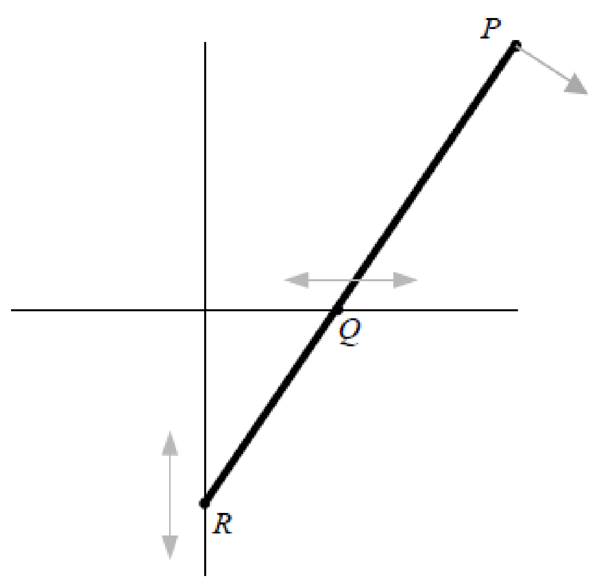
b) If F is the origin and D is the line $y = -3$, what is the equation of this ellipse?



9. Logan Graphics Products Inc. makes an oval Mat Cutter used in picture framing. I was curious whether the oval it produced was an ellipse or not. Pictures of the tool did not reveal how it worked but designer Curt Logan was gracious enough to send me a free mat cutter to examine for myself. Although the cutting head is a patented part, inside the oval base I discovered that its mechanism was similar to a trammel for drawing ellipses designed by Archimedes!



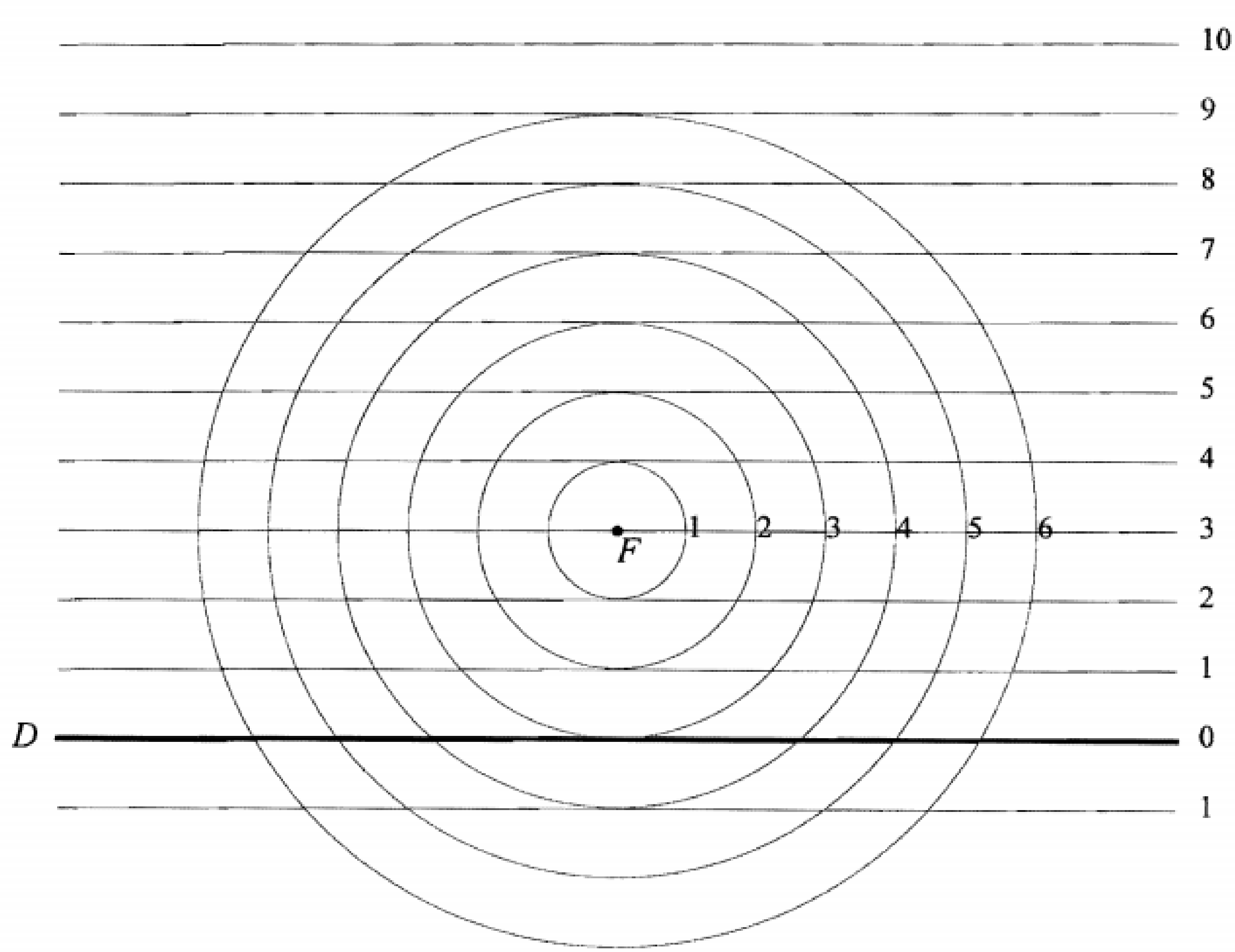
As point P is rotated around the origin, point Q moves back and forth in a horizontal (x-axis) channel while point R moves up and down in a vertical (y-axis) channel. If $RQ = a, QP = b, Q = (t, 0)$, and $P = (x, y)$, show that the locus of all possible points P determine an ellipse by determining the equation of the ellipse.



ARML Power Contest
2009–2010
Special Answer Sheet

Problem 8

a)



b)

The Solutions

1. a) Let $L =$ the length of the loop of string.

   $L = F_1P + F_2P + F_1F_2$
   $L - F_1F_2 = F_1P + F_2P$
   $\text{constant} = F_1P + F_2P$

   Therefore, P is on an ellipse.

   b) $L = F_1P + F_2P + F_1F_2$
   $L = 2a + 2c$
   $L = 12 + 2c$

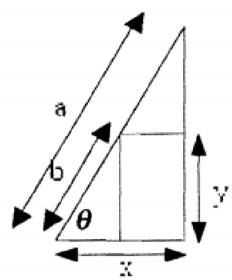
   $a^2 - b^2 = c^2$
   $36 - 16 = c^2$
   $2\sqrt{5} = c$

   $F_1F_2 = 2c = 4\sqrt{5} \approx 8.94$
   $L = 12 + 4\sqrt{5} \approx 20.94$

2. a) For any point P_1 on the circle, P_1 is folded onto F forming l_1, the perpendicular bisector of segment FP_1. Therefore, $Q_1F = Q_1P$. Since OP_1 is the radius of the circle and $OP_1 = OQ_1 + Q_1P_1$, $OQ_1 + Q_1P_1$ is constant and therefore, $OQ_1 + Q_1F_1$ is constant. Therefore, Q is a point on an ellipse with foci at O and F_1.

   b) $2a = 10$ and $2c = 8$. Therefore, $a = 5$ and $c = 4$ and since $a^2 - b^2 = c^2$, $b = 3$. Assume F is $(8, 0)$, then the center is at $(4, 0)$. Therefore, the equation of the ellipse is $\dfrac{(x-4)^2}{25} + \dfrac{y^2}{9} = 1$.

3. a) $\cos(\theta) = \dfrac{x}{a} \Rightarrow \cos^2(\theta) = \dfrac{x^2}{a^2}$ and $\sin(\theta) = \dfrac{y}{b} \Rightarrow \sin^2(\theta) = \dfrac{y^2}{b^2}$. Adding these two equations produces $1 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.

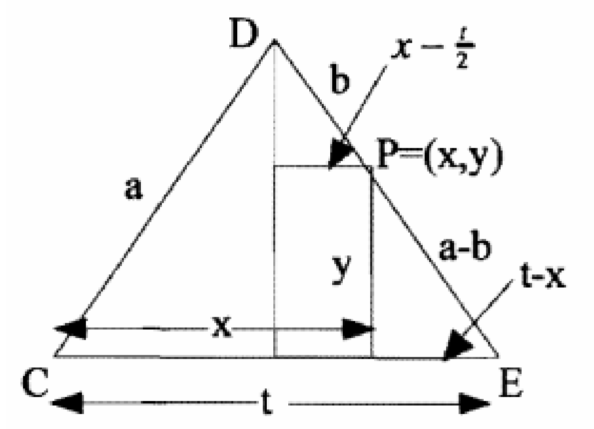


   b) The area of the ellipse $= ab\pi$ and the area of the annulus $= a^2\pi - b^2\pi$. Therefore,

$$\begin{aligned}
ab &= a^2 - b^2 \\
\frac{ab}{b^2} &= \frac{a^2}{b^2} - \frac{b^2}{b^2} \\
\frac{a}{b} &= \frac{a^2}{b^2} - 1 \\
0 &= \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 \\
\frac{a}{b} &= \frac{1+\sqrt{5}}{2}, \text{ the Golden Ratio!}
\end{aligned}$$

4. a) When $\overline{CD}$ is horizontal $CP = a + B$ and when $\overline{CD}$ is vertical $CP = a - b$. Therefore, $\dfrac{x^2}{(a+b)^2} + \dfrac{y^2}{(a-b)^2} = 1$.

b) From similar right triangles, $\dfrac{b}{x-\frac{t}{2}} = \dfrac{a-b}{t-x} \Rightarrow t = \dfrac{2ax}{a+b}$ and from the Pythagorean Theorem, $(t-x)^2 + y^2 = (a-b)^2$. Substituting results in $\left(\dfrac{2ax}{a+b} - x\right)^2 + y^2 = (a-b)^2$. And so $\left(\dfrac{2ax}{a+b} - \dfrac{x(a+b)}{a+b}\right)^2 + y^2 = a - b^2 \Rightarrow \left(\dfrac{x(a-b)}{a+b}\right)^2 + y^2 = (a-b)^2$. This results in $\dfrac{x^2(a-b)^2}{(a+b)^2} + y^2 = (a-b)^2$, which implies



$$\frac{x^2}{(a+b)^2} + \frac{y^2}{(a-b)^2} = 1.$$

5. If $R = 2r$ then $x = r\sin\left(\dfrac{\theta}{2}\right) + d\sin\left(\dfrac{-\theta}{2}\right)$ and $y = r\cos\left(\dfrac{\theta}{2}\right) + d\cos\left(\dfrac{-\theta}{2}\right)$. This simplifies to $x = (r-d)\sin\left(\dfrac{\theta}{2}\right)$ and $y = (r+d)\cos\left(\dfrac{\theta}{2}\right)$. Squaring results in $x^2 = (r-d)^2\sin^2\left(\dfrac{\theta}{2}\right)$ and $y = (r+d)^2\cos^2\left(\dfrac{\theta}{2}\right)$, and dividing, produces $\dfrac{x^2}{(r-d)^2} = \sin^2\left(\dfrac{\theta}{2}\right)$ and $\dfrac{y^2}{(r+d)^2} = \cos^2\left(\dfrac{\theta}{2}\right)$. Summing results in $\dfrac{x^2}{(r-d)^2} + \dfrac{y^2}{(r+d)^2} = 1$.

6. First, find the coordinates of A_k and D_k.

   $A_k = \big((1 - \frac{k}{n})a, 0\big)$ and $D_k = \big(a, (\frac{k}{n})b\big)$.

   Second, find the equation of $\overleftrightarrow{CA_k}$ and $\overleftrightarrow{BD_k}$.

   $\overleftrightarrow{CA_k} : nbx + (k-n)ay = ab(n-k)$ and $\overleftrightarrow{BD_k} : (n-k)bx + any = abn$.

   Square each:

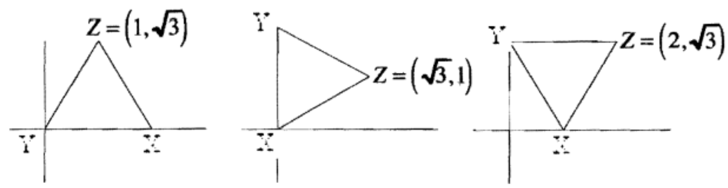
   $n^2b^2x^2 + 2nbx(k-n)ay + (k-n)^2a^2y^2 = a^2b^2(n-k)^2$ and

   $(n-k)^2b^2x^2 + 2(n-k)bxany + a^2n^2y^2 = a^2b^2n^2$.

   Add them, producing: $(n^2 + (n-k)^2)\,b^2x^2 + (n^2 + (k-n)^2)\,a^2y^2 = a^2b^2\left((n-k)^2 + n^2\right)$.

   Divide by $a^2b^2\left((n-k)^2 + n^2\right)$, resulting in $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (N.B. $(n-k)^2 = (k-n)^2$.)

7. a) Let the equation of the ellipse be: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Since it is centered at the origin, $D = E = 0$. To avoid answers that are just multiples of each other, let $A = 1$. Three points are necessary to solve the equation $x^2 + Bxy + Cy^2 + F = 0$. Positioning $\triangle XYZ$ as in the diagrams produces these three points:

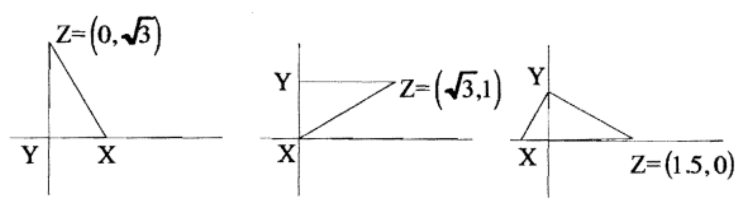


Solving the system of equations

$$\begin{cases} 1 + B(\sqrt{3}) + C(3) + F = 0 \\ 3 + B(\sqrt{3}) + C(1) + F = 0 \\ 4 + B(2\sqrt{3}) + C(3) + F = 0 \end{cases}$$

results in $x^2 - \sqrt{3}xy + y^2 - 1 = 0$ as the equation of the ellipse.

b) As in solution 7a, three points are needed to satisfy the equation $x^2 + Bxy + Cy^2 + F = 0$. Positioning $\triangle XYZ$ as in the diagrams produces these three points:

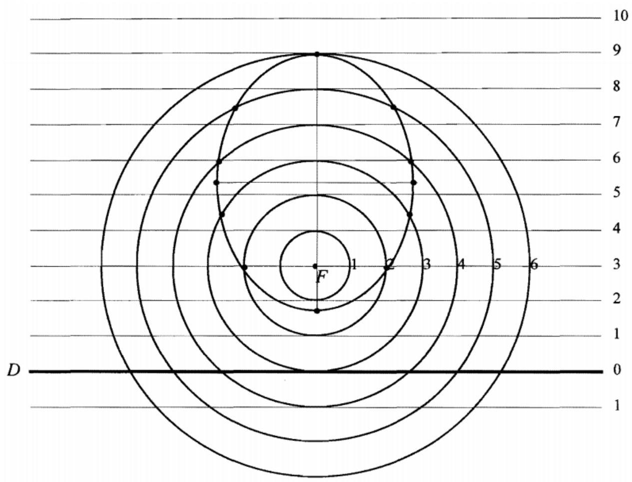


Solving the system of equations

$$\begin{cases} 0 + B(0) + C(3) + F = 0 \\ 3 + B(\sqrt{3}) + C(1) + F = 0 \\ \frac{9}{4} + B(0) + C(0) + F = 0 \end{cases}$$

results in $x^2 - \dfrac{\sqrt{3}}{2}xy + \dfrac{3}{4}y^2 - \dfrac{9}{4} = 0$ or $4x^2 - 2\sqrt{3}xy + 3y^2 - 9 = 0$ as the equation of the ellipse.

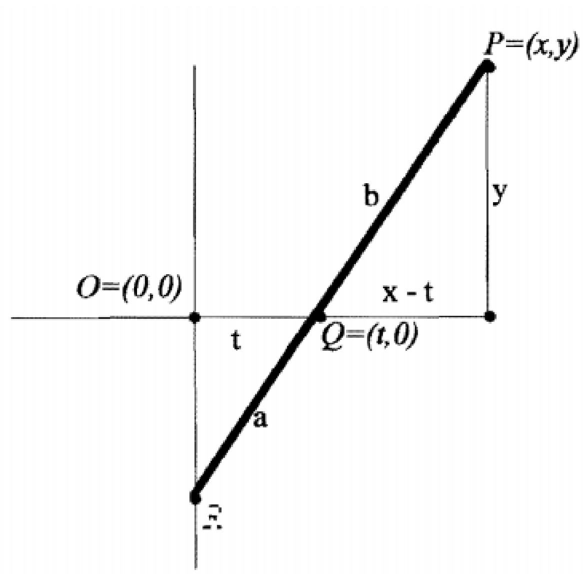
8. a)



b) The length of the major axis is $6+1.2 = 7.2$ and so $b = 3.6$, the center is $(0, 2.4)$, and $c = 2.4$. Since $a^2 = b^2 - c^2$, $a^2 = 7.2$. Therefore, the equation of the ellipse is $\dfrac{x^2}{7.2} + \dfrac{(y-2.4)^2}{12.96} = 1$.

9. From the diagram below, $(x-t)^2 + y^2 = b^2$ and $\dfrac{a}{t} = \dfrac{b}{x-t} \Rightarrow t = \dfrac{ax}{a+b}$. Substituting,

$$\left(x - \frac{ax}{a+b}\right)^2 + y^2 = b^2$$
$$\Rightarrow \left(\frac{x(a+b) - ax}{a+b}\right)^2 + y^2 = b^2$$
$$\Rightarrow \left(\frac{bx}{a+b}\right)^2 + y^2 = b^2$$
$$\Rightarrow \left(\frac{x(a+b) - ax}{a+b}\right)^2 + y^2 = b^2$$
$$\Rightarrow \frac{b^2x^2}{(a+b)^2} + y^2 = b^2$$
$$\Rightarrow \frac{x^2}{(a+b)^2} + \frac{y^2}{b^2} = 1.$$



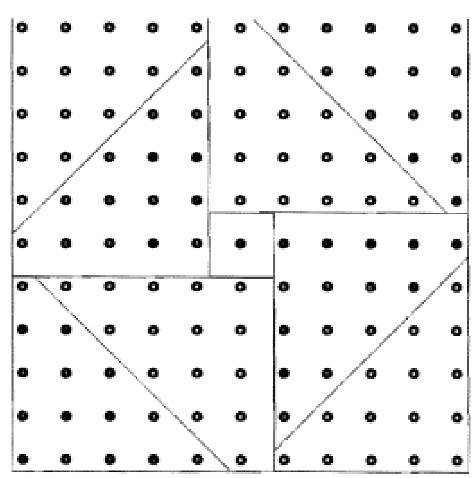
Deltorials

The Background

The n^{th} triangular number is usually defined as the sum of the positive integers from 1 to n. But viewed in reverse $n+(n-1)+(n-2)+\ldots+3+2+1$, one can see that the triangular numbers are the additive analogs of the factorials. In light of this, in this contest the triangular numbers will be called **deltorials** and Δ_n will be the symbol for the n^{th} deltorial. Like factorials, deltorials pop up in many places throughout mathematics.

Although often associated with a story from Gauss' early life, the formula for the n^{th} deltorial, namely $\Delta_n = \dfrac{n(n+1)}{2}$, was known to the ancient Greeks. However, Gauss at age 19 recorded in his diary, "ΕΥΡΗΚΑ! num $= \Delta + \Delta + \Delta$", after proving that every natural number is the sum of (at most) three deltorials.

Deltorials appear in Pascal's Triangle in the second diagonal and so $\Delta_k = \dbinom{k+1}{2}$. The sums of the deltorials appear in the next diagonal in the triangle and so $\displaystyle\sum_{k=1}^{n} \Delta_k = \binom{n+2}{3}$.



The array of dots at the right shows that $121 = 11^2 = 8(15)+1$, or in general, $(2n+1)^2 = 8(\Delta_n)+1$. Richard Guy, a prominent Canadian mathematician, calls this Theorem Zero of number theory.

Theorem 0: "All odd squares are congruent to 1 mod 8."

Here is a very curious pattern involving deltorials:

$$1+3+6=10$$
$$15+21+28+36=45+55$$
$$66+78+91+105+120=136+153+171$$

Will it always be true that you can find n consecutive deltorials whose sum is the sum of the next $n-2$ consecutive deltorials? I have verified it is true for the first 58 deltorials! "I have found a marvelous proof of this curiosity but the margins of this paper are too small to contain it."

Deltorials and their associated formulas appear in all the problems and solutions in this first round of the contest. **Be sure to justify all answers.** Enjoy!

$\Delta_1=1$; $\Delta_2=3$; $\Delta_3=6$; $\Delta_4=10$; $\Delta_5=15$; $\Delta_6=21$; $\Delta_7=28$; $\Delta_8=36$; $\Delta_9=45$; $\Delta_{10}=55$; $\Delta_{11}=66$; $\Delta_{12}=78$; $\Delta_{13}=91$; $\Delta_{14}=105$; $\Delta_{15}=120$; $\Delta_{16}=136$; $\Delta_{17}=153$; $\Delta_{18}=171$; $\Delta_{19}=190$

<u>**The Problems**</u>

Part A: Make a conjecture suggested by each of these patterns, express it in a suitable mathematical notation, using Δ_n for the n^{th} deltorial, and then prove each conjecture.

1.

$$\begin{aligned} 9(1)+1&=10\\ 9(3)+1&=28\\ 9(6)+1&=55\\ 9(10)+1&=92\ \ldots \end{aligned}$$

2.

$$\begin{aligned} 1+3&=4\\ 3+6&=9\\ 6+10&=16\\ 10+15&=25\ \ldots \end{aligned}$$

3.

$$\begin{aligned} 1(6)+3&=9\\ 3(10)+6&=36\\ 6(15)+10&=100\\ 10(21)+15&=225\ \ldots \end{aligned}$$

4.

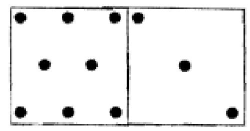
$$\begin{aligned} 1+0&=1\\ 9+1&=10\\ 36+9&=45\\ 100+36&=136\ \ldots \end{aligned}$$

5.

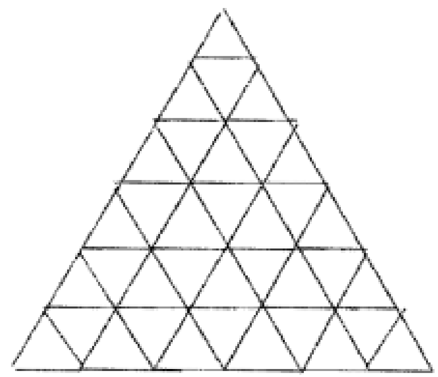
$$\begin{aligned} 1+15&=16\\ 15+66&=81\\ 66+190&=256\\ 190+435&=625\ \ldots \end{aligned}$$

Part B: Counting

6. Dominoes are rectangular tiles and each tile is divided into two squares with a number of dots in each square. A generic set of dominoes has from 0 to n dots in each square. How many different dominoes are possible in a generic set?

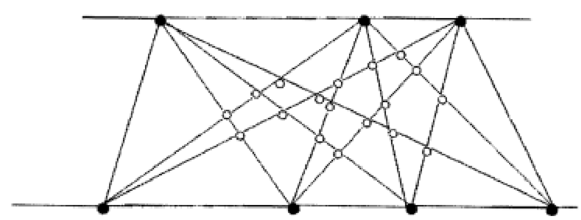


7. An equilateral triangle is divided into n^2 congruent non-overlapping equilateral triangles. (In the diagram at the right $n = 6$.) If n is an **even integer**, find the number of triangles in the entire figure.



8. A square is divided into n^2 congruent non-overlapping squares. Find the number of rectangles in the entire figure. (N.B. All squares are rectangles and the rectangles may overlap.)

9. Draw two parallel lines, l_1 and l_2, and place three points on l_1 and four points on l_2. Draw segments connecting each point on l_1 to each point on l_2. (You must be sure that the original points are placed so that no three of these segments intersect at one point.) There should be eighteen points of intersection in your diagram.

   If you started with m points on l_1 and n points on l_2 and connected each point on l_1 to each point on l_2 with a segment (being sure no three segments intersect at one point), how many points of intersection will there be?

10. How many non-negative integer solutions (x, y, z) are there to the equation $x + y + z = n$, where n is a positive integer?

11. Prove: Any positive integer n is a deltorial if and only if $8n + 1$ is a perfect square.

12. Find the value of $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\Delta_k}$. Algebraically justify your answer. (Hint: Express $\dfrac{1}{\Delta_k}$ as the sum of two fractions.)

13. Consider the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, ..., where the integer k occurs exactly k times.

    a. Find the 2010^{th} number in the pattern.
    b. Find the sum of the first 2010 numbers in the pattern.
    c. Find a formula for the n^{th} number in the pattern.

14. $1+9+9^2+9^3+\ldots+9^n=\Delta_m$. Express m as a function of n in closed form.

15. Prove that there are infinitely many deltorials that are perfect squares. (Hint: Show that if Δ_n is a perfect square then so is $\Delta_{4n(n+1)}$.)

Not Part of the Contest

Another curiosity!

The continued fraction for $\sqrt{2}$ is $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}$.

Its partial fractions are $1, \frac{3}{2}, \frac{7}{5}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{m}{n}$. For every fraction $\frac{m}{n}$ in this sequence, $(mn)^2$ is a deltorial that is a perfect square and these are the only square deltorials!
From the fractions above, we get:

$$\begin{aligned}
1^2 &= 1 = \Delta_1\\
(3\cdot 2)^2 &= 36 = \Delta_8\\
(7\cdot 5)^2 &= 1225 = \Delta_{49}\\
(17\cdot 12)^2 &= 41616 = \Delta_{288}, \text{ etc.}
\end{aligned}$$

The Solutions

1. $9(\Delta_n) + 1 = \Delta_{3n+1}$

$$\begin{aligned} 9\left(\frac{n(n+1)}{2}\right) + 1 &= \frac{9n^2 + 9n + 2}{2} \\ &= \frac{(3n+1)(3n+2)}{2} \\ &= \Delta_{3n+1} \end{aligned}$$

2. $\Delta_n + \Delta n + 1 = (n+1)^2$

$$\begin{aligned} \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} &= \frac{(n+1)(n+n+2)}{2} \\ &= (n+1)\frac{(2n+2)}{2} \\ &= (n+1)^2 \end{aligned}$$

3. $\Delta_n(\Delta_{n+2}) + \Delta_{n+1} = (\Delta_{n+1})^2$

$$\begin{aligned} \frac{n(n+1}{2} \cdot \frac{(n+2)(n+3)}{2} + \frac{(n+1)(n+2)}{2} &= \frac{(n+1)(n+2)}{4} \cdot (n(n+3)+2) \\ &= \frac{(n+1)(n+2)}{4} \cdot ((n+1)(n+2)) \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\ &= (\Delta_{n+1})^2 \end{aligned}$$

4. $(\Delta_n)^2 + (\Delta_{n-1})^2 = \Delta_{n^2}$

$$\begin{aligned} \left(\frac{n(n+1)}{2}\right)^2 + \left(\frac{n(n-1)}{2}\right)^2 &= \frac{n^2}{4}\left((n+1)^2 + (n-1)^2\right) \\ &= \frac{n^2}{4}\left(n^2 + 2n + 1 + n^2 - 2n + 1\right) \\ &= \frac{n^2}{4}\left(2n^2 + 2\right) \\ &= \frac{n^2(n^2+1)}{2} \\ &= \Delta_{n^2} \end{aligned}$$

5. $\Delta_{n^2+n-1} + \Delta_{n^2+3n+1} = (n+1)^4$

$$\begin{aligned}
&\frac{(n^2+n-1)(n^2+n)}{2} + \frac{(n^2+3n+1)(n^2+3n+2)}{2} \\
&= \frac{n^4+n^3+n^3+n^2-n^2-n+n^4+6n^3+12n^2+9n+2}{2} \\
&= \frac{2n^4+8n^3+12n^2+8n+2}{2} \\
&= n^4+4n^3+6n^2+4n+1 \\
&= (n+1)^4
\end{aligned}$$

6. There are two types of dominos: either the two numbers on the domino are the same or they are different. So if a domino had numbers from 0 to n, there would be $\binom{n+1}{1}$ of the first type and $\binom{n+1}{2}$ of the second type. By Pascal's identity, $\binom{n+1}{1} + \binom{n+1}{2} = \binom{n+2}{2} = \Delta_{n+1}$.

7. In each triangle of side length n (where n is even), there are $\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \Delta_n$ *upright triangles* ($\triangle$) of side lengths $n, n-1, n-2, \ldots, 1$ and $\Delta_1 + \Delta_2 + \Delta_3 + \ldots + \Delta_n = \binom{n+2}{3} = \frac{(n+2)(n+1)(n)}{3!} = \frac{n^3+3n^2+2n}{6}$. Also there are $\Delta_1+\Delta_2+\Delta_3+\ldots+\Delta_n$ *upside down triangles* ($\triangledown$) of lengths $\frac{n}{2}, \frac{n-2}{2}, \frac{n-4}{2}, \ldots, 1$. Using finite differences, $\sum_{k=1}^{n/2} \Delta_{2k-1} = \frac{2n^3+3n^2-2n}{24} = \frac{n(n+2)(2n-1)}{24}$. Therefore, the total number of triangles is the sum of these two functions, or $\frac{n(n+2)(2n+1)}{8} = \frac{2n^3+5n^2+2n}{8}$.

8. Consider a 1×8 rectangle make up of 8 squares. In it there is one 1×8 rectangle, two 1×7 rectangles, three 1×6 rectangles, ..., and eight 1×1 rectangles, for a total of Δ_8 rectangles. Generalizing, there would be Δ_n rectangles in a $1 \times n$ rectangle. Think of the horizontal segments making up these rectangles. These segments represent all the possible horizontal sides of all rectangles in the $n \times n$ array of squares. Rotating these segments $90°$ would represent all the vertical sides of all the possible rectangles in the $n \times n$ array. Therefore, there are $(\Delta_n)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4}$ rectangles in a $n \times n$ array.

9. Pick any two points on l_1 and connect them to any two points on l_2 forming four segments. Two of these segments intersect, forming a point of intersection. Therefore, the number of points of intersection when m points of l_1 are connected to n points of l_2 would be equal to the number of ways of selecting two points on l_1 times the number of ways of selecting two points on l_2. This would be $\binom{m}{2} \cdot \binom{n}{2}$. But this is equal to $\Delta_{m-1} \cdot \Delta_{n-1}$.

10. $1+1+1+1+1+1+1+1 = 8$. Let $11 \circ 11111 \circ 1$ represent $2+5+1$ and $111 \circ \circ 11111$ represent $3+0+5$. Therefore, the number of non-negative solutions to the equation, $x+y+z=8$ is

the number of ways of arranging eight 1's and two $\circ$'s or $\dfrac{(8+2)!}{8!2!}$. Generalizing, the number of non-negative solutions to $x+y+z=n$ is $\dfrac{(n+2)!}{n!2!}$. But this is equal to $\dbinom{n+1}{2}$, which is Δ_{n+1}.

11. $\Rightarrow$ (A positive integer n is a deltorial.) Then there exists a $k \in \mathbb{Z}^+$, such that $n = \dfrac{k(k+1)}{2}$. Thus

$$\begin{aligned} 8n+1 &= 4k(k+1)+1 \\ &= 4k^2 \cdot 4k+1 \\ &= (2k+1)^2, \text{ a perfect square.} \end{aligned}$$

$\Leftarrow$ ($8n+1$ is a perfect square.) Then there exists a $m \in \mathbb{Z}^+$, such that $8n+1 = m^2$ and $n = \dfrac{(m-1)(m+1)}{8}$. m^2 is odd and therefore, so is m. Let $m = 2j+1$, where j is a non-negative integer. Then $n = \dfrac{(2j)(2j+2)}{8} = \dfrac{j(j+1)}{2}$. So n is a deltorial.

12. We have $\dfrac{1}{\Delta_k} = \dfrac{2}{k(k+1)} = \dfrac{2}{k} - \dfrac{2}{k+1}$. Thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\Delta_k} = \sum_{k=1}^{\infty} \left(\frac{2}{k} - \frac{2}{k+1} \right) = \sum_{k=1}^{\infty} \frac{2}{k} - \sum_{k=1}^{\infty} \frac{2}{k+1}$
$\displaystyle = \sum_{k=1}^{\infty} \frac{2}{k} - \sum_{k=2}^{\infty} \frac{2}{k} = 2.$

13. a. Notice that $a_{\Delta_n} = n$ and determine that $\Delta_{62} = 1954$ and $\Delta 63 = 2016$. Therefore, $a_{1954} = 62$ and $a_{2016} = 63$, and $a_{2010} = 63$.

b. From above $\Delta_{63} = 2016$ and notice that $1+2+2+3+3+3+4+4+4+4+\ldots = 1+2(2)+3(3)+4(4)+\ldots = 1^2+2^2+3^2+4^2+\ldots$. Recall that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and so $\displaystyle\sum_{k=1}^{63} k^2 = 85344$. Because 2016 is 6 more than 2010, therefore, 85344 has six too many 63's, so the answer is $85344 - 6(63) = 84966$.

c. Let $a_n = y = f(x)$. Notice that the last number in every integer run of the y-values of this function has an x-value that is Δ_y. For these numbers, this function is the inverse of the function for finding deltorials and so $x = \dfrac{y(y+1)}{2}$. Therefore, $2x = y^2+y$ and $y^2+y-2x = 0$. Using the quadratic formula produces $y = \dfrac{-1 \pm \sqrt{1+8x}}{2}$. Since this is the last number in each integer run, adding the ceiling function accomplishes the goal and so $a_n = \left\lceil \dfrac{-1 \pm \sqrt{1+8n}}{2} \right\rceil$.

14. $1+9+9^2+9^3+\ldots+9^n$ is a geometric series and its sum is $\dfrac{9^{n+1}-1}{9-1} = \dfrac{(3^2)^{n+1}-1}{8} = \dfrac{(3^{n+1})^2-1}{8} = \dfrac{(3^{n+1}-1)(3^{n+1}+1)}{8}$. The two numbers in the numerator differ by 2, and therefore, if they are both divided by 2, they will still be integers whose difference is now 1. $\dfrac{(3^{n+1})^2-1}{8} = \dfrac{\frac{3^{n+1}-1}{2} \cdot \frac{3^{n+1}+1}{2}}{2}$. And this is $\Delta_{\frac{3^{n+1}-1}{2}}$.

15. If Δ_n is a square number, then there exists an $m \in \mathbb{Z}^+$, such that $\dfrac{n(n+1)}{2} = m^2$. Then:

$$\begin{aligned}
\Delta_{4n(n+1)} &= \frac{4n(n+1)(4n(n+1)+1)}{2} \\
&= \frac{n(n+1)}{2} \cdot 16n^2 + 16n + 4 \\
&= \frac{n(n+1)}{2} \cdot (4n+2)^2 \\
&= (m(4n+2))^2, \text{ a perfect square.}
\end{aligned}$$

Very curious conjecture: For any positive integer n,

$$\sum_{k=n^2+2n-1}^{n^2+2n} \Delta_k = \sum_{k=n^2+2n+1}^{n^2+3n} \Delta_k.$$

Proof:

Left side:

$$\begin{aligned}
\sum_{k=n^2+2n-1}^{n^2+2n} \Delta_k &= \sum_{k=1}^{n^2+2n} \Delta_k - \sum_{k=1}^{n^2+2n-2} \Delta_k \\
&= \binom{n^2+2n+2}{3} - \binom{n^2+n}{3} \\
&= \frac{(n^2+2n+2)(n^2+2n+1)(n^2+2n)}{6} - \frac{(n^2+n)(n^2+n-1)(n^2+n-2)}{6} \\
&= \frac{3n^5+15n^4+25n^3+15n^2+2n}{6}
\end{aligned}$$

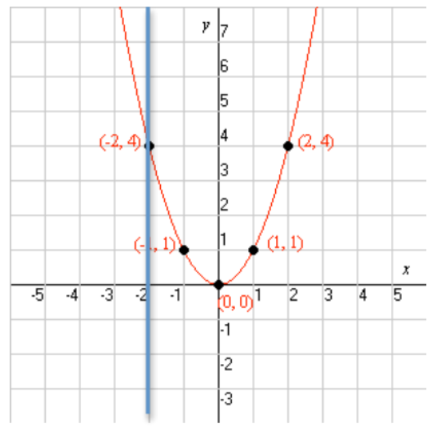
Right side:

$$\begin{aligned}
\sum_{k=n^2+2n+1}^{n^2+3n} \Delta_k &= \sum_{k=1}^{n^2+3n} \Delta_k - \sum_{k=1}^{n^2+2n} \Delta_k \\
&= \binom{n^2+3n+2}{3} - \binom{n^2+2n}{3} \\
&= \frac{(n^2+3n+2)(n^2+3n+1)(n^2+3n)}{6} - \frac{(n^2+2n+2)(n^2+2n+1)(n^2+2n)}{6} \\
&= \frac{3n^5+15n^4+13n^3-15n^2+2n}{6}
\end{aligned}$$

A Geometry with Straight and Curved Lines

The Background

I always find it redundant and a bit amusing when my students use expressions like "a linear line" or "a straight line" when describing a graph. But in today's problem set we will be exploring a geometry in which two types of lines are defined, one straight and one curved!



Definition 1. A *point* is any ordered pair (x, y) in $\mathbb{R}^2$.

Definition 2. A *line* is the set of points which satisfies either of the following conditions:

a) $x = a$ for some $a \in \mathbb{R}$, or
b) $y = x^2 + bx + c$ for some $b, c \in \mathbb{R}$.

For simplicity, the first type of line will be called *straight* and the second will be called *curved*.

Theorem 1. Given any two points, there is one and only one line that contains both of them, i.e., any two points determine a line.

Theorem 2. If two lines intersect they have only one point in common.

Definition 3. Two lines are *parallel* if and only if they have no points in common.

Theorem 3. If two lines are parallel, they both must be straight lines or both must be curved lines, i.e., every straight line intersects every curved line.

Theorem 4. Given a line and a point not on the line, there is one and only one line parallel to the given line that contains the given point.

Definition 4. Given the curved line, $y = x^2 + bc + c$, the *slope* of this line at $x = m$ is $2m + b$.

Definition 5. The straight line, $x = a$, and the curved line, $y = x^2 + bx + c$, are *perpendicular* if and only if $b = -2a$. Two curved lines, l_1 and l_2, intersecting at (m, n), are *perpendicular* if and only if $s_1 \cdot s_2 = -1$, where s_i is the slope of l_i at (m, n). Two straight lines are never *perpendicular*.

Theorem 5. Given a line and a point on the line, there is only one line perpendicular to the given line containing the given point.

Theorem 6. Given a line and a point not on the line, there are one, two, or three lines perpendicular to the given line containing the given point!

<u>The Problems</u>

(All curved lines should be written in the form $y = x^2 + bx + c$.)

Reminder: You cannot use results from Problem n to solve Problem m, if $n > m$.

1. (Theorem 1) Determine the equation of the line containing each pair of points:

   a) $(-5, 3)$ and $(-5, -2)$.

   b) $(-3, 4)$ and $(5, -2)$.

   c) (m, n) and (p, q).

2. (Theorem 2) For each pair of lines, determine the point that is contained in both of them:

   a) $x = 2$ and $y = x^2 - 5x + 6$.

   b) $y = x^2 + 3x - 5$ and $y = x^2 - 5x - 1$.

   c) $x = a$ and $y = x^2 + bx + c$.

3. Show that the point contained in both lines $y = x^2 + b_1x + c_1$ and $y = x^2b_2x + c_2$ is

$$\left(-\frac{c_1 - c_2}{b_1 - b_2}, \frac{(c_1 - c_2)^2 - b_1b_2(c_1 + c_2) + c_1(b_2)^2 + c_2(b_1)^2}{(b_1 - b_2)^2}\right).$$

   Notice the beautiful symmetry!

4. (Theorem 4) Determine the equation of the line containing the given point and parallel to the given line. (In each problem the given point is <u>not</u> on the given line.)

   a) $(-3, 5)$ and $x = 2$

   b) $(-3, 5)$ and $y = x^2 - 2x + 3$

5. Given line l_1 and point (m, n) not on line l_1, determine the equation of the line containing (m, n) and parallel to l_1.

6. (Theorem 5) Determine the equation of the line containing the given point and perpendicular to the given line. (In each problem the given point is on the given line.)

   a) $(3, 2)$ and $x = 3$

   b) $(-1, -9)$ and $y = x^2 + 2x - 8$

   c) $(1, -5)$ and $y = x^2 + 2x - 8$

7. Show that if $b \neq -2m$, the lines

$$y = x^2 + bx + c \quad \text{and} \quad y = x^2 - \left(\frac{4m^2 + 2mb + 1}{2m + b}\right)x + \left(\frac{b(m^2 + n) + m(2m^2 + 2n + 1)}{2m + b}\right)$$

   are perpendicular at (m, n).

8. (Theorem 6) Determine the equation of the line containing the given point and perpendicular to the given line. (In each problem the given point is <u>not</u> on the given line.)

   a) $(5, 2)$ and $x = 3$

   b) (m, n) and $x = a$

   c) $(3, -3)$ and $y = x^2 - 4x + 1$

9. Given $y_1 = x^2 - 6x + 8$ and $P = (3, 0)$. Verify that $y_2 = x^2 - 8x + 15$, $y_3 = x^2 - 4x + 3$, and $x = 3$ each contain P and are perpendicular to y_1.

10. a) Find the equations of the three lines that are perpendicular to $y = x^2$ and contain $(3, 4)$.

    b) Find the equations of the three lines that are perpendicular to $y = x^2 - 4x - 3$ and contain $(5, -3)$.

11. a) $ABCD$ is a parallelogram (quadrilateral with two pairs of parallel sides) in this geometry. If $A = (1, 3)$, $B = (3, 5)$, and $C = (1, 7)$, determine the possible coordinates of vertex D.

    b) $ABCD$ is a trapezoid (quadrilateral with only one pair of parallel sides) in this geometry with two right angles. If $A = (5, 1)$, $B = (2, 10)$, and $C = (2, 12)$, determine the possible coordinates of vertex D.

<u>Extensions (not part of the contest)</u>

When triangles and quadrilaterals are defined with straight and curved sides, then triangles can have two right angles and rectangles have only two right angles! In addition, if a distance metric is defined for curved segments (using arc lengths as defined in calculus), there can be equilateral trapezoids and the SAS Triangle Congruence Theorem is no longer a theorem! If $A = (1, 1)$, $B = \left(\frac{1}{4}, \frac{31}{16}\right)$, $C = \left(-\frac{1}{4}, \frac{31}{16}\right)$, and $D = (-1, 1)$, then quadrilateral $ABCD$ is an isosceles trapezoid with four right angles! Check it out!

The Solutions

1. a) $x = -5$

   b) $\left\{\begin{array}{r} (-3,4) \Rightarrow 9-3b+c=4 \\ (5,-2) \Rightarrow 25+5b+c=-2 \end{array}\right\} \Rightarrow \left\{\begin{array}{r} -3b+c=5 \\ 5b+c=-27 \end{array}\right\} \Rightarrow b=-\frac{11}{4}$ and $c=-\frac{53}{4}$.

   Therefore, $y = x^2 - \frac{11}{4}x - \frac{53}{4}$.

   c) If $m = p$, then $x = m$ or $x = p$. If $m \neq p$, then

$$\begin{aligned} \left\{\begin{array}{r} n = m^2+mb+c \\ q = p^2+pb+c \end{array}\right\} \quad &\Rightarrow \quad \left\{\begin{array}{r} mb+c = n-m^2 \\ pb+c = q-p^2 \end{array}\right\} \\ &\Rightarrow \quad b = \frac{q-n+m^2-p^2}{p-m} \quad \text{and} \\ & \quad c = \frac{p^2m-m^2p+pn-qm}{p-m}. \end{aligned}$$

   Therefore, $y = x^2 + \dfrac{q-n+m^2-p^2}{p-m}x + \dfrac{p^2m-m^2p+pn-qm}{p-m}$.

2. a) If $x = 2$, $y = 4 - 10 + 6 = 0$. So the point of intersection is $(2, 0)$.

   b) $x^2 + 3x - 5 = x^2 - 5x - 1 \Rightarrow 8x = 4 \Rightarrow x = \frac{1}{2}$ and $y = -\frac{13}{4}$. So the point is $\left(\frac{1}{2}, -\frac{13}{4}\right)$.

   c) If $x = a$, $y = a^2 + ba + c$. So the point of intersection is $(a, a^2 + ba + c)$.

3.

$$\begin{aligned} x^2+b_1x+c_1 &= x^2+b_2x+c_2 \\ (b_1-b_2)x &= c_2-c_1 \\ x &= \frac{c_2-c_1}{b_1-b_2} = -\frac{c_1-c_2}{b_1-b_2} \end{aligned}$$

$$\begin{aligned} y &= \left(-\frac{c_1-c_2}{b_1-b_2}\right)^2 + b_1\left(-\frac{c_1-c_2}{b_1-b_2}\right) + c_1 \\ &= \frac{(c_1-c_2)^2 - b_1(c_1-c_2)(b_1-b_2) + c_1(b_1-b_2)^2}{(b_1-b_2)^2} \\ &= \frac{(c_1-c_2)^2 - {b_1}^2c_1 + b_1b_2c_1 + {b_1}^2c_2 - b_1b_2c_1 + c_1{b_1}^2 - 2c_1b_1b_2 + c_1{b_2}^2}{(b_1-b_2)^2} \\ &= \frac{(c_1-c_2)^2 + b_1b_2(c_1+c_2) + c_1{b_2}^2 + c_2{b_1}^2}{(b_1-b_2)^2} \end{aligned}$$

4. a) $x = -3$.

   b) We need $x^2 - 2x + c = y$ and $(x, y) = (-3, 5)$. $9 + 6 + c = 5 \Rightarrow c = -10$. Therefore, $y = x^2 - 2x - 10$.

5. <u>Case 1:</u> Suppose l_1 is a straight line with the equation $x = a$. Since the point (m, n) is not on l_1, we know $m \neq a$. Hence, the equation of the line parallel to l_1 containing (m, n) is $x = m$.

   <u>Case 2:</u> Suppose l_1 is a curved line with the equation $y = x^2+bx+c$. We need $y = x^2+bx+c_1$ with $(x, y) = (m, n)$. So $n = m^2 + bm + c_1$ and $c_1 = n - m^2 - bm$. Therefore, the line will be $y = x^2 + bx + (n - m^2 - bm)$.

6. a) Since $x = 3$ is a straight line, a line perpendicular to it must be of the form $y = x^2-6x+c$, and $2 = 9 - 18 + c$, implying $c = 11$. Therefore, the desired line is $y = x^2 - 6x + 11$.

   b) Since $2 = -2(-1)$, the desired line is a straight line and hence $x = -1$ is the line.

   c) Since $2 \neq -2(1)$, the desired line is a curved line of the form, $y = x^2 + bx + c$, and $-5 = 1+b+c$, implying $c = -6-b$. Since the lines are perpendicular at $x = 1$, $(2\cdot1+1)(2\cdot1+b) = -1$. Solving this system yields: $y = x^2 - \frac{9}{4}x - \frac{15}{4}$.

7. Show (m, n) is on both lines. If $x = m$, $y_1 = m^2 + bm + n - m^2 - bm = n$.

$$\begin{aligned} y_2 &= m^2 - \left(\frac{4m^2+2mb+1}{2m+b}\right) m + \left(\frac{b(m^2+n)+m(2m^2+2n+1)}{2m+b}\right) \\ &= \frac{2m^3+mb}{2m+b} + \frac{-4m^3-2m^2b-m}{2m+b} + \frac{bm^2+bn+2m^3+2mn+m}{2m+b} \\ &= \frac{2m^3+mb-4m^3-2m^2b-m+bm^2+bn+2m^3+2mn+m}{2m+b} \\ &= \frac{2mn+bn}{2m+b} = n \end{aligned}$$

   Show the product of the slopes of the lines at (m, n) equals -1.

$$\begin{aligned} s_1 \cdot s_2 &= (2m+b)\left(2m - \frac{4m^2+2mb+1}{2m+b}\right) \\ &= 4m^2 + 2mb - (4m^2+2mb+1) \\ &= -1 \end{aligned}$$

8. a) The line must be of the form, $y = x^2 + bx + c$. Because it contains $(5, 2)$, $2 = 25 + 5b + c$ and because it is perpendicular to $x = 3$, $b = -2(3) = -6$. Solving this system yields: $y = x^2 - 6x + 7$.

   b) Again the line must be of the form, $y = x^2 + bx + c$. Because it contains (m, n), $n = m^2 - 2am + c$ and because it is perpendicular to $x = a$, $b = -2(a)$. Solving this system yields: $y = x^2 - 2ax + n - m^2 + 2am$.

c) Because $-4 \neq -2(3)$, the line must be of the form $y_2 = x^2 + bx + c$. Because y_2 contains $(3, -3)$, $-3 = 9 + 3b + c$ and therefore, $y_2 = x^2 + bx - 12 - 3b$. y_1 and y_1 intersect when $x^2 - 4x + 1 = x^2 + bx - 12 - 3b$, they intersect when $x = \frac{13+3b}{b+4}$. Since y_1 and y_2 are perpendicular at $x = \frac{13+3b}{b+4}$, $\left(2\left(\frac{13+3b}{b+4}\right) + b\right)\left(2\left(\frac{13+3b}{b+4}\right) - 4\right) = -1$. Expanding this equation yields the cubic, $2b^3 + 31b^2 + 160b + 276 = 0$, which has only one real zero, $b = -6$. Since $c = -12 - 3b = -12 - 3(-6) = 6$, $y_2 = x^2 - 6x + 6$.

9. $P = (3, 0)$ is on each line because: $y_2 : 3^2 - 8(3) + 15 = 0$, $y_3 : 3^2 - 4(3) + 3 = 0$, and $x = 3$. $y - 1$ is perpendicular to $x = 3$, because $-6 = -2(3)$. y_1 intersects y_2 at $\left(\frac{7}{2}, -\frac{3}{4}\right)$ and y_1 is perpendicular to y_2 because at $x = \frac{7}{2}$, $s_1 \cdot s_2 = (7-6)(7-8) = -1$. $-6 = -2(3)$. y_1 intersects y_3 at $\left(\frac{5}{2}, -\frac{3}{4}\right)$ and y_1 is perpendicular to y_3 because at $x = \frac{5}{2}$, $s_1 \cdot s_3 = (5-6)(5-4) = -1$.

10. a) $y_1 = x^2$ and $(3, 4)$ is not on y_1 and $0 \neq -2(3)$. Therefore, the line perpendicular to y_1 must be curved lines, $y_2 = x^2 + bx + c$. Because $(3, 4)$ is on these lines, $4 = 9 + 3b + c$, implying $c = -5 - 3b$. y_1 and y_2 intersect when $x^2 = x^2 - bx - 5 - 3b$ or when $x = \frac{5+3b}{b}$. The slope of y_1 at $x = \frac{5+3b}{b}$ is $2 \cdot \left(\frac{5+3b}{b}\right) = \frac{10+6b}{b}$ and the slope of y_2 at $x = \frac{5+3b}{b}$ is $2 \cdot \left(\frac{5+3b}{b}\right) + b = \frac{10+6b+b^2}{b}$. y_1 is perpendicular to y_2 when $\left(\frac{10+6b}{b}\right) \cdot \left(\frac{10+6b+b^2}{b}\right) = -1$. This simplifies to $6b^3 + 47b^2 + 120b + 100 = 0$. This cubic has three zeros: $x = -2$, $x = -\frac{5}{2}$, and $x = -\frac{10}{3}$. These three values, along with the fact $c = -5 - 3b$, produce the following three lines: $y_2 = x^2 - 2x + 1$, $y_2 = x^2 - \frac{5}{2}x + \frac{5}{2}$, and $y_2 = x^2 - \frac{10}{3} + 5$.

b) This problem can be solved just like 10a or one can notice that the line and point in 10b is the line and point in 10a translated 2 units to the right and 7 down. Therefore, the solutions from 10a just need to also be translated 2 units to the right and 7 down:

$$\begin{aligned} y_2 &= x^2 - 2x + 1 \text{ becomes} \\ &= \left[(x-2)^2 - 2(x-2) + 1\right] - 7 \\ &= x^2 - 4x + 4 - 2x + 4 + 1 - 7 \\ &= x^2 - 6x + 2, \end{aligned}$$

$$\begin{aligned} y_2 &= x^2 - \tfrac{5}{2}x + \tfrac{5}{2} \text{ becomes} \\ &= \left[(x-2)^2 - \tfrac{5}{2}(x-2) + \tfrac{5}{2}\right] - 7 \\ &= x^2 - 4x + 4 - \tfrac{5}{2}x + 5 + \tfrac{5}{2} - 7 \\ &= x^2 - \tfrac{13}{2}x + \tfrac{9}{2}, \text{ and} \end{aligned}$$

$$\begin{aligned} y_2 &= x^2 = \tfrac{10}{3}x + 5 \text{ becomes} \\ &= \left[(x-2)^2 - \tfrac{10}{3}(x-2) + 5\right] - 7 \\ &= x^2 - 4x + 4 - \tfrac{10}{3}x + \tfrac{20}{3} + 5 - 7 \\ &= x^2 - \tfrac{22}{3}x + \tfrac{26}{3}. \end{aligned}$$

11. a) By definition, in parallelogram $ABCD$, $AB \parallel CD$ and $BC \parallel AD$. Solving systems of equations, line AB is $y = x^2 - 3x + 5$ and line BC is $y = x^2 - 5b + 11$. Since lines CD and AB are parallel, line CD must be $y = x^2 - 3x + 9$. Since lines AD and BC are parallel, line AD must be $y = x^2 - 5x + 7$. But lines AD and CD intersect when $x^2 - 3x + 9 = x^2 - 5x + 7$ or when point D is $(-1, 13)$.

    b) By definition, in trapezoid $ABCD$, $AB \parallel CD$ or $BC \parallel AD$. Solving systems of equations, line AB is $y = x^2 - 10x + 26$ and line BC is $x = 2$.

    Case 1: $AB \parallel CD$. Line CD must be of the form $y = x^2 - 10x + c$ and contain point C $(2, 12)$. Therefore, line CD is $y = x^2 - 10x + 28$. Line BC is perpendicular to neither line AB nor line CD. Therefore, line AD must be perpendicular to line AB at point A. But the line perpendicular to line AB at A is the line $x = 5$ and therefore lines AD and BC would be parallel and $ABCD$ would not be a trapezoid.

    Case 2: $BC \parallel AD$. Since line BC is a straight line, line AD must be the straight line, $x = 5$. Therefore, line AD is perpendicular to line AB. Line BC is not perpendicular to line AB and cannot be perpendicular to line CD because then $ABCD$ would be a parallelogram. Therefore, lines BC and CD must be perpendicular, implying line CD is of the form $y = x^2 - 4x + c$ and plugging in point C, gives $y = x^2 - 4x + 16$. Since lines AD and CD intersect at D, point D must be $(5, 21)$.

Number Puzzles

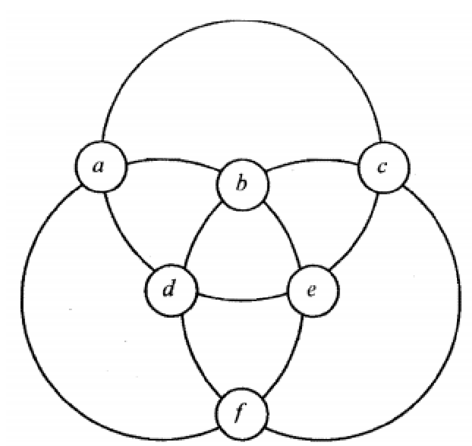
The Background

Number puzzles have been mathematical curiosities for centuries. Solving them often requires skills in arithmetic, algebra, number theory, and logic. Enjoy!

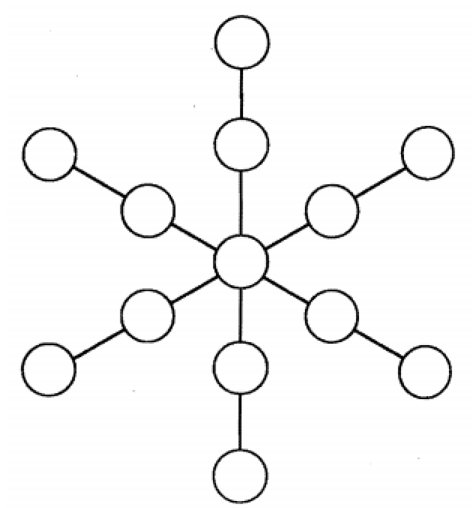
Throughout this problem set, a solution that is obtained by merely rotating or reflecting a known solution is NOT considered another unique solution.

The Problems

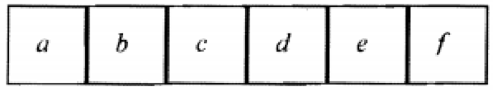
1. Place the integers from 1 through 6 in the small circles so that the sums of the four numbers connected by each of the large circles are equal. How many unique solutions to this puzzle are there? Justify your answer.



2. Place the integers from 1 through 13 in each of the circles so that the sums of the five numbers connected by each of the three lines are equal.
   a. What numbers can go in the center circle? Justify.
   b. If only the odd numbers from 1 through 25 were used, what numbers could go in the center circle? Justify.



3. Replace a, b, c, d, e, and f with numbers such that $a < b < c < d < e < f$ and the fifteen sums of *all* pairs of these numbers are in the set $\{6, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. (Note: The values 9, 10, 11, 12, and 14 are the sums of two different pairs!)



4. a. Place the integers from 1 through 8 in the circles in Figure 4a so that the sums, S, of the three numbers along each edge are equal. How many unique values for S are there? For each value of S, find a solution to the puzzle.

   b. Place the integers from 1 through 12 in the circles in Figure 4b so that the sums, S, of the four numbers along each edge **and** the sum of the numbers in the four corners is also equal to S. How many unique values of S are there? Find one of the many solutions to this puzzle.

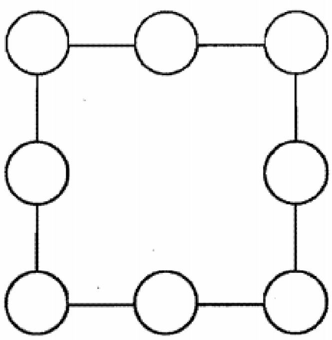


Figure 4a

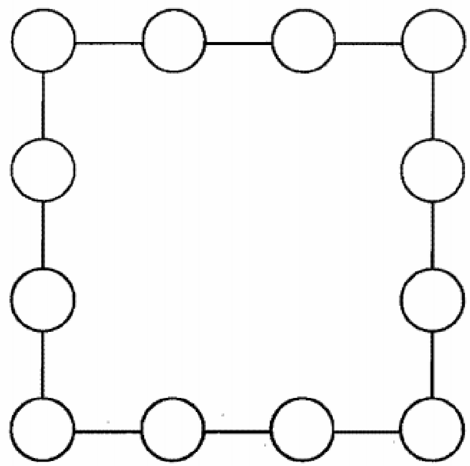
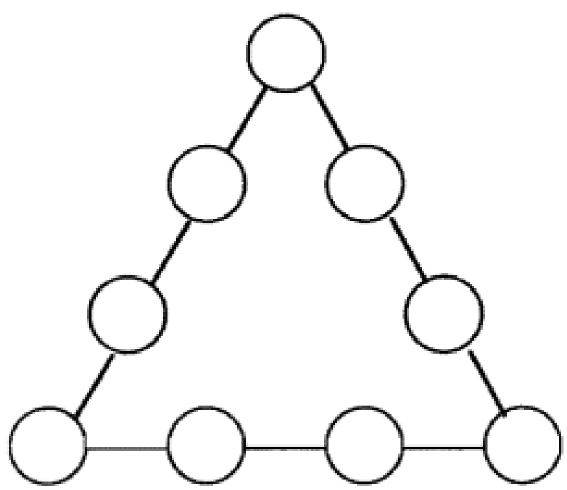


Figure 4b

5. The integers from 1 through 9 are placed in the circles in the figure to the right so that the sums of the four numbers along each edge are equal to S.
   a. Find a solution for $S = 21$.
   b. Prove there is no solution for $S = 22$.
   c. For what values of S does the puzzle to the right have a solution? Justify.



6. a. Place the integers from 1 through 9 in the squares of the diagram to the right so that the 3-digit number formed in the second row is twice the 3-digit number formed in the first row and the 3-digit number formed in the third row is three times the 3-digit number formed in the first row.
   b. How many unique solutions are there to this puzzle? Justify.

a	b	c
d	e	f
g	h	i

7. Magic Multiplicative Square: The numbers a, b, c, d, e, f, g, h, and i are unique (but not necessarily consecutive) integers with the products, $abc = def = ghi = adg = beh = cfi = aei = ceg = P$.

   a. Show that P must be a perfect cube.
   b. Find a solution to the puzzle with $P = 216$.
   c. For $P = 216$, how many unique solutions to the puzzle are there? Justify.
   d. Find a solution to the puzzle if $a, b, c, \ldots, i$ are nine consecutive powers of 2.

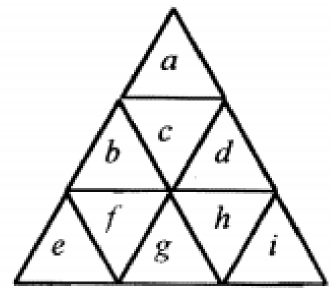
a	b	c
d	e	f
g	h	i

8. An Anti-Magic Square: An anti-magic square is an n by n array of integers from 1 to n^2 such that the n numbers in each row, column, and diagonal sum to a *different* sum and the sums form a sequence of $2n + 2$ *consecutive* integers. The following anti-magic square has been started for you.

   a. Determine the set of possible sums for this puzzle.
   b. Find a solution to the puzzle.
   c. How many solutions are there? Justify.

15			4
1	14	10	5
	9	3	
11			7

9. Replace $a, b, c, d, e, f, g, h,$ and i with digits 1 through 9 and place a minus sign in front of some of them so that the products $acbfe = 320$, $dhg = 162$, $acdhi = 6048$, $bfg = -60$, $bcd = -240$, and $efghi = 756$.
   a. Find a solution to this puzzle.
   b. How many unique solutions are there? Justify.



The Solutions

1. $a+b+c+d+e+f=21$. $a+b+e+f=b+c+f+d=a+d+e+c \Rightarrow a+e=c+d=b+f=7$. Because of rotation and reflection properties of the diagram, we can let $a,e \in \{1,6\}$, $b,f \in \{2,5\}$, and $c,d \in \{3,4\}$. Each of these pairs can be interchanged, i.e., $a=1, e=6$ or $a=6, e=1$, so there are $2 \cdot 2 \cdot 2 = 8$ unique solutions to this puzzle.

2. a. $1+2+3+\ldots+13=91$. Therefore, if x is the value in the middle circle, $91+2x=3S$ and so $91-x \equiv 0 \pmod 3$ and $x \equiv 1 \pmod 3$, giving $x = 1, 4, 7, 10$, or 13.

   b. Using the same reasoning as in part a, $1+3+5+\ldots+25=169$ and so $169-x \equiv 0 \pmod 3$. But x is odd and $x \equiv 1 \pmod 3$, so $x \equiv 1 \pmod 6$, giving $x = 1, 7, 13, 19$, and 25.

3. $(a+b)+(a+c)+(a+d)+(a+e)+(a+f)+(b+c)+(b+d)+(b+e)+(b+f)+(c+d)+(c+e)+(c+f)+(d+e)+(d+f)+(e+f) = 6+8+9+9+10+10+11+11+12+12+13+14+14+15+16$. Therefore, $5(a+b+c+d+e+f)=170$, implying $a+b+c+d+e+f=34$ and $6+c+d+16=34$ implies $c+d=12$, giving $d=6.5$. $d+f=15$, the second largest sum. So $f=8.5$ and this implies $e=7.5$.

4. a. Let $S =$ edge sum. $a+b+c+d+e+f+g+h=36$, the sum of the first eight counting numbers. $(a+b+c)+(c+d+e)+(e+f+g)+(g+h+a)=4S$, implying $c+e+g+a+36=4S$. Therefore, $S = 9+\dfrac{c+e+g+a}{4}$, implying the sum of the four corners must be a multiple of 4. Therefore, the only possible values for $c+e+g+a$ are 12, 16, 20, 24, or 28.

   <u>Case 1:</u> Corners sum to 12 and $S=12$. The only possible candidates for the corners are $(1,2,3,6)$ or $(1,2,4,5)$. The only unique solution for this case is:

1	8	3
5		7
6	4	2

   <u>Case 2:</u> Corners sum to 16 and $S=13$. By systematic counting, the only candidates for the corners are: $(1,2,5,8)$, $(1,2,6,7)$, $(1,3,4,8)$, $(1,3,5,7)$, or $(1,4,5,6)$. There are two unique solutions for this case:

1	7	5		1	8	4
4		6		7		3
8	3	2		5	2	6

   <u>Case 3:</u> Corners sum to 20 and $S=14$. Now there are seven possible sets of corner numbers: $(8,7,4,1)$, $(8,7,3,2)$, $(8,6,5,1)$, $(8,6,4,2)$, $(8,5,4,3)$, $(7,6,5,2)$, or $(7,6,4,3)$. There are two unique solutions for this case:

1	6	7		3	7	4
5		3		6		2
8	2	4		5	1	8

   <u>Case 4:</u> Corners sum to 24 and $S=15$. Only two possible candidates for the corners: $(8,7,6,3)$ or $(8,7,5,4)$. The only unique solution for this case is:

8	4	3
1		5
6	2	7

<u>Case 5:</u> Corners sum to 28 and $S = 16$. The only candidate for the corners is $(8, 7, 6, 5)$. Of the three unique arrangements of these corner numbers, one can quickly see there are no solutions for this case.

b. $(a+b+c+d)+(d+e+f+g)+(g+h+i+j)+(j+k+l+a) = 4S$ and $1+2+3+\ldots+12 = a+b+c+d+e+f+g+h+ui+j+k = 78$. Therefore, $78+2(a+d+g+j) = 5S$, implying $78+2(S) = 5S$ or $S = 26$. Two solutions to this puzzle:

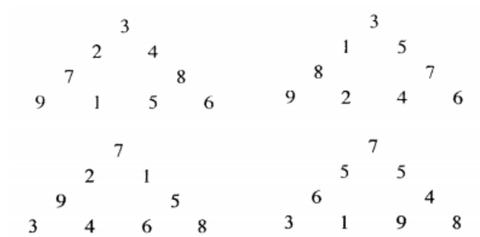
11	3	5	7
9			12
4			1
2	8	10	6

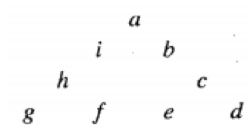
,

10	3	5	8
1			4
9			12
6	11	7	2

.

5. a. There are four different possibilities; each can be represented in eight unique ways because the interior numbers along each side can be interchanged:



b. 

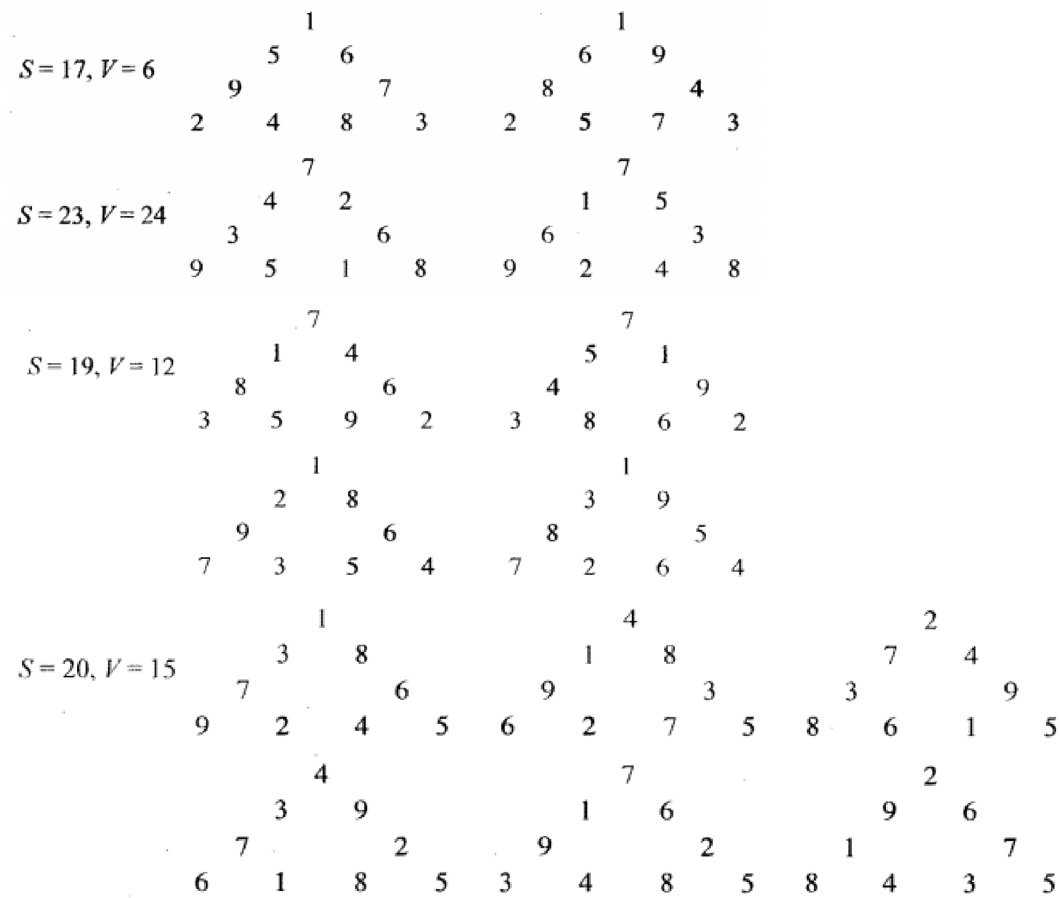
$a+d+g+45 = 3(S)$, implying $a+d+g = 21$. There are only three possibilities to check: $(a, d, g) = (9, 4, 8), (9, 5, 7),$ or $(8, 7, 6)$.

<u>Case 1:</u> $(a, d, g) = (9, 4, 8)$. Then $h+i = 5$, implying $h = 2$ and $i = 3$. Likewise, $b+c$ must equal 9, implying $b+c = 1+8, 2+7, 3+6,$ or $4+5$. But 8, 2, 3, and 4 have already been used; thus eliminating all possibilities for this case.

<u>Case 2:</u> $(a, d, g) = (9, 5, 7)$. Then $b+c = 8$, implying $b = 2$ and $c = 6$. Likewise, $f+e$ must equal 10, implying $e+f = 1+9, 2+8, 3+7,$ or $4+6$. But 9, 2, 7, and 6 have already been used; thus eliminating all possibilities for this case.

<u>Case 3:</u> $(a, d, g) = (8, 7, 6)$. Then $f+e = 9$, implying $f = 4$ and $e = 5$. Likewise, $b+c$ must equal 6, implying $b+c = 1+5$ or $2+4$. But 4 and 5 have already been used; thus eliminating all possibilities for this case.

c. $a+d+g+45 = 3(S)$, implying $a+d+g$ is a multiple of 3. $1+2+3 = 6$ is the smallest possible multiple of 3, so 17 is the smallest possible value for S. $7+8+9 = 24$ is the largest possible multiple of 3, so 23 is the largest possible value for S. From 5a and 5b we know 21 works and 22 does not work, so 17, 18, 19, 20, and 23 must be checked. For every possible value of S, let $V = 3S - 45$, the sum of the three vertices. Then:



$S = 18, V = 9$: There are only three possibilities to check: $(a, d, g) = (1, 2, 6), (1, 3, 5)$, or $(2, 3, 4)$.

<u>Case 1:</u> $(a, d, g) = (1, 2, 6)$. Then $h+i = 15$, implying $h = 7$ and $i = 8$. Likewise, $b + c$ must equal 11, implying $b + c = 2 + 9, 3 + 8, 4 + 7,$ or $5 + 6$. But 2, 8, 7, and 6 have already been used; thus eliminating all possibilities for this case.

<u>Case 2:</u> $(a, d, g) = (1, 3, 5)$. Then $b+c = 14$, implying $b = 8$ and $c = 6$. Likewise, $h + i$ must equal 12, implying $h + i = 3 + 9, 4 + 9,$ or $5 + 7$. But 3, 8, and 5 have already been used; thus eliminating all possibilities for this case.

<u>Case 3:</u> $(a, d, g) = (2, 3, 4)$. Then $b + c = 13$, $f + e = 11$, and $h + i = 12$, leaving no place for 1.

6. a. Consider the table below of possible values for the last column, c, f, i. The value of c cannot be 5 because 0 is not used in this puzzle. The value of c cannot be 1 because a must be 1, 2, or 3.

c	1	2	3	4	5	6	7	8	9
f	2	4	6	8	0	2	4	6	8
i	3	6	9	2	5	8	1	4	7

<u>Case 1:</u> The last column is $(2, 4, 6)$ and so the middle column must also come from the table above but cannot contain 2, 4, or 6. The only possibility for the middle column is $(9, 8, 7)$, leaving 1, 3, and 5 for the first column. Note that $192 \cdot 2 = 384$ and $192 \cdot 3 = 576$, a solution!

<u>Case 2:</u> Because the double and triple of 2 are still single digit numbers, the last column can be moved to the first column, shifting the first and middle columns to the right. Note that $219 \cdot 2 = 438$ and $219 \cdot 3 = 657$, a solution, representing the last column in the table above.

<u>Case 3:</u> Try the next set of numbers from the table, $(4, 6, 9)$. Since the double and triple of 3 are still single digit numbers, the middle column must come the table and not contain 3, 6, or 9. There are two possibilities: $(4, 8, 2)$ or $(7, 4, 1)$, producing the two "solutions" below. Unfortunately, in the first case $143 \cdot 2 = 286$, not 386. However, in the second case, $273 \cdot 2 = 546$ and $273 \cdot 3 = 819$, a correct solution!

$$\begin{array}{ccc} 1 & 4 & 3 \\ 3 & 8 & 6 \\ 5 & 2 & 9 \end{array} \quad \text{and} \quad \begin{array}{ccc} 2 & 7 & 3 \\ 5 & 4 & 6 \\ 8 & 1 & 9 \end{array}$$

<u>Case 4:</u> As in the first solution the columns can be shifted, producing another solution $327 \cdot 2 = 654$ and $327 \cdot 3 = 981$, representing the column $(7, 4, 1)$ in the table.

<u>Case 5:</u> Consider the next column $(4, 8, 2)$ from the table. Cell b can only be a 1 or a 3. If b is a 1, the middle column is $(1, 2, 5)$ but 2 has already been used. If b is a 3, the middle column is $(3, 6, 0)$ but 0 is not allowed.

<u>Case 6:</u> Consider the next column $(6, 2, 8)$. Of the six possible remaining values for b, only two, 1 and 4, produce a valid middle column. But the first column cannot be completed in either case with the remaining digits.

$$\begin{array}{ccc} a & 1 & 6 \\ d & 3 & 2 \\ g & 4 & 8 \end{array} \quad \text{and} \quad \begin{array}{ccc} a & 4 & 6 \\ d & 9 & 2 \\ g & 3 & 8 \end{array}$$

<u>Case 7:</u> Consider the last remaining column, $(8, 6, 4)$. Of the six possible remaining values for b, four of them produce valid middle columns. In the first, a must be 2, but $218 \cdot 2 = 436$ and 4 has already been used. In the second, a must be 2, but $238 \cdot 2 = 476$ and 4 has already been used. In the third, a must be 2, but $258 \cdot 2 = 516$ and 5 has already been used. Finally in the last case, a must be 1, but $178 \cdot 2 = 356$ and 3 has already been used.

$$\begin{array}{ccc} a & 1 & 8 \\ d & 3 & 6 \\ g & 5 & 4 \end{array}, \begin{array}{ccc} a & 3 & 8 \\ d & 7 & 6 \\ g & 1 & 4 \end{array}, \begin{array}{ccc} a & 5 & 8 \\ d & 1 & 6 \\ g & 7 & 4 \end{array}, \text{and} \quad \begin{array}{ccc} a & 7 & 8 \\ d & 5 & 6 \\ g & 3 & 4 \end{array}$$

b. There are four solutions to this puzzle:

1	9	2		2	1	9		2	7	3		3	2	7
3	8	4		4	3	8		5	4	6		6	5	4
5	7	6		6	5	7		8	1	9		9	8	1

7. a. $a \cdot b \cdot c = P$ and $a = \dfrac{P}{ei}$, $b = \dfrac{P}{eh}$, and $c = \dfrac{P}{eg}$. Therefore, $P = \dfrac{P}{ei} \cdot \dfrac{P}{eh} \cdot \dfrac{P}{eg} = \dfrac{P^3}{e^3(ihg)} = \dfrac{P^3}{e^3 \cdot P}$. Therefore, $P = e^3$.

b.

3	4	18
36	6	1
2	9	12

c. From 7a, e must be 6 and therefore must be surrounded by pairs of numbers whose products are 36, namely 1 and 36, 2 and 18, 3 and 12, and 4 and 9. There are only four triples of these numbers whose products are 216, namely $(1, 12, 18)$, $(2, 3, 36)$, $(2, 9, 12)$, and $(3, 4, 18)$. Notice the numbers 2, 3, 12, and 18 each occur twice in these triplets, while the numbers 1, 4, 9, and 36 only occur once. Therefore, the numbers 2, 3, 12, and 18 must occur in the corners of the puzzle and 1, 4, 9, and 36 must occur on the sides. Once the first number is placed in the grid, the positions of the other numbers are predetermined. Therefore, barring rotations and reflections, there is only one solution to this puzzle.

d. The traditional 3 by 3 magic square is below to the left. Since you add exponents when multiplying powers of two, the resulting magic multiplicative square is below to the right.

8	1	6
3	5	7
4	9	2

$$\begin{matrix} 2^8 & 2^1 & 2^6 \\ 2^3 & 2^5 & 2^7 \\ 2^4 & 2^9 & 2^2 \end{matrix} = \begin{matrix} 256 & 2 & 64 \\ 8 & 32 & 128 \\ 16 & 512 & 4 \end{matrix}$$

8. a. Since the sums must be ten consecutive numbers and we know one sum is 30 and another is 39, the sums must be the numbers $30, 31, 32, \ldots, 39$.

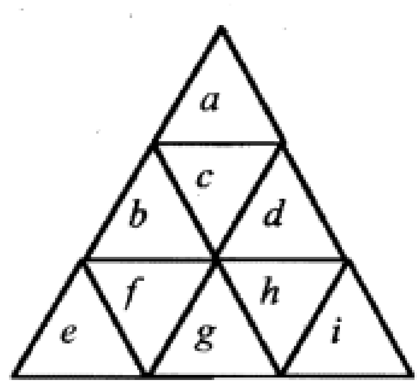
b.

$$\begin{matrix} 15 & 2 & 12 & 4 \\ 1 & 14 & 10 & 5 \\ 8 & 9 & 3 & 16 \\ 11 & 13 & 6 & 7 \end{matrix}, \quad \begin{matrix} 15 & 2 & 12 & 4 \\ 1 & 14 & 10 & 5 \\ 8 & 9 & 3 & 16 \\ 11 & 6 & 13 & 7 \end{matrix}, \text{ and } \begin{matrix} 15 & 6 & 12 & 4 \\ 1 & 14 & 10 & 5 \\ 8 & 9 & 3 & 16 \\ 11 & 2 & 13 & 7 \end{matrix}$$

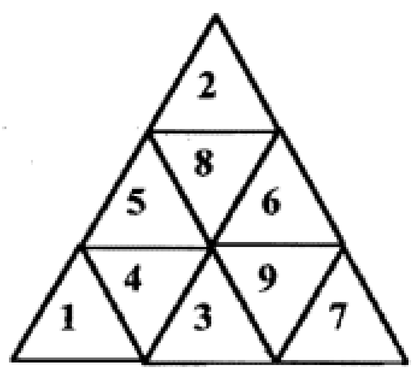
$$\begin{matrix} 15 & a & b & 4 \\ 1 & 14 & 10 & 5 \\ c & 9 & 3 & d \\ 11 & e & f & 7 \end{matrix}$$

c. Using the diagram to the left for reference, we currently have three sums, namely 30, 39, and 34. $S_4 = 4 + 5 + d + 7 = d + 16$, implying $d = 15$ or 16. But 15 is already in the array so $d = 16$ and $S_4 = 32$. $S_5 = 15 + 1 + c + 11$ and $S_6 = c + 9 + 3 + 16$. Therefore, $S_5 = 27 + c$ and $S_6 = 28 + c$, implying S_5 and S_6 are consecutive numbers and therefore must be 35, 36, 37, or 38, thus making c an 8, 9, or 10. But 9 and 10 are already in the array, so $c = 8$, implying $S_5 = 35$ and $S_6 = 36$. This leaves $a, b, e, f \in \{2, 6, 12, 13\}$ and $S_7, S_8, S_9, S_{10} \in \{31, 33, 37, 38\}$. Considering the first row, $a + b + 10 = 31, 33, 37,$ or 38, implying $a + b = 12, 14, 18,$ or 19, implying $a, b \in \{2, 6, 12\}$ and 12 is either a or b. Considering the last row, $e + f + 18 = 31, 33, 37,$ or 39, implying $e + f = 13, 15, 19,$ or 21, implying $e, f \in \{2, 6, 13\}$ and 13 is either e or f. Considering the third column, $b + f + 13 = 31, 33, 37,$ or 38, implying $b + f = 18, 20, 24,$ or 26, implying $b, f \in \{6, 12, 13\}$ and 12 is either b or f. Therefore, $b = 12$ and a is 2 or 6 and f is 6 or 13. If $a = 2$, then $e = 2$ and $b = 13$ or $e = 13$ and $b = 2$. If $a = 6$, then $e = 2$ and $f = 13$. Therefore there are three solutions.

9. a. Ignoring the negatives for the time being and looking at the prime factorizations of the products, the only possible positions for 5 and 7 are b and i, respectively. It follows that h must be 9. Then 3 and 6 can be g or d. But 2^4 is too big for c, so c must be 8 and d is 6 and g is 3. Putting 4, 1, and 2 in f, e, and a, respectively completes the triangle.



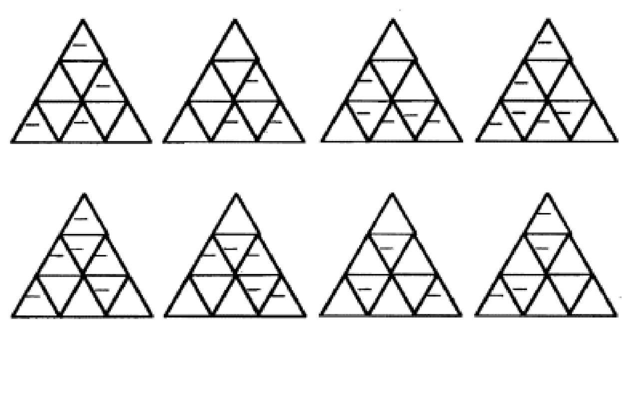
b. 1) Cells $b, c, d, f,$ and g are factors of the two negative products. Therefore, changing the signs in cells a, e, and i will not affect the signs of any of the products.



2) Also, changing the signs of the numbers in cells $b, d, f,$ and h will not affect the signs of any of the products since exactly two of these factors occur in each of the six products.

3) Likewise, changing the signs of the numbers in cells $b, c, g,$ and h will not affect the signs of any of the products since exactly two of these factors occur in each of the six products.

Therefore there are eight solutions to this puzzle because for each of the three changes above, you can either make the change or not make the change and $2 \cdot 2 \cdot 2 = 8$. Starting with the first diagram, $\Rightarrow 1 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 1 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$ cycles through the changing of signs for the eight solutions.



The Power(s) of Fibonacci

The Background

Fibonacci numbers have been a fascinating topic in mathematics since Leonardo of Pisa, the son (filius) of Bonacci, discussed the sequence of numbers in his book *Liber Abaci* in 1202. This sequence, originally associated with pairs of reproducing rabbits, is $1, 1, 2, 3, 5, 8, 13, \ldots$. The simplest property of this sequence is that each term is the sum of the two previous terms. This was surely known to Fibonacci, though he nowhere states it. Nevertheless, the sequence has been studies by mathematicians ever since, with the sequence popping up in many unexpected places. This contest will look at the relationships among the powers of the Fibonacci numbers, leading to the development of "Fibonacci's Triangle," an array of numbers similar to Pascal's Triangle.

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
$(F_n)^2$	1	1	4	9	25	64	169	441	1156	3025
$(F_n)^3$	1	1	8	27	125	512	2197	9261	39304	166375

The Problems

1. Notice: $F_n = F_{n-1}+F_{n-2}$ and $F_{n-1} = F_{n-2}+F_{n-3}$. Substituting produces $F_n = 2\cdot F_{n-1}+F_{n-3}$. But $F_{n-2} = F_{n-3} + F_{n-4}$. Again, substituting produces $F_n = 2(F_{n-3} + F_{n-4}) + F_{n-3} = 3 \cdot F_{n-3} + 2 \cdot F_{n-4}$.

   a. Continue this process and make a generalization by writing a, b, and c in terms of n and k: $F_n = F_k \cdot F_a + F_b \cdot F_c$.

   b. Using mathematical induction, prove that your generalization is true. Call this Theorem 1. (Hint: You may need to use this theorem in problems 5 and 7.)

2. The relation $F_n = F_{n-1} + F_{n-2}$ can be rewritten as $F_n - F_{n-1} - F_{n-2} = 0$, or $aF_n + bF_{n-1} + cF_{n-2} = 0$ with $a = 1, b = -1$, and $c = -1$.

   a. If $a = 1$, find b, c, and d so that $a(F_n)^2 + b(F_{n-1})^2 + c(F_{n-2})^2 + d(F_{n-3})^2 = 0$.

   b. If $a = 1$, find b, c, d, and e so that $a(F_n)^3 + b(F_{n-1})^3 + c(F_{n-2})^3 + d(F_{n-3})^3 + e(F_{n-4})^3 = 0$.

   c. If $a = 1$, find b, c, d, e, and f so that $a(F_n)^4 + b(F_{n-1})^4 + c(F_{n-2})^4 + d(F_{n-3})^4 + e(F_{n-4})^4 + f(F_{n-5})^4 = 0$.

3. For the time being, **ignore the negative signs of these coefficients** and form a triangle similar to Pascal's Triangle (see the following page). We will call the new triangle "Fibonacci's Triangle". The entries in Pascal's Triangle are called "binomial coefficients", or simply, "binomials". The symbol $\binom{n}{k}$ is used to represent the binomial found in row n and column k in Pascal's Triangle. We will call the entries in Fibonacci's Triangle "**fibonomials**" and use

the symbol $\begin{bmatrix} n \\ k \end{bmatrix}$ to represent the **fibonomial** in row n and column k. Notice that similar to binomials, $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ n \end{bmatrix} = 1$, but while $\binom{n}{1} = n$, $\begin{bmatrix} n \\ 1 \end{bmatrix} = F_n$.

Pascal's Triangle

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1
5	1	5	5	...	

Fibonacci's Triangle

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	1	1		
3	1	2			
4					
5					

a. Make a copy of Fibonacci's Triangle on your Answer Sheet and complete rows 3, 4, and 5 of the triangle, using the coefficients you found in the Fibonacci power recurrences in problem 2. (N.B. The 3$^{\text{rd}}$ row has the coefficients of the recurrence relations of the squares of the Fibonacci numbers, the 4$^{\text{th}}$ row has the coefficients of the recurrence relations of the cubes of the Fibonacci numbers, etc.)

b. Note: $1(F_n)^5 - 8(F_{n-1})^5 - 40(F_{n-2})^5 + 60(F_{n-3})^5 + 40(F_{n-4})^5 - 8(F_{n-5})^5 + 1(F_{n-6})^5 = 0$ and $1(F_n)^6 - 13(F_{n-1})^6 - 104(F_{n-2})^6 + 260(F_{n-3})^6 + 260(F_{n-4})^6 - 104(F_{n-5})^6 - 13(F_{n-6})^6 + 1(F_{n-7})^6 = 0$. Use these recurrence relations to complete rows 6 and 7 of the Fibonacci Triangle.

4. In Pascal's Triangle an important relation is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, known as Pascal's Identity. The identity shows that a binomial can be derived by adding two adjacent binomials from the previous row. But actually, any binomial can be derived from the single binomial on its left (Theorem 2) or from the single binomial directly above it (Theorem 3).

a. [Theorem 2] Find p and q in terms of n and k such that $\binom{n}{k} = \frac{p}{q} \cdot \binom{n}{k-1}$.

b. [Theorem 3] Find r and s in terms of n and k such that $\binom{n}{k} = \frac{r}{s} \cdot \binom{n-1}{k}$.

c. [Theorem 4] **Use Theorem 2 and Theorem 3** to prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(Hint: Write both sides in terms of $\binom{n-1}{k}$.)

5. Fibonacci's Triangle Identities Using the values found in rows 6 and 7 of Fibonacci's Triangle,

   a. [Theorem 5] Find p and q such that $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{p}{q} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}$. (No proof is necessary.)
   (Hint: p and q will be Fibonacci numbers with subscripts involving n and k.)

   b. [Theorem 6] Find r and s such that $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{r}{s} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}$. (No proof is necessary.)
   (Hint: r and s will be Fibonacci numbers with subscripts involving n and k.)

   c. [Theorem 7] **Use Theorem 5 and Theorem 6** to prove that

   $$\begin{bmatrix} n \\ k \end{bmatrix} = F_{n-k+1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + F_{k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

   (Hint: Write both sides in terms of $\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$.)

   d. **Use Theorem 7** to find the value of $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$.

6. Factorials are often associated with binomials. Here are three definitions for $nFactorial, n!$:

   i) $n! = (n)(n-1)(n-2)\ldots(3)(2)(1)$,

   ii) $n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{if } n > 0, \end{cases}$

   iii) $n! =$ the number of ways of arranging n distinct things.

   Likewise, "**Fibtorials**" are associated with Fibonomials!
   Here are two definitions for $nFibtorial, F_n^!$:

   i) $F_n^! = (F_n)(F_{n-1})(F_{n-2})\ldots(F_3)(F_2)(F_1) = (F_n)(F_{n-1})(F_{n-2})\ldots(2)(1)(1)$,

   ii) $n! = \begin{cases} 1 & \text{if } n = 0 \\ F_n(F_{n-1}^!) & \text{if } n > 0. \end{cases}$

   a. What is the value of $F_8^!$?

   b. What is the value of $\dfrac{F_8^!}{F_3^! \cdot F_5^!}$?

7. Amazingly, just as the binomial $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$, the Fibonomial $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{F_n^!}{F_{n-k}^! \cdot F_k^!}$.
   Using this definition, prove Theorem 7. (Be legible!)

Not part of the contest:

Theorems 5 and 6 can now be easily proven with this definition.

Just as $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ is used for determining the powers of binomials,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{\left\lceil \frac{k}{2} \right\rceil} (F_{n-k})^{n-1} = 0$$

is used to determine the recurrence relation for powers of Fibonacci numbers!

The Solutions

1. a. $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-2} + F_{n-3}$. Therefore, $F_n = 2F_{n-2} + F_{n-3}$.

   But $F_{n-2} = F_{n-3} + F_{n-4}$. Therefore, $F_n = 2(F_{n-3} + F_{n-4}) + F_{n-3} = 3F_{n-3} + 2F_{n-4}$.
   But $F_{n-3} = F_{n-4} + F_{n-5}$ and so $F_n = 3(F_{n-4} + F_{n-5}) + 2F_{n-4} = 5F_{n-4} + 3F_{n-5}$.
   Continuing the pattern, $F_n = 8F_{n-5} + 5F_{n-6}$ and $F_n = 13F_{n-6} + 8F_{n-7} = F_7F_{n-6} + F_6F_{n-7}$.
   Generalizing, for any $n > k > 0$, $F_n = F_kF_{n-k+1} + F_{k-1}F_{n-k}$.

   b. Starts: If $n = 2$ and $k = 1$, $F_2 = F_1F_2 + F_0F_1 \Rightarrow 1 = 1 \cdot 1 + 0 \cdot 1$, which is true.

   Continues: Given $F_n = F_jF_{n-j+1} + F_{j-1}F_{n-j}$, we must show $F_n = F_{j+1}F_{n-j} + F_jF_{n-j-1}$.

$$\begin{aligned}
F_n &= F_jF_{n-j+1} + F_{j-1}F_{n-j} \\
&= F_j(F_{n-j} + F_{n-j-1} + F_{j-1}F_{n-j} \\
&= F_jF_{n-j} + F_jF_{n-j-1} + F_{j-1}F_{n-j} \\
&= F_jF_{n-j} + F_{j-1}F_{n-j} + F_jF_{n-j-1} \\
&= (F_j + F_{j-1})F_{n-j} + F_jF_{n-j-1} \\
&= F_{j+1}F_{n-j} + F_jF_{n-j-1}
\end{aligned}$$

2. a. $1 \cdot (F_n)^2 + b \cdot (F_{n-1})^2 + c \cdot (F_{n-2})^2 + d \cdot (F_{n-3})^2 = 0$ produces the following system:

$$\begin{cases}
1 \cdot 9 + b \cdot 4 + c \cdot 1 + d \cdot 1 = 0 \\
1 \cdot 25 + b \cdot 9 + c \cdot 4 + d \cdot 1 = 0 \\
1 \cdot 64 + b \cdot 25 + c \cdot 9 + d \cdot 4 = 0.
\end{cases}$$

   Solving this system produces $b = -2, c = -2$, and $d = 1$. Therefore, $1 \cdot (F_n)^2 - 2 \cdot (F_{n-1})^2 - 2 \cdot (F_{n-2})^2 + 1 \cdot (F_{n-3})^2 = 0$.

   b. $1 \cdot (F_n)^3 + b \cdot (F_{n-1})^3 + c \cdot (F_{n-2})^3 + d \cdot (F_{n-3})^3 + e \cdot (F_{n-4})^3 = 0$ produces the following system:

$$\begin{cases}
1 \cdot 125 + b \cdot 27 + c \cdot 8 + d \cdot 1 + e \cdot 1 = 0 \\
1 \cdot 512 + b \cdot 125 + c \cdot 27 + d \cdot 8 + e \cdot 1 = 0 \\
1 \cdot 2197 + b \cdot 512 + c \cdot 125 + d \cdot 27 + e \cdot 8 = 0 \\
1 \cdot 9261 + b \cdot 2197 + c \cdot 512 + d \cdot 125 + e \cdot 27 = 0.
\end{cases}$$

   Solving this system produces $b = -3, c = -6, d = 3$, and $e = 1$. Therefore, $1 \cdot (F_n)^3 - 3 \cdot (F_{n-1})^3 - 6 \cdot (F_{n-2})^3 + 3 \cdot (F_{n-3})^3 + 1 \cdot (F_{n-4})^3 = 0$.

c. $1 \cdot (F_n)^4 + b \cdot (F_{n-1})^4 + c \cdot (F_{n-2})^4 + d \cdot (F_{n-3})^4 + e \cdot (F_{n-4})^4 + f \cdot (F_{n-5})^4 = 0$ generates:

$$\begin{cases} 1 \cdot 4096 + b \cdot 625 + c \cdot 81 + d \cdot 16 + e \cdot 1 + f \cdot 1 = 0 \\ 1 \cdot 28561 + b \cdot 4096 + c \cdot 625 + d \cdot 81 + e \cdot 16 + f \cdot 1 = 0 \\ 1 \cdot 194481 + b \cdot 28561 + c \cdot 4096 + d \cdot 625 + e \cdot 81 + f \cdot 16 = 0 \\ 1 \cdot 1336336 + b \cdot 194481 + c \cdot 28561 + d \cdot 4096 + e \cdot 625 + f \cdot 81 = 0 \\ 1 \cdot 9150625 + b \cdot 1336336 + c \cdot 194481 + d \cdot 28561 + e \cdot 4096 + f \cdot 625 = 0. \end{cases}$$

Solving this system produces $b = -5, c = -15, d = 15, e = 5$, and $f = -1$. Therefore, $1 \cdot (F_n)^4 - 5 \cdot (F_{n-1})^4 - 15 \cdot (F_{n-2})^4 + 15 \cdot (F_{n-3})^4 + 5 \cdot (F_{n-4})^4 - 1 \cdot (F_{n-5})^4 = 0$.

3.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	1	1					
3	1	2	2	1				
4	1	3	6	3	1			
5	1	5	15	15	5	1		
6	1	8	40	60	40	8	1	
7	1	13	104	260	260	104	13	1

4. a. $\dfrac{p}{q} = \dfrac{\binom{n}{k}}{\binom{n}{k-1}} = \dfrac{\frac{n!}{(n-k)!k!}}{\frac{n!}{(n-(k-1))!(k-1)!}} = \dfrac{n!}{(n-k)!k!} \cdot \dfrac{(n-k+1)!(k-1)!}{n!}$

$= \dfrac{n!}{(n-k)!k(k-1)!} \cdot \dfrac{(n-k+1)!(n-k)!(k-1)!}{n!} = \dfrac{n-k+1}{k}$.

Therefore, $\dbinom{n}{k} = \dfrac{n-k+1}{k} \cdot \dbinom{n}{k-1}$. [Theorem 2]

b. $\dfrac{r}{s} = \dfrac{\binom{n}{k}}{\binom{n-1}{k}} = \dfrac{\frac{n!}{(n-k)!k!}}{\frac{(n-1)!}{(n-1-k))!k!}} = \dfrac{n!}{(n-k)!k!} \cdot \dfrac{(n-k-1)!k!}{(n-1)!} = \dfrac{n(n-1)!}{(n-k)(n-k-1)!k!} \cdot \dfrac{(n-k+1)!k!}{(n-1)!}$

$= \dfrac{n}{n-k}$. Therefore, $\dbinom{n}{k} = \dfrac{n}{n-k} \cdot \dbinom{n-1}{k}$. [Theorem 3]

c. Theorem 3 revised:

$$\binom{n}{k} = \frac{n-k+1}{k} \cdot \binom{n}{k-1}$$

$$\binom{n}{k-1} = \frac{k}{n-k+1} \cdot \binom{n}{k}$$

$$\binom{n-1}{k-1} = \frac{k}{n-1-k+1} \cdot \binom{n-1}{k}$$

Therefore, $\dbinom{n-1}{k-1} = \dfrac{k}{n-k} \cdot \dbinom{n-1}{k}$. [Theorem 3 revised]

To prove $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, use Theorem 2 on the left side and the revised Theorem 3 on the right side:

$$\begin{aligned}
\binom{n}{k} &= \frac{n-1}{k-1} + \binom{n-1}{k} \\
\frac{n}{n-k} \cdot \binom{n-1}{k} &= \frac{k}{n-k}\,\frac{n-1}{k} + \binom{n-1}{k} \\
\frac{n}{n-k} \cdot \binom{n-1}{k} &= \left(\frac{k}{n-k} + 1\right) \cdot \binom{n-1}{k} \\
\frac{n}{n-k} \cdot \binom{n-1}{k} &= \left(\frac{k}{n-k} + \frac{n-k}{n-k}\right) \cdot \binom{n-1}{k} \\
\frac{n}{n-k} \cdot \binom{n-1}{k} &= \frac{k}{n-k} \cdot \binom{n-1}{k}.
\end{aligned}$$

5. a. $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{p}{q} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix} \Rightarrow \frac{p}{q} = \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]}$.

n	6	6	6	6	6	6	7	7	7	7	7	7	7
k	1	2	3	4	5	6	1	2	3	4	5	6	7
$\frac{p}{q}$	$\frac{8}{1}$	$\frac{5}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{13}{1}$	$\frac{8}{1}$	$\frac{5}{5}$	$\frac{1}{1}$	$\frac{2}{5}$	$\frac{1}{8}$	$\frac{1}{13}$
F	$\frac{F_6}{F_1}$	$\frac{F_5}{F_2}$	$\frac{F_4}{F_3}$	$\frac{F_3}{F_4}$	$\frac{F_2}{F_5}$	$\frac{F_1}{F_6}$	$\frac{F_7}{F_1}$	$\frac{F_6}{F_2}$	$\frac{F_5}{F_3}$	$\frac{F_4}{F_4}$	$\frac{F_3}{F_5}$	$\frac{F_2}{F_6}$	$\frac{F_1}{F_7}$

It appears that $\frac{p}{q} = \frac{F_{n-k+1}}{F_k}$. Therefore, $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_{n-k+1}}{F_k} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}$. [Theorem 5]

b. $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{r}{s} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} \Rightarrow \frac{r}{s} = \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]}$.

n	6	6	6	6	6	6	7	7	7	7	7	7	7
k	0	1	2	3	4	5	0	1	2	3	4	5	6
$\frac{r}{s}$	$\frac{8}{8}$	$\frac{8}{5}$	$\frac{8}{3}$	$\frac{8}{2}$	$\frac{8}{1}$	$\frac{8}{1}$	$\frac{13}{13}$	$\frac{13}{8}$	$\frac{13}{5}$	$\frac{13}{3}$	$\frac{13}{2}$	$\frac{13}{1}$	$\frac{13}{1}$
F	$\frac{F_6}{F_6}$	$\frac{F_6}{F_5}$	$\frac{F_6}{F_4}$	$\frac{F_6}{F_3}$	$\frac{F_6}{F_2}$	$\frac{F_6}{F_1}$	$\frac{F_7}{F_7}$	$\frac{F_7}{F_6}$	$\frac{F_7}{F_5}$	$\frac{F_7}{F_4}$	$\frac{F_7}{F_3}$	$\frac{F_7}{F_2}$	$\frac{F_7}{F_1}$

It appears that $\frac{r}{s} = \frac{F_n}{F_{n-k}}$. Therefore, $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}$. [Theorem 6]

c. Theorem 5 revised:

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{F_{n-k+1}}{F_k} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix} \\
\frac{F_k}{F_{n-k+1}} \cdot \begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n \\ k-1 \end{bmatrix} \\
\frac{F_k}{F_{n-1-k+1}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
\text{Therefore, } \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} &= \frac{F_k}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}. \text{ [Theorem 5 revised]}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= F_{n-k+1} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + F_{k-1} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
\frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= F_{n-k+1} \cdot \frac{F_k}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} + F_{k-1} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} && \text{Thm. 6 and revised Thm. 5} \\
\frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \left(F_{n-k+1} \cdot \frac{F_k}{F_{n-k}} + F_{k-1} \right) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} && \text{Factor} \\
\frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \left(\frac{F_{n-k+1} \cdot F_k}{F_{n-k}} + \frac{F_{k-1} \cdot F_{n-k}}{F_{n-k}} \right) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} && \text{Common Denominator} \\
\frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \left(\frac{F_{n-k+1} \cdot F_k + F_{k-1} \cdot F_{n-k}}{F_{n-k}} \right) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} && \text{Addition} \\
\frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} &= \frac{F_n}{F_{n-k}} \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} && \text{Theorem 1}
\end{aligned}$$

d. $\begin{bmatrix} 8 \\ 5 \end{bmatrix} = F_4 \begin{bmatrix} 7 \\ 4 \end{bmatrix} + F_4 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = 3(260) + 3(104) = 1092.$

6. a. $F_8^! = F_8 \cdot F_7 \cdot F_6 \cdot F_5 \cdot F_4 \cdot F_3 \cdot F_2 \cdot F_1 = 21 \cdot 13 \cdot 8 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 65520.$

b. $\dfrac{F_8^!}{F_3^! \cdot F_5^!} = \dfrac{65520}{(2 \cdot 1 \cdot 1) \cdot (5 \cdot 3 \cdot 2 \cdot 1 \cdot 1)} = \dfrac{65520}{2 \cdot 30} = 1092.$ (Should be the same as your answer in 5d.)

7.

$$
\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= F_{n-k+1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + F_{k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
\frac{F_n^!}{F_{n-k}^! \cdot F_k^!} &= F_{n-k+1} \cdot \frac{F_{n-1}^!}{F_{n-1-(k-1)}^! \cdot F_{k-1}^!} + F_{k-1} \cdot \frac{F_{n-1}^!}{F_{n-1-k}^! \cdot F_k^!} && \text{Fibonomial Definition} \\
&= F_{n-k+1} \cdot \frac{F_{n-1}^!}{F_{n-k}^! \cdot F_{k-1}^!} + F_{k-1} \cdot \frac{F_{n-1}^!}{F_{n-1-k}^! \cdot F_k^!} && \text{Simplify} \\
&= F_{n-k+1} \cdot \frac{F_{n-1}^!}{F_{n-k} \cdot F_{n-k-1}^! \cdot F_{k-1}^!} + F_{k-1} \cdot \frac{F_{n-1}^!}{F_{n-1-k}^! \cdot F_k \cdot F_{k-1}^!} && \text{Fibtorial Definition} \\
&= F_{n-k+1} \cdot \frac{F_k \cdot F_{n-1}^!}{F_{n-k} \cdot F_{n-k-1}^! \cdot F_k \cdot F_{k-1}^!} \\
&\quad + F_{k-1} \cdot \frac{F_{n-k} \cdot F_{n-1}^!}{F_{n-k} \cdot F_{n-k-1}^! \cdot F_k \cdot F_{k-1}^!} && \text{Common Denominator} \\
&= \frac{F_{n-k+1} \cdot F_k \cdot F_{n-1}^! + F_{k-1} \cdot F_{n-k} \cdot F_{n-1}^!}{F_{n-k} \cdot F_{n-k-1}^! \cdot F_k \cdot F_{k-1}^!} && \text{Addition} \\
&= \frac{(F_{n-k+1} \cdot F_k + F_{k-1} \cdot F_{n-k}) \cdot F_{n-1}^!}{F_{n-k}^! \cdot F_k^!} && \text{Factor} \\
&= \frac{F_n \cdot F_{n-1}^!}{F_{n-k}^! \cdot F_k^!} && \text{Theorem 1} \\
&= \frac{F_n^!}{F_{n-k}^! \cdot F_k^!} && \text{Fibtorial Definition}
\end{aligned}
$$

Brahmagupta's Cyclic Quadrilaterals

The Background

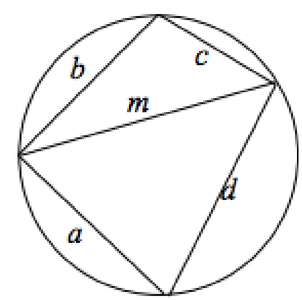
Brahmagupta (ca 598 AD – 668 AD) was an Indian mathematician and astronomer. His best-known work, *Brāhmasphuṭasiddhānta* ("The Correctly Established Doctrine of Brahma"), was written in 628. It is the very first book that uses zero as a number, defining it to be the result when a number is subtracted from itself. He gives rules for computing with zero, including division by zero. But the part for which Brahmagupta's name is remembered the most is his chapter that deals with cyclic quadrilaterals, four-sided polygons whose vertices lie on a circle. In this chapter he states a formula to determine the area and length(s) of the diagonal(s) of a cyclic quadrilateral, knowing the length of its four sides.

Brahmagupta's Area Formula:

$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \dfrac{a+b+c+d}{2}$.

Brahmagupta's Diagonal Formula:

$$m^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}$$



It also contains a beautiful theorem regarding cyclic quadrilaterals with perpendicular diagonals.

Brahmagupta's Theorem

In a cyclic quadrilateral with perpendicular diagonals, a line drawn perpendicular to one side of the quadrilateral through the intersection of the diagonals bisects the opposite side.

In this problem set, we will prove and use these results. The following theorems may be useful:

1. Quadrilateral $ABCD$ is cyclic if and only if $\angle A + \angle C = \angle B + \angle D = 180°$.
2. A trapezoid is cyclic if and only if it is isosceles.
3. (Ptolemy's Theorem) In a cyclic quadrilateral, with side lengths a, b, c, d and diagonal lengths m and n, $mn = ac + bd$.
4. The length of a median drawn to the hypotenuse of a right triangle is half the length of the hypotenuse.

5. The vertex centroid of a quadrilateral can be found in three ways, i.e., all three methods produce the same point:

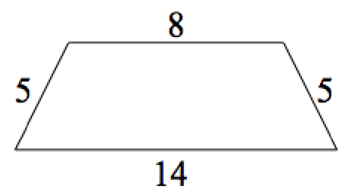
    a. The coordinates of the centroid: $(x, y) =$ (the sum of the x-coordinates of the vertices divided by four, the sum of the y-coordinates divided by four).

    b. Form two segments by connecting the midpoints of the opposite sides of the quadrilateral. Where they intersect is the centroid.

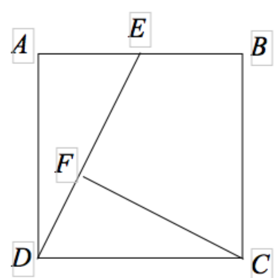
    c. Form a segment by connecting the midpoints of the diagonals of the quadrilateral. The midpoint of this segment is the centroid. (This theorem is easily proven using coordinate geometry.)

<u>The Problems</u> (In this problem set, $[ABCD]$ = the area of polygon $ABCD$.)

1. a. Verify Brahmagupta's Area Formula by finding the area of this trapezoid in two ways. (One way should be Brahmagupta's Area Formula.)



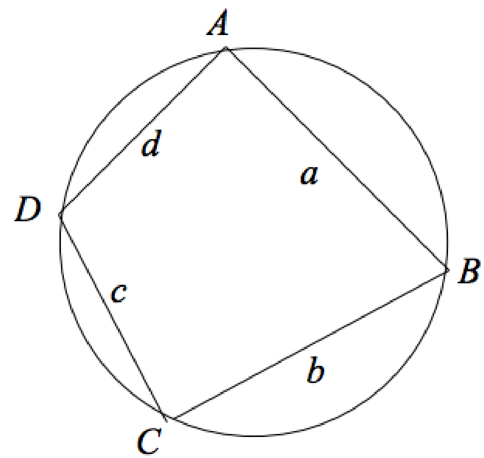
b. Verify Brahmagupta's Area Formula by finding $[BCFE]$ in two ways. $ABCD$ is a square with $AB = 2$, E is the midpoint of $\overline{AB}$, and F is a point on $\overline{DE}$ such that $\overline{CF} \perp \overline{DE}$. (One way should be Brahmagupta's Area Formula.)



c. $ABCD$ is a convex quadrilateral with integral sides, diagonal $AC = 25$, and $\angle B = \angle D = 90°$. How many different cyclic quadrilaterals are there with these conditions? Find the area of each.

2. Derive Brahmagupta's Area Formula:

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \text{ where } s = \frac{a+b+c+d}{2}.$$



You may use your own approach or you might want to use the following outline:

a) Begin by showing that $s - a = \dfrac{-a+b+c+d}{2}$, $s - b = \dfrac{a-b+c+d}{2}$, etc.

b) Start the derivation with $[ABCD] = [ABD] + [BCD]$ and show that

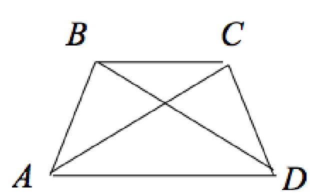
$$[ABCD] = \frac{1}{2}\sin A(ad+bc).$$

c) Continue by showing that

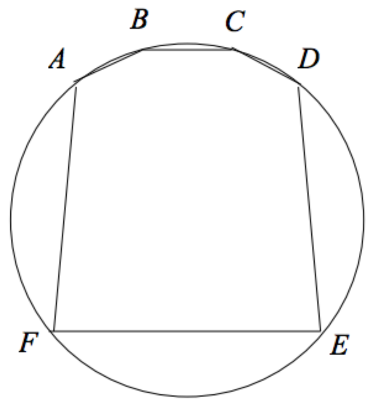
$$16 \cdot [ABCD]^2 = 4(ad+bc)^2(1-\cos^2 A) = 4(ad+bc)^2 - \left(a^2+d^2-b^2-c^2\right)^2.$$

d) Factor, using the difference of squares three times, to complete the derivation.

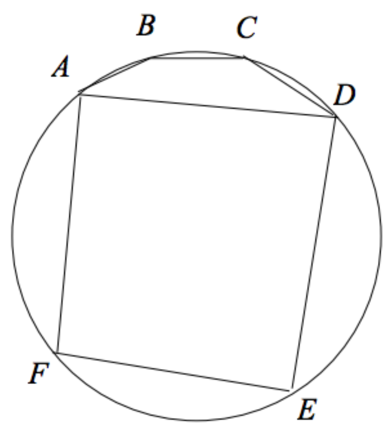
3. Compute the area of the isosceles trapezoid $ABCD$ where $AB = BC = CD = 1$ and $AD = AC = BD$. Write your final, simplified answer in the form $\dfrac{\sqrt{a+b\sqrt{5}}}{c}$.



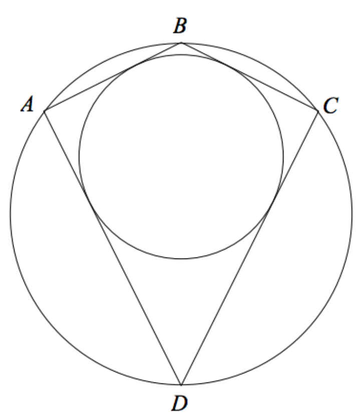
4. Hexagon $ABCDEF$ is inscribed in a circle with $AB = BC = CD = 2$ and $DE = EF = FA = 6$. Compute $[ABCDEF]$.



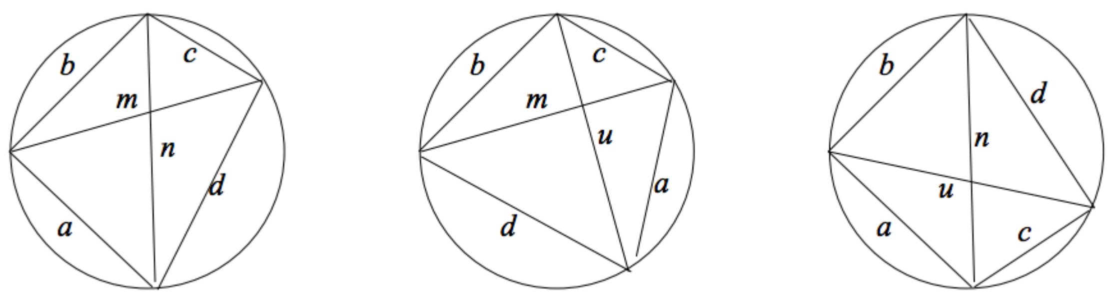
5. Cyclic hexagon $ABCDEF$ is formed by trapezoid $ABCD$ and quadrilateral $DEFA$. If $AB = BC = 4$, $DE = 8$, $EF = 7$, $FA = 9$, and $[ADEF] = 12\sqrt{35}$, compute $[ABCD]$.



6. If cyclic quadrilateral $ABCD$, with side lengths a, b, c, and d, circumscribes a circle, show that $[ABCD] = \sqrt{abcd}$.



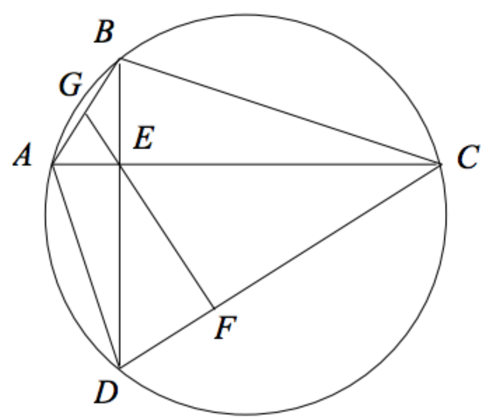
7. Brahmagupta's Diagonal Formula states that $m^2 = \dfrac{(ab+cd)(ac+bd)}{ad+bc}$. It can be derived using the Law of Cosines twice or very elegantly using Ptolemy's Theorem three times with the figures below and the identity $m^2 = \dfrac{(mu)(mn)}{(un)}$. Each of the three cyclic quadrilaterals has sides a, b, c, and d, but pairs of sides have been switched in adjacent figures. Derive Brahmagupta's Diagonal Formula using BOTH methods.



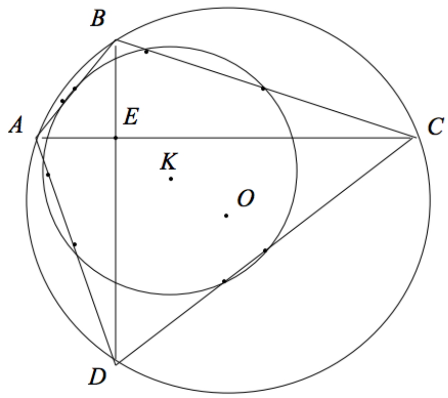
8. Cyclic quadrilateral $ABCD$ has consecutive side lengths of 25, 39, 52, and 60. Find its area, the lengths of its two diagonals, and the diameter of the circumscribed circle.

9. Brahmagupta's Theorem states that in a cyclic quadrilateral with perpendicular diagonals, a line drawn perpendicular to one side of the quadrilateral through the intersection of the diagonals bisects the opposite side.

   In the figure, if $ABCD$ is a cyclic quadrilateral with $\overline{AC} \perp \overline{BD}$ and line FEG is drawn perpendicular to $\overline{CD}$, then $AG = GB$. This beautiful theorem can easily be proven as a result of labeling the congruent angles in the figure. Do so.



10. Brahmagupta's Corollary to his theorem. (Even more beautiful!)



If the diagonals AC and BD of a quadrilateral $ABCD$, which is inscribed in a circle centered at point O, are perpendicular at point E, then the midpoints of the sides of the quadrilateral, M_a, M_b, M_c, and M_d, and the feet of the perpendiculars drawn from E to each of the sides, P_a, P_b, P_c, and P_d, all lie on a circle whose center is K, the midpoint of $\overline{EO}$.

Prove M_a, M_b, M_c, and M_d all lie on a circle centered at K. Then show P_a, P_b, P_c, and P_d lie on this same circle. And finally, show that K is the midpoint of $\overline{EO}$.

11. Brahmagupta's Trapezium Construction

Take any two Pythagorean Triples, say (r, s, t) and (x, y, z), where $r^2+s^2 = t^2$ and $x^2+y^2 = z^2$, and show there exists a quadrilateral $ABCD$ with side lengths $a = rz$, $b = tx$, $c = sz$, and $d = ty$ which

i) is cyclic,
ii) has perpendicular diagonals with lengths $m = sx + ry$ and $n = rx + sy$, and
iii) $[ABCD] = \dfrac{mn}{2}$.

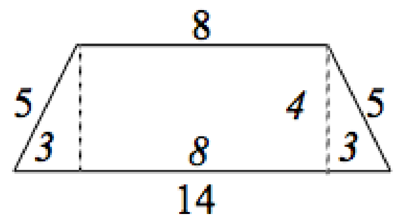
a. Show the construction works when you use the triples $(3, 4, 5)$ and $(8, 15, 17)$.

b. Prove the three conclusions in general are true for any pair of Pythagorean triples. (Hint: Make the construction.)

The Solutions

1. a.

i) Drop altitudes, dividing lower base into segments of lengths 3, 8, and 3. Use the Pythagorean Theorem to determine a height of 4. Using the trapezoid area formula, $[\text{Area}] = h\left(\dfrac{b_1+b_2}{2}\right) = 4\left(\dfrac{8+14}{2}\right) = 44$.



ii) Since all isosceles trapezoids are cyclic (Theorem 2), Brahmagupta's Area Formula (abbreviated "B.A.F.") can be used with $s = 1$. Thus

$$[\text{Area}] = \sqrt{(16-8)(16-5)(16-14)(15-5)} = 44.$$

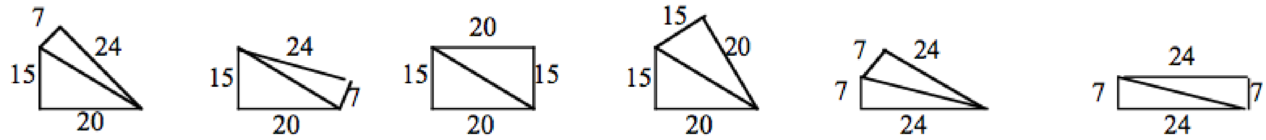
b. $AE = 1$ and $DE = \sqrt{5}$. $\angle AED \cong \angle EDC$ and $\angle EAD \cong \angle DFC$, therefore, $\triangle AED \approx \triangle FDC$. $FD = \dfrac{2\sqrt{5}}{5}$, $FC = \dfrac{4\sqrt{5}}{5}$, and $FE = \dfrac{3\sqrt{5}}{5}$.

i) $[BCFE] = [ABCD] - [AED] - [DFC] = 4 - \frac{1}{2}(1)(2) - \frac{1}{2}\left(\dfrac{2\sqrt{5}}{5}\right)\left(\dfrac{4\sqrt{5}}{5}\right) = \dfrac{11}{5}$.

ii) Since $\angle B = \angle D = 90°$, $\angle A + \angle C = 180°$ and so $BCFE$ is cyclic and B.A.F. can be used with $s = \dfrac{1+2+\frac{4\sqrt{5}}{5}+\frac{3\sqrt{5}}{5}}{2} = \dfrac{7\sqrt{5}+15}{10}$.

$$\begin{aligned}[BCFE] &= \sqrt{\left(\frac{7\sqrt{5}+15}{10}-1\right)\left(\frac{7\sqrt{5}+15}{10}-2\right)\left(\frac{7\sqrt{5}+15}{10}-\frac{3\sqrt{5}}{5}\right)\left(\frac{7\sqrt{5}+15}{10}-\frac{4\sqrt{5}}{5}\right)} \\ &= \sqrt{\left(\frac{7\sqrt{5}+5}{10}\right)\left(\frac{7\sqrt{5}-5}{10}\right)\left(\frac{\sqrt{5}+15}{10}\right)\left(\frac{-\sqrt{5}+15}{10}\right)} \\ &= \sqrt{\left(\frac{245-25}{100}\right)\left(\frac{225-5}{100}\right)} = \frac{11}{5}\end{aligned}$$

c. Since $25 = 5^2$, right triangle ABC is either a primitive Pythagorean triangle with $c = 25$ or a primitive Pythagorean triangle with $c = 5$ and a scale factor of 5. Therefore, $m^2 + n^2 = 25$ or 5. The first gives $m = 4$ and $n = 3$, producing $2mn = 24$ and $m^2 - n^2 = 7$, while the second gives $m = 2$ and $n = 1$, producing $2mn = 4$ and $m^2 - n^2 = 3$. Therefore, triangles ABC and ADC have either sides 7, 24, 25 or 15, 20, 25. They can be arranged in the following six ways:



[234] [234] [300] [300] [168] [168]

2. $s = \dfrac{a+b+c+d}{2}$. $s-a = \dfrac{a+b+c+d}{2} - a = \dfrac{a+b+c+d}{2} - \dfrac{2a}{2} = \dfrac{-a+b+c+d}{2}$. Likewise, $s-b = \dfrac{a-b+c+d}{2}$, $s-c = \dfrac{a+b-c+d}{2}$, and $s-d = \dfrac{a+b+c-d}{2}$.

   $[ABCD] = [ABC] + [BCD] = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C$. Since $ABCD$ is cyclic, $A + C = 180°$ and, therefore, $\sin A = \sin C$. Therefore, $[ABCD] = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin A = \frac{1}{2}\sin A(ad+bc)$.

   Therefore, $[ABCD]^2 = \frac{1}{4}\sin^2 A(ad+bc)^2$, $4[ABCD]^2 = (1-\cos^2 A)(ad+bc)^2$, and $4[ABCD]^2 = (ad+bc)^2 - (ad+bc)^2\cos^2 A$. But $BD^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$. Since $A + C = 180°$, $\cos C = -\cos A$, and so $a^2 + d^2 - b^2 - c^2 = 2(ad+bc)\cos A$ and $\left(a^2+d^2-b^2-c^2\right)^2 = 4(ad+bc)\cos^2 A$. $4[ABCD] = (ad+bc)^2 - \frac{1}{4}\left(a^2+d^2-b^2-c^2\right)^2$.

   Therefore,

$$
\begin{aligned}
16[ABCD]^2 &= 4(ad+bc)^2 - \left(a^2+d^2-b^2-c^2\right)^2 \\
&= \left(2(ad+bc) - (a^2+d^2-b^2-c^2)\right)\left(2(ad+bc) + (a^2+d^2-b^2-c^2)\right) \\
&= \left(-a^2+2ad-d^2+b^2+2bc+c^2\right)\left(a^2+2ad+d^2-(b^2-2bc+c^2)\right) \\
&= \left((b+c)^2-(a-d)^2\right)\left((a+d)^2-(b-c)^2\right) \\
&= (b+c-(a-d))\,(b+c+(a-d))\,(a+d-(b-c))\,(a+d+(b-c)) \\
&= (-a+b+c+d)(a+b+c-d)(a-b+c+d)(a+b+c+d) \\
\Rightarrow [ABCD]^2 &= \frac{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}{16} \\
&= \left(\frac{-a+b+c+d}{2}\right)\left(\frac{a-b+c+d}{2}\right)\left(\frac{a+b-c+d}{2}\right)\left(\frac{a+b+c-d}{2}\right) \\
&= (s-a)(s-b)(s-c)(s-d).
\end{aligned}
$$

   Therefore, $[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

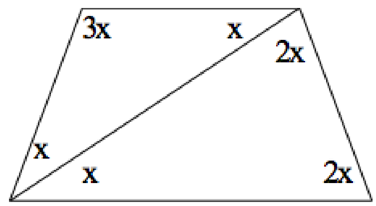
3. The lengths of the diagonals can be established in two ways: i) Because base angles of isosceles triangles are equal and base angles of isosceles trapezoids are also equal, $10x = 360°$. So $x = 36°$ and $\triangle ACD$ is a $36°-72°-72°$ triangle. Bisecting a $72°$ angle of this triangle produces a pair of similar triangles. Thus

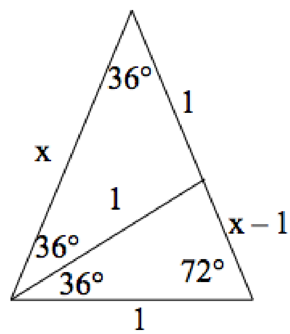
$$\frac{x}{1} = \frac{1}{x-1} \Rightarrow x(x-1) = 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2}.$$

   ii) Since $ABCD$ is an isosceles trapezoid, it is cyclic and Ptolemy's Theorem applies, giving

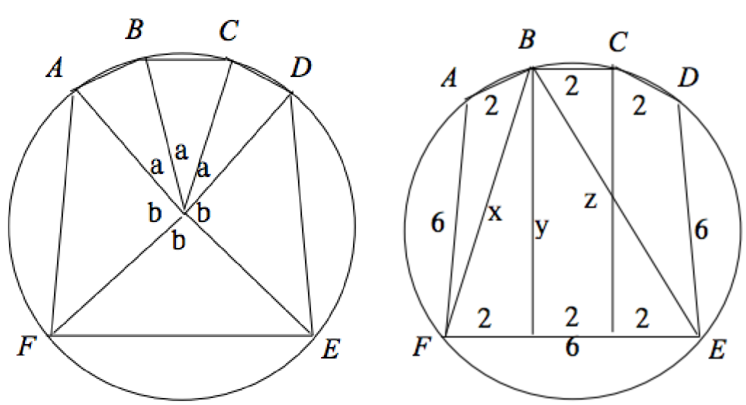
$1 \cdot x + 1 \cdot 1 = x \cdot x$ or $x^2 - x - 1 = 0$ as above. Therefore, $s = \dfrac{1+1+1+\frac{1+\sqrt{5}}{2}}{2} = \dfrac{7+\sqrt{5}}{4}$ and

$$[ABCD] = \sqrt{\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{5-\sqrt{5}}{4}\right)} = \frac{\sqrt{200+88\sqrt{5}}}{16} = \frac{\sqrt{50+22\sqrt{5}}}{8}.$$





4. Draw radii coming from the center of the circle.



Then $3a + 3b = 360° \Rightarrow a + b = 120°$. Thus the measure of minor arc FB is $120°$ and the measure of major arc FB is $240°$, making $\angle FAB = 120°$. Therefore,

$$\begin{aligned}
x^2 &= 2^2 + 6^2 - 2(2)(6)\cos 120° \\
&= 4 + 36 + 12 \\
&= 52 \\
x &= 2\sqrt{13}, \\
y^2 + 2^2 &= \left(2\sqrt{13}\right)^2 \\
y^2 &= 48 \\
y &= 4\sqrt{3}, \\
z^2 &= 4^2 + \left(4\sqrt{3}\right)^2 \\
&= 64 \\
z &= 8.
\end{aligned}$$

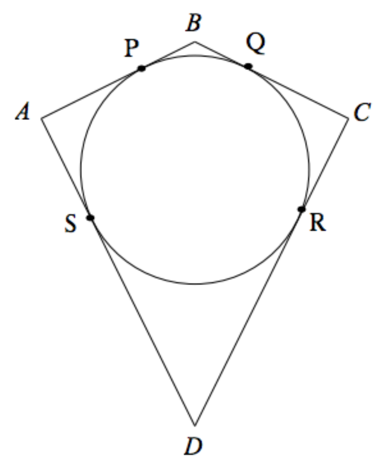
$BCDE$ is a cyclic quadrilateral with $s = 9$, thus $[BCDE] = \sqrt{(7)(7)(3)(1)} = 7\sqrt{3}$. $ABEF$ is a cyclic quadrilateral with $s = 11$, thus $[ABEF] = \sqrt{(9)(3)(5)(5)} = 15\sqrt{3}$. Therefore, $[ABCDEF] = [BCDE] + [ABEF] = 22\sqrt{3}$.

5. Let $x = AD$ in cyclic quadrilateral $ADEF$. $s = \dfrac{x+8+7+9}{2} = \dfrac{24+x}{2} = 12 + \dfrac{x}{2}$. By B.A.F.,

$$\begin{aligned}
12\sqrt{35} &= \sqrt{\left(12 + \frac{x}{2} - x\right)\left(12 + \frac{x}{2} - 8\right)\left(12 + \frac{x}{2} - 7\right)\left(12 + \frac{x}{2} - 9\right)} \\
&= \sqrt{\left(12 - \frac{x}{2}\right)\left(4 + \frac{x}{2}\right)\left(5 + \frac{x}{2}\right)\left(3 + \frac{x}{2}\right)} \\
(144)(35) &= \left(48 + 4x - \frac{x^2}{4}\right)\left(15 + 4x - \frac{x^2}{4}\right) \\
(16)(144)(35) &= (192 + 16x - x^2)(60 + 16x + x^2) \\
0 &= x^4 - 388x^2 - 4032x + 69120.
\end{aligned}$$

This quartic has a zero at $x = 10$. Now consider cyclic quadrilateral $ABCD$. It has $s = \dfrac{4+4+4+10}{2} + 11$ and so $[ABCD] = \sqrt{(7)(7)(7)(1)} = 7\sqrt{7}$.

6. Let P, Q, R, and S be the points where $ABCD$ touches the inscribed circle. By equal tangents, $AP = AS$, $BP = BQ$, $CQ = CR$, and $DR = DS$.



Let $AB = a$, $BC = b$, $CD = c$, and $DA = d$. If $AP = x$, then $BP = a - x$. If $BQ = a - x$, then $CQ = b - (a - x) = b - a + x$. If $CR = b - a + x$, then $DR = c - (b - a + x) = c - b + a - x$. If $DS = c - b + a - x$, then $AS = d - (c - b + a - x) = d - c + b - a + x$. Since $AP = AS$, $x = d - c + b - a + x$, implying $b + d = a + c$. Since $s = \dfrac{a+b+c+d}{2}$, $s = \dfrac{2(a+c)}{2} = a + c$ OR $s = \dfrac{2(b+d)}{2} = b + d$. Therefore,

$$\begin{aligned} [ABCD] &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ &= \sqrt{(a+c-a)(b+d-b)(a+c-c)(b+d-d)} \\ &= \sqrt{c \cdot d \cdot a \cdot b}. \end{aligned}$$

7. i) $\cos A = \dfrac{a^2 + d^2 - m^2}{2ad}$ and $\cos C = \dfrac{b^2 + c^2 - m^2}{2bc}$. Since $A + C = 180°$, $\cos C = -\cos A$. Therefore,

$$\begin{aligned} \frac{b^2 + c^2 - m^2}{2bc} &= -\frac{a^+ d^2 - m^2}{2ad} \\ ad(b^2 + c^2 - m^2) &= bc(m^2 - a^2 - d^2) \\ ad(b^2 + c^2) + bc(a^2 + d^2) &= bcm^2 + adm^2 \\ adb^2 + adc^2 + bca^2 + bcd^2 &= (bc + ad)m^2 \\ ab(bd + ac) + cd(ac + bd) &= (bc + ad)m^2 \\ \frac{(ab + cd)(ac + bd)}{(ad + bc)} &= m^2. \end{aligned}$$

ii) By Ptolemy's Theorem, the first diagram has $ac + bd = mn$, the second diagram has $ab + cd = mu$, and the third diagram has $ad + bc = nu$. But $m^2 = \dfrac{(mu)(mn)}{(un)}$ and substituting, gives: $m^2 = \dfrac{(ab+cd)(ac+bd)}{(ad+bc)}$.

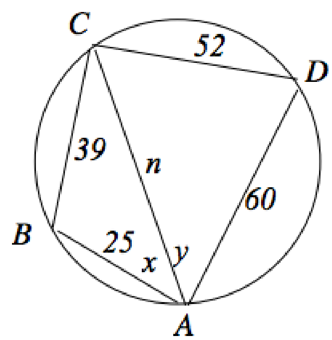
8. $s = \dfrac{25+39+52+60}{2} = 88$. By B.A.F., $[ABCD] = \sqrt{(63)(49)(36)(28)} = 1764$. By B.D.F., $m = \sqrt{\dfrac{(25\cdot 39+52\cdot 60)(25\cdot 52+39\cdot 60)}{(25\cdot 60+39\cdot 52)}} = 65$. By Ptolemy's Theorem, $n = \dfrac{ac+bd}{m} = \dfrac{25\cdot 52+39\cdot 60}{65} = 56$.

The diameter of the circumscribed circle can be found in two ways:

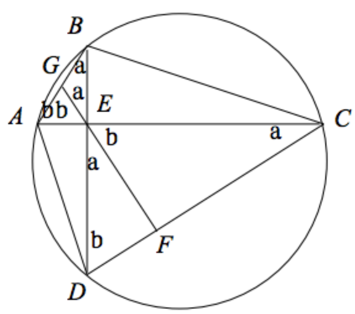
i) Notice that $39^2+52^2 = 25^2+60^2 = 65^2$. Therefore, ABD and BCD are right triangles and right triangles are inscribed in circles where the diameter is the hypotenuse. So $BD = 65$ is the diameter.

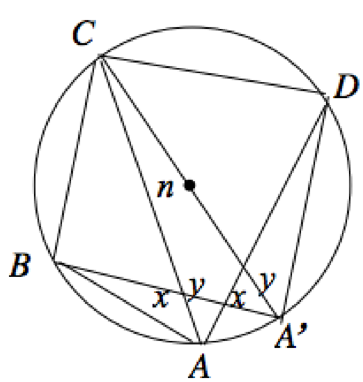
ii)

$$\begin{aligned}
[ABCD] &= [ABC] + [ACD] \\
1764 &= \frac{1}{2}(25)(56)\sin x + \frac{1}{2}(60)(56)\sin y \\
&= \frac{1}{2}(56)(25\sin x + 60\sin y) \\
&= \frac{1}{2}(56)\left(25\cdot\frac{39}{AC'} + 60\cdot\frac{52}{AC'}\right) \\
AC' &= \frac{\frac{1}{2}(56)(25\cdot 39+60\cdot 52)}{1764} = 65
\end{aligned}$$



9. The altitude drawn to the hypotenuse of any right triangle, divides the triangle into two right triangles, similar to triangle EDC. The measures of vertical angles are equal. Inscribed angles intercepting the same arc have equal measure. Therefore, triangle EGB is isosceles with $EG = GB$ and triangle AGE is also isosceles with $EG = GA$. Therefore, $GA = GB$.





10. (See Figure 1) The segment formed by connecting M_a and M_b is the midsegment of $\triangle ABC$ and, therefore, parallel to $\overline{AC}$. Likewise, the segment formed by connecting M_c and M_d is the midsegment of $\triangle ADC$ and is also parallel to $\overline{AC}$. Likewise, the segment connecting M_b and M_c and the segment connecting M_d and M_a are parallel to $\overline{BD}$. Therefore, quadrilateral $M_aM_bM_cM_d$ is a parallelogram. Because $\overline{AC} \perp \overline{BD}$, $M_aM_bM_cM_d$ is a rectangle. Since the diagonals of a rectangle are congruent and bisect each other at K. Therefore, M_a, M_b, M_c, and M_d lie on a circle centered at K. By Theorem 5b, K must be the vertex centroid of $ABCD$.

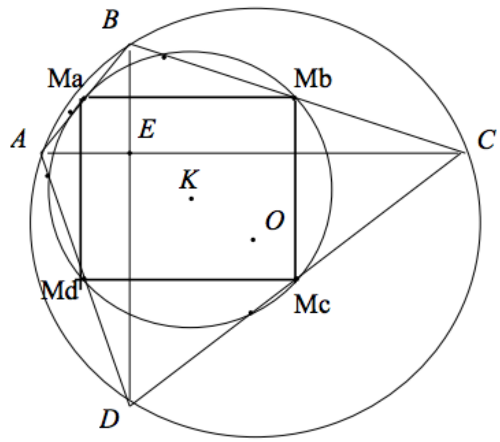


Figure 1

(See Figure 2) By Brahmagupta's Theorem above, M_d, E, and P_b are collinear and so $\triangle M_dP_bM_b$ is a right triangle. By Theorem 5b, $M_dK = M_bK = P_bK$ and so M_d, M_b, and P_b all lie on a circle centered at K. The same argument can be used to show that P_a, P_c, and P_d also lie on the same circle.

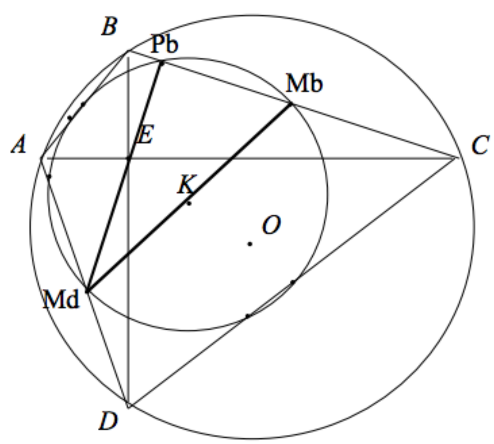


Figure 2

(See Figure 3) Construct $\overline{OF}$ perpendicular to $\overline{AC}$. Since $\overline{AC}$ is a chord of the circle centered at O, $\overline{OF}$ bisects $\overline{AC}$ at F. Likewise, $\overline{OG}$, constructed perpendicular to $\overline{BD}$, bisects $\overline{BD}$

at C. Quadrilateral $EFOG$ must be a rectangle. The midpoint of $\overline{FG}$ must be the vertex centroid of $ABCD$ by Theorem 5c. But K has already been determined to be the centroid of $ABCD$. Since the diagonals of a rectangle bisect each other, K must be the midpoint of $\overline{OE}$.

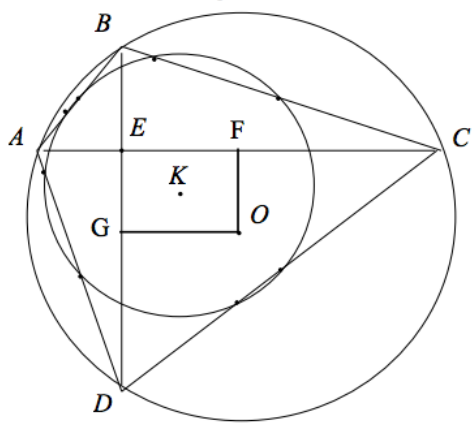


Figure 3

11. a. (See Figure 11a)

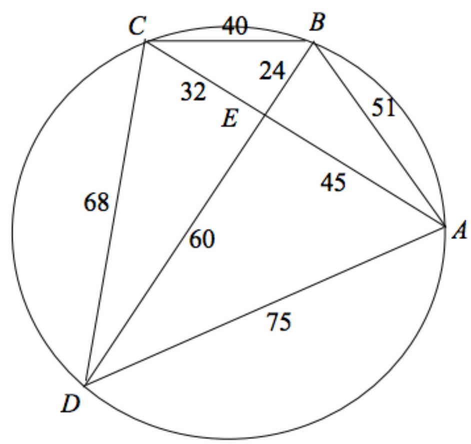


Figure 11a

$$(24, 32, 40) = 8(3, 4, 5)$$
$$(45, 60, 75) = 15(3, 4, 5)$$
$$(24, 45, 51) = 3(8, 15, 17)$$
$$(32, 60, 68) = 4(8, 15, 17)$$

$$
\begin{aligned}
[ABCD] &= \frac{1}{2}(AC \cdot EB) + \frac{1}{2}(AC \cdot ED)\\
&= \frac{1}{2}AC \cdot (EB + ED)\\
&= \frac{1}{2}AC \cdot BD\\
&= \frac{1}{2}(77)(84)\\
&= 3234
\end{aligned}
$$

Using B.A.F., $s = \dfrac{40+51+68+75}{2} = 117$. So

$$[ABCD] = \sqrt{(117-40)(117-51)(117-68)(117-75)} = \sqrt{(77)(66)(49)(42)} = 3234.$$

b. Four right triangles can be formed and arranged as in Figure 11b.1.

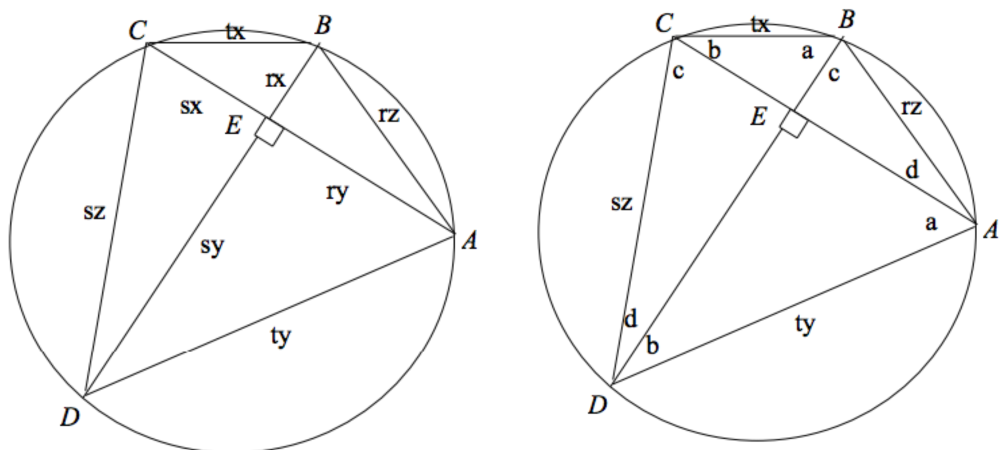


Figure 11b.1 Figure 11b.2

Since vertex angles intercepting the same arc must have equal measures, the angles of $ABCD$ can be labeled as in Figure 11b.2.

Because the acute angles in any right triangle must be complementary, $a + b = 90°$ and $c + d = 90°$. Therefore,

$$
\begin{aligned}
\angle CBA + \angle CDA &= a + c + d + b\\
&= a + b + c + d\\
&= 180°
\end{aligned}
$$

and

$$\begin{aligned}\angle DCB + \angle DAB &= c + b + a + d \\ &= a + b + c + d \\ &= 180^\circ.\end{aligned}$$

Since opposite angles are complementary, by Theorem 1, $ABCD$ is cyclic.

It's obvious from the construction that the diagonals are perpendicular. Thus

$$\begin{aligned}[ABCD] &= \frac{1}{2}(AC \cdot EB) + \frac{1}{2}(AC \cdot ED) \\ &= \frac{1}{2}AC \cdot (EB + ED) \\ &= \frac{1}{2}AC \cdot BD \\ &= \frac{1}{2}(sx + ry)(rx + sy) \\ &= \frac{1}{2}mn.\end{aligned}$$

Rational Trigonometry

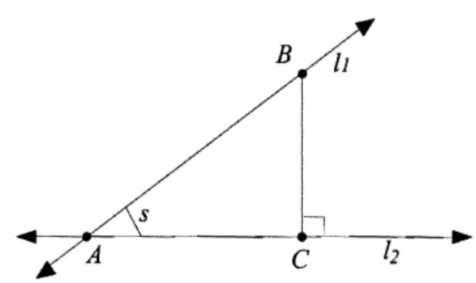
The Background

In his book, *Divine Proportions: Rational Trigonometry to Universal Geometry*, N.J. Wildberger states that the key concepts of rational trigonometry are far simpler and mathematically more natural than those of classic trigonometry, which uses concepts like $\cos(\theta)$ and $\sin^{-1}(\frac{1}{3})$. Classical trigonometry requires the use of a calculator or a table of trigonometric values to solve even the simplest problems and most of its solutions are approximations. But the study of triangles, the most fundamental of geometric topics as evident from the study of the ancient Greeks, should not require the use of advanced mathematics and transcendental functions. Professor Wildberger develops a new "rational trigonometry" in which the difficult concepts of length, requiring an understanding of irrational numbers, and angle measurement, based on the arbitrary number 360, are replaced by two much simpler and exact measurements called *quadrance* and *spread*.

The *quadrance* between two points A and B, symbolically written as $Q(A, B)$, measures the separation of two points. *Quadrance* is defined as the area of the square whose side length is AB, i.e., it is the square of the length of $\overline{AB}$. In Euclid's *Elements*, the Pythagorean Theorem is stated in terms of quadrances: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle." (Book I, Prop. 47)

In this problem set, three different symbols will be used for quadrance: $Q(A, B) = Q(\overline{AB}) = \hat{c}$. (The last is a shorthand notation often used with triangles.)

The measure of the separation of two lines, known as the *spread*, is a dimensionless number in the range of $[0, 1]$. Suppose two lines l_1 and l_2 intersect at point A. Choose a point B on l_1 ($\neq A$) and let C be the foot of the perpendicular from B to l_2. Then the spread s is defined as $s(l_1, l_2) = \dfrac{Q(B, C)}{Q(A, B)}$, a ratio of two areas. If $l_1 \perp l_2$, then $s = 1$. If $l_1 \parallel l_2$, then $s = 0$. In this problem set, three different symbols will be used for spread: $s(l_1, l_2) = s(\overleftrightarrow{AB}, \overleftrightarrow{AC}) = s_a$. (The last is a shorthand notation for the spread of two lines intersecting at point A.)



Five Basic Laws of Rational Trigonometry: Given any three points A_1, A_2, and A_3, define $Q_1(A_2, A_3), Q_2(A_1, A_3)$, and $Q_3(A_1, A_2)$.

> Triple Quadrance Formula: Three points A_1, A_2, A_3 are collinear if and only if $(Q_1 + Q_2 + Q_3)^2 = 2({Q_1}^2 + {Q_2}^2 + {Q_3}^2)$. This can also be rewritten in an asymmetrical (but useful) form: $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2$.

<u>The Pythagorean Theorem</u>: Two lines $\overleftrightarrow{A_1A_3}$ and $\overleftrightarrow{A_2A_3}$ are perpendicular if and only if $Q_1 + Q_2 = Q_3$.

Given $\triangle ABC$, let $\hat{a} = Q(B,C)$, $\hat{b} = Q(A,C)$, $\hat{c} = Q(A,B)$, $s_a = s(\overline{AB}, \overline{AC})$, $s_b = s(\overline{BA}, \overline{BC})$, and $s_c = s(\overline{CA}, \overline{CB})$.

<u>The Spread Law</u>: In any $\triangle ABC$, $\dfrac{s_a}{\hat{a}} = \dfrac{s_b}{\hat{b}} = \dfrac{s_c}{\hat{c}}$.

<u>The Cross Law</u>: In any $\triangle ABC$, $(\hat{a}+\hat{b}-\hat{c})^2 = 4\hat{a}\hat{b}(1-s_c)$. (Because of its asymmetry, the Cross Law can be written in three different ways.)

<u>The Triple Spread Formula</u>: In any $\triangle ABC$, $(s_a+s_b+s_c)^2 = 2({s_a}^2+{s_b}^2+{s_c}^2)+4s_as_bs_c$.

(N.B. These last three laws are analogs of the Law of Sines, the Law of Cosines, and Triangle Sum equals 180° and, therefore, are useful in solving triangles.)

Because all these laws are at most quadratic, all quadrances and spreads are exact values.

Although the concepts of quadrance and spread are easily understood, some caution is advised.
In *figure i*, when dealing with distances,
$XY + YZ = XZ$, but $Q(X,Y) + Q(Y,Z) \neq Q(X,Z)$.

figure i

In *figure ii*, when dealing with angles,
$\angle XPY + \angle YPZ = \angle XPZ$, but
$s(\overleftrightarrow{PX}, \overleftrightarrow{PY}) + s(\overleftrightarrow{PY}, \overleftrightarrow{PZ}) \neq s(\overleftrightarrow{PX}, \overleftrightarrow{PZ})$.

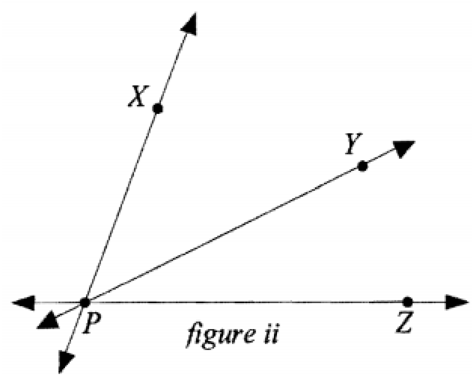


figure ii

In this problem set, you must think rationally even though it might drive you crazy because the irrational ideas of lengths and angle measures are so ingrained in our thinking!

Although thinking outside the box and clever approaches are usually encouraged, in this problem set, no solutions can use classical trigonometric functions or measures of angles. In addition, no credit will be given for any answer not supported by work.

The Problems

1. A rectangular piece of paper is folded in half to form a smaller rectangle similar to the original rectangle. If the original rectangle is $ABCD$ (with $AB > BC$) and $Q(\overline{BC}) = 1$, compute $Q(\overline{AB})$, $Q(\overline{AC})$, $s(\overline{AB}, \overline{AC})$, and $s(\overline{CB}, \overline{CA})$.

2. a. In $\triangle ABC$, $\hat{a} = 5$, $\hat{b} = 13$, and $\hat{c} = 10$. Compute s_a, s_b, and s_c.

   b. In $\triangle ABC$, $s_a = \frac{4}{5}$, $\hat{b} = 45$, and $\hat{c} = 25$. Compute s_b, s_c, and $\hat{a}$ for two different triangles.

   c. In $\triangle ABC$, $s_a = \frac{4}{5}$, $s_b = \frac{4}{13}$, and $\hat{c} = 16$. Compute $s_c, \hat{a}$, and $\hat{b}$ for two different triangles.

   d. In $\triangle ABC$, $s_a = \frac{1}{4}$, $\hat{a} = 2$, and $\hat{c} = 4$. Compute s_b, s_c, and $\hat{b}$ for two different triangles.

3. Using the notation defined in the Triple Quadrance Formula, prove the following theorem.

   The Midpoint Quadrance Theorem: If A_2 is the midpoint of $\overline{A_1A_3}$, then $Q_1 = \dfrac{Q_2}{4}$.

4. Triangle ABC is given, with $\hat{a} = 25, \hat{b} = 36$, and $\hat{c} = 16$.

   a. Compute $Q(\overline{BN})$, where $\overline{BN}$ is the altitude from vertex B to side $\overline{AC}$.

   b. Compute $Q(\overline{BM})$, where $\overline{BM}$ is the median from vertex B to side $\overline{AC}$.

   c. Compute $Q(\overline{BP})$, where $\overline{BP}$ is the vertex bisector from vertex B to side $\overline{AC}$. ($\overline{BP}$ is the vertex bisector of vertex B if and only if $s(\overleftrightarrow{BA}, \overleftrightarrow{BP}) = s(\overleftrightarrow{BP}, \overleftrightarrow{BC})$.)

5. Prove Stewart's Formula: In any $\triangle ABC$, if $\overline{BD}$ is any cevian from vertex B to side $\overline{AC}$, then $\hat{n}(\hat{x} + \hat{m} - \hat{c})^2 = \hat{m}(\hat{x} + \hat{n} - \hat{a})^2$.

   $\hat{a} = Q(B, C) \quad \hat{x} = Q(B, D)$
   $\hat{b} = Q(A, C) \quad \hat{m} = Q(A, D)$
   $\hat{c} = Q(A, B) \quad \hat{n} = Q(C, D)$

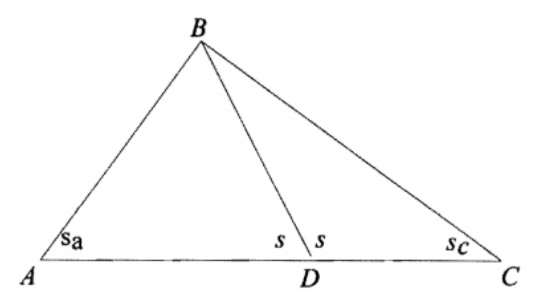


figure 5

6. a. Use *figure 6a* to prove half of the Spread Law: $\dfrac{s_a}{\hat{a}} = \dfrac{s_c}{\hat{c}}$.

   $\hat{a} = Q(B, C)$
   $\hat{b} = Q(A, C)$
   $\hat{c} = Q(A, B)$
   $\hat{x} = Q(B, D)$

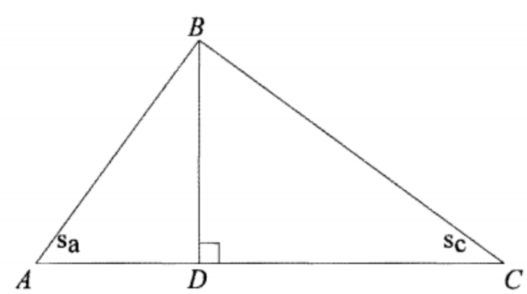


figure 6a

b. Use *figure 6b* along with the Pythagorean Theorem and the Triple Quadrance Law to prove the Cross Law.

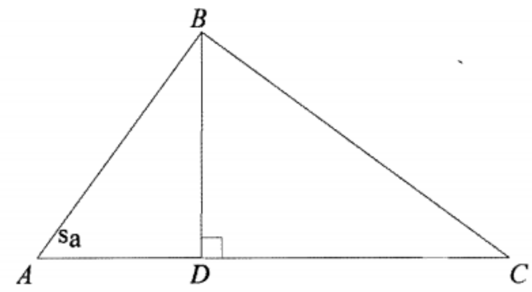


figure 6b

$\hat{a} = Q(B, C)$ $\quad \hat{x} = Q(B, D)$

$\hat{b} = Q(A, C)$ $\quad \hat{m} = Q(A, D)$

$\hat{c} = Q(A, B)$ $\quad \hat{n} = Q(C, D)$

c. The Spread Law can be extended: $\dfrac{s_a}{\hat{a}} = \dfrac{s_b}{\hat{b}} = \dfrac{s_c}{\hat{c}} = \dfrac{1}{D}$, so that $\hat{a} = s_a D$, $\hat{b} = s_b D$, and $\hat{c} = s_c D$. Use this fact and the Cross Law and prove the Triple Spread Law. (Hint: Begin by showing that the Cross Law can be rewritten as $(\hat{a} + \hat{b} + \hat{c})^2 = 2(\hat{a}^2 + \hat{b}^2 + \hat{c}^2) + 4\hat{a}\hat{b}s_c$.)

Coordinate Geometry and Rational Trigonometry

Quadrance: If $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then $Q(A, B) = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

Spread: A line in coordinate geometry is defined as $ax + by + c = 0$. Its slope is determined solely by the coefficients a and b. Therefore, the spread between the two lines, $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, is determined solely by the coefficients a_1, a_2, b_1, and b_2.

7. If l_1 is defined by $a_1x + b_1y + c_1 = 0$ and l_2 is defined by $a_2x + b_2y + c_2 = 0$, prove that

$$s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}.$$

Hint: Let l_1 and l_2 intersect at the origin. Then c's are eliminated from the equations. Then

a) Let $A = (0, 0)$, $B = (-b_1, a_1)$, and $C = (-\lambda b_2, \lambda a_2)$. (N.B. Point C can be any point on l_2 depending on the choice of λ. For one choice of λ, l_1 and l_2 will be perpendicular.)

b) Determine $Q(A, B)$, $Q(A, C)$, and $Q(B, C)$.

c) Use the (Rational) Pythagorean Theorem to determine the conditions on λ ($\lambda \neq 0$) necessary to make $\overline{BC} \perp \overline{AC}$, i.e., express λ in terms of a_1, a_2, b_1, and b_2.

d) Replace λ in $Q(B, C)$ with this equivalent expression to derive the spread formula.

The Solutions

1. Let $AB = x$. Then $\dfrac{x}{1} = \dfrac{1}{x/2} \Rightarrow x^2 = 2$ and $Q(\overline{AB}) = 2$ and, by the (Rational) Pythagorean Theorem, $Q(\overline{AC}) = 3$. Therefore, $s(\overline{AB}, \overline{AC}) = \dfrac{Q(\overline{BC})}{Q(\overline{AC})} = \dfrac{1}{3}$ and $s(\overline{CA}, \overline{CB}) = \dfrac{Q(\overline{AB})}{Q(\overline{AC})} = \dfrac{2}{3}$.

2. a. By the Cross Law, $(12+10-5)^2 = 4(13)(10) - 4(13)(10)s_a$. Solving for s_a gives $s_a = \dfrac{49}{130}$. Similarly $s_b = \dfrac{49}{50}$ and $s_c = \dfrac{49}{65}$.

   b. By the Cross Law, $(45+25-\hat{a})^2 = 4(45)(25)\left(1 - \dfrac{4}{5}\right)$. Solving for $\hat{a}$ gives $\hat{a} = 40$ or 100. By the Spread Law with $\hat{a} = 40$, $\dfrac{4/5}{40} = \dfrac{s_b}{45} = \dfrac{s_c}{25}$. Solving, this gives $s_b = \dfrac{9}{10}$ and $s_c = \dfrac{1}{2}$. Using $\hat{a} = 100$, results in $s_b = \dfrac{9}{25}$ and $s_c = \dfrac{1}{5}$.

   c. By the Triple Spread Formula, $\left(\dfrac{4}{5} + \dfrac{4}{13} + s_c\right)^2 = 2\left(\left(\dfrac{4}{5}\right)^2 + \left(\dfrac{4}{13}\right)^2 + (s_c)^2\right) + 4\left(\dfrac{4}{5}\right)\left(\dfrac{4}{13}\right)$. This simplifies to $0 = (s_c)^2 - \dfrac{80}{65}(s_c) + \dfrac{1024}{4225}$. Solving this quadratic, gives $s_c = \dfrac{64}{65}$ or $s_c = \dfrac{16}{65}$. By the Spread Law with $s_c = \dfrac{64}{65}$, $\dfrac{4/5}{\hat{a}} = \dfrac{4/13}{\hat{b}} = \dfrac{64/65}{16}$ and solving, this gives $\hat{a} = 13$ and $\hat{b} = 5$. Using $s_c = \dfrac{16}{65}$ and the Spread Law, results in $\hat{a} = 52$ and $\hat{b} = 20$.

   d. By the Cross Law, $(\hat{b}+4-2)^2 = 4(\hat{b})(4)\left(1 - \dfrac{1}{4}\right)$. This simplifies to $(\hat{b})^2 - 8(\hat{b}) + 4 = 0$ and solving, this gives $\hat{b} = 4 \pm 2\sqrt{3}$. By the Spread Law with $\hat{b} = 4 + 2\sqrt{3}$, $\dfrac{1/4}{2} = \dfrac{s_b}{4+2\sqrt{3}} = \dfrac{s_c}{4}$ and $\hat{b} = 4 - 2\sqrt{3}$, the result is $s_b = \dfrac{2-\sqrt{3}}{4}$ and $s_c = \dfrac{1}{2}$.

3. Since A_2 is the midpoint, $Q_1 = Q_2$. By the Triple Quadrance Formula,

$$(Q_1 + Q_1 + Q_3)^2 = 2\left((Q_1)^2 + (Q_1)^2 + (Q_3)^2\right) \text{ and}$$
$$4\left(Q_1\right)^2 + 4Q_1Q_3 + \left(Q_3\right)^2 = 4(Q_1)^2 + 2(Q_3)^2.$$

   This simplifies to $4Q_1Q_3 = (Q_3)^2$, implying $Q_1 = \dfrac{Q_3}{4}$.

4. a. Let $\hat{x}$ be the quadrance of the altitude from vertex B to side AC. By the Cross Law, $(16+36-25)^2 = 4(16)(36)(1-s_a)$. Solving for s_a results in $s_a = \dfrac{175}{256}$. But $s_a = \dfrac{\hat{x}}{16}$ and so $\hat{x} = \dfrac{175}{16}$.

   b. Let $\hat{x}$ be the quadrance of the median from vertex B to side AC. From #3, $Q(\overline{AM}) = 9$ and from #4a, $s_a = \dfrac{175}{256}$, and using the Cross Law, $(16+9-\hat{x})^2 = 4(16)(9)\left(1 - \dfrac{175}{256}\right)$. Solving for $\hat{x}$, gives $\hat{x} = \dfrac{77}{2}$ or $\hat{x} = \dfrac{23}{2}$. But $\hat{x} = \dfrac{77}{2}$ is too large for this triangle and so is $\hat{x} = \dfrac{23}{2}$. (In general, the median $\hat{x} = \dfrac{2\hat{a} + 2\hat{b} - \hat{c}}{4}$. This is easily proven by Stewart's Theorem — see #5.)

c. Let $\hat{x}$ be the quadrance of the vertex bisector (BP) from vertex B to side AC.

Lemma: Let $\hat{m} = Q(\overline{AP})$ and $\hat{n} = Q(\overline{CP})$, then, using the Spread Law twice, $\dfrac{s_a}{\hat{x}} = \dfrac{s(BA, BP)}{\hat{m}}$ and $\dfrac{s_c}{\hat{x}} = \dfrac{s(BC, BP)}{\hat{n}}$. Therefore, $\dfrac{s_a \cdot \hat{m}}{\hat{x}} = s(BA, BP)$ and $\dfrac{s_c \cdot \hat{n}}{\hat{x}} = s(BC, BP)$. Since $s(BA, BP) = s(BC, BP)$, $s_a \cdot \hat{m} = s_c \cdot \hat{n}$. By the Cross Law, $(36+25-16)^2 = 4(36)(25)(1-s_c)$ and so $\hat{n} = \dfrac{s_a \cdot \hat{m}}{s_c} = \dfrac{175/256}{7/16} \cdot \hat{m} = \dfrac{25}{16} \cdot \hat{m}$. (Note the similarity of this result to the classical angle bisector theorem.)

Since A, P, and C are collinear, $\left(\hat{m} + \dfrac{25}{16}\hat{m} + 36\right)^2 = 2\left((\hat{m})^2 + \left(\dfrac{25}{16}\hat{m}\right)^2 + 36^2\right)$. This simplifies to the quadratic

$$0 = 81(\hat{m})^2 - 47232\hat{m} + 331776,$$

whose roots are $\hat{m} = 576$ or $\hat{m} = \dfrac{64}{9}$. $\hat{m} = 576$ is too large for this triangle and so is $\hat{m} = \dfrac{64}{9}$. Using the Cross Law on $\triangle ABP$, $\left(16 + \dfrac{64}{9} - \hat{x}\right)^2 = 4(16)\left(\dfrac{64}{9}\right)\left(1 - \dfrac{175}{256}\right)$ and solving for $\hat{x}$, results in $\hat{x} = \dfrac{100}{9}$.

5. Using the Cross Law twice, gives $(\hat{x} + \hat{m} - \hat{c})^2 = 4(\hat{x})(\hat{m})(1-s)$ and $(\hat{x} + \hat{n} - \hat{a})^2 = 4(\hat{x})(\hat{n})(1-s)$. Therefore, $\dfrac{(\hat{x} + \hat{n} - \hat{a})^2}{\hat{n}} = 4(\hat{x})(1-s)$ and $\dfrac{(\hat{x} + \hat{m} - \hat{c})^2}{\hat{m}} = 4(\hat{x})(1-s)$. Therefore, $\dfrac{(\hat{x} + \hat{m} - \hat{c})^2}{\hat{m}} = \dfrac{(\hat{x} + \hat{n} - \hat{a})^2}{\hat{n}}$ and $\hat{n}(\hat{x} + \hat{m} - \hat{c})^2 = \hat{m}(\hat{x} + \hat{n} - \hat{a})^2$.

6. a. $s_a = \dfrac{\hat{x}}{\hat{c}}$ and $s_c = \dfrac{\hat{x}}{\hat{a}}$. Therefore, $\hat{c}s_a = \hat{x}$ and $\hat{a}s_c = \hat{x}$. Therefore, $\hat{c}s_a = \hat{a}s_c$ and $\dfrac{s_a}{\hat{a}} = \dfrac{s_c}{\hat{c}}$.

b. $s_c = \dfrac{\hat{x}}{\hat{a}}$ and $\hat{a}s_c = \hat{x}$. By the Pythagorean Theorem, $\hat{m} + \hat{x} = \hat{c}$ and $\hat{n} + \hat{x} = \hat{a}$. Therefore, $\hat{m} = \hat{c} - \hat{x} = \hat{c} - \hat{a}s_c$ and $\hat{n} = \hat{a} - \hat{x} = \hat{a} - \hat{a}s_c = \hat{a}(1 - s_c)$. Using the Triple Quadrance Law, $(\hat{n} + \hat{b} - \hat{m})^2 = 4\hat{n}\hat{b}$, and substituting for $\hat{m}$ and $\hat{n}$, produces $\left(\hat{a}(1 - s_c) + \hat{b} - (\hat{c} - \hat{a}s_c)\right)^2 = 4\hat{a}(1 - s_c)\hat{b}$, which simplifies to $(\hat{a} + \hat{b} - \hat{c})^2 = 4\hat{a}\hat{b}(1 - s_c)$.

c. Starting with the Cross Law,

$$\begin{aligned}
(\hat{a}+\hat{b}-\hat{c})^2 &= 4\hat{a}\hat{b}(1-s_c)\\
(\hat{a}+\hat{b})^2-2(\hat{a}+\hat{b})\hat{c}+(\hat{c})^2 &= 4\hat{a}\hat{b}(1-s_c)\\
(\hat{a}+\hat{b})^2-2(\hat{a}+\hat{b})\hat{c}+(\hat{c})^2+4(\hat{a}+\hat{b})\hat{c} &= 4\hat{a}\hat{b}(1-s_c)+4(\hat{a}+\hat{b})\hat{c}\\
(\hat{a}+\hat{b})^2+2(\hat{a}+\hat{b})\hat{c}+(\hat{c})^2 &= 4\hat{a}\hat{b}(1-s_c)+4\hat{a}\hat{c}+4\hat{b}\hat{c}\\
(\hat{a}+\hat{b}+\hat{c})^2 &= (\hat{a}+\hat{b}-\hat{c})^2+4\hat{a}\hat{c}+4\hat{b}\hat{c}\\
&= (\hat{a})^2+2\hat{a}\hat{b}+(\hat{b})^2-2\hat{a}\hat{c}-2\hat{b}\hat{c}+(\hat{c})^2+4\hat{a}\hat{c}+4\hat{b}\hat{c}\\
&= (\hat{a})^2+(\hat{b})^2+(\hat{c})^2+2\hat{a}\hat{b}+2\hat{a}\hat{c}+2\hat{b}\hat{c}\\
&= (\hat{a})^2+(\hat{b})^2+(\hat{c})^2+4\hat{a}\hat{b}-2\hat{a}\hat{b}+2\hat{a}\hat{c}+2\hat{b}\hat{c}+4\hat{a}\hat{b}s_c-4\hat{a}\hat{b}s_c\\
&= (\hat{a})^2+(\hat{b})^2+(\hat{c})^2 2\hat{a}\hat{b}+2\hat{a}\hat{c}+2\hat{b}\hat{c}+4\hat{a}\hat{b}s_c+4\hat{a}\hat{b}(1-s_c)\\
&= (\hat{a})^2+(\hat{b})^2+(\hat{c})^2 2\hat{a}\hat{b}+2\hat{a}\hat{c}+2\hat{b}\hat{c}+4\hat{a}\hat{b}s_c+(\hat{a}+\hat{b}+\hat{c})^2\\
&= 2(\hat{a})^2+2(\hat{b})^2+2(\hat{c})^2+4\hat{a}\hat{b}s_c.
\end{aligned}$$

Extended Spread Law: $\dfrac{s_a}{\hat{a}}=\dfrac{s_b}{\hat{b}}=\dfrac{s_c}{\hat{c}}=\dfrac{1}{D}$, so that $\hat{a}=s_aD$, $\hat{b}=s_bD$, and $\hat{c}=s_cD$. Substitute these expressions for $\hat{a},\hat{b}$, and $\hat{c}$ into the equations above to yield:

$$\begin{aligned}
(s_aD+s_bD+s_cD)^2 &= 2(s_aD)^2+2(s_bD)^2+2(s_cD)^2+4s_aDs_bDs_c\\
D^2(s_a+s_b+s_c)^2 &= 2D^2\left((s_a)^2+(s_b)^2+(s_c)^2\right)+4D^2s_as_bs_c\\
(s_a+s_b+s_c)^2 &= 2\left((s_a)^2+(s_b)^2+(s_c)^2\right)+4s_as_bs_c.
\end{aligned}$$

7. Let l_1 and l_2 intersect at $A(0,0)$. Therefore, $l_1=a_1x+b_1y=0$ and $l_2=a_2x+b_2y=0$. Now let $B=(-b_1,a_1)$ and $C=(-\lambda b_2,\lambda a_2)$. Clearly A is on l_1 and B is on l_2. $Q(\overline{AB})={a_1}^2+{b_1}^2$, $Q(\overline{AC})=\lambda^2({a_2}^2+{b_2}^2)$, and $Q(\overline{BC})=(-\lambda b_2+b_1)^2+(\lambda a_2-a_1)^2$. To determine the value of λ that will make $\overleftrightarrow{BC}\perp\overleftrightarrow{AC}$, the Pythagorean Theorem must hold, i.e., $Q(\overline{AB})=Q(\overline{AC})+Q(\overline{BC})$. Thus

$$\begin{aligned}
{a_1}^2+{b_1}^2 &= \lambda^2({a_2}^2+{b_2}^2)+(-\lambda b_2+b_1)^2+(\lambda a_2-a_1)^2\\
&= \lambda^2{a_2}^2+\lambda^2{b_2}^2+(-\lambda b_2)^2-2\lambda b_1b_2+{b_1}^2+(\lambda a_2)^2-2\lambda a_1a_2+{a_1}^2
\end{aligned}$$

which implies

$$\begin{aligned}
0 &= 2\lambda^2{a_2}^2+2\lambda^2{b_2}^2-2\lambda b_1b_2-2\lambda a_1a_2\\
&= 2\lambda(\lambda{a_2}^2+\lambda{b_2}^2-b_1b_2-a_1a_2).
\end{aligned}$$

Therefore, $\lambda=0$ (which implies $A=C$, and therefore, $l_1\perp l_2$) or $\lambda{a_2}^2+\lambda{b_2}^2-b_1b_2-a_1a_2=0$, implying $\lambda = \dfrac{a_1a_2+b_1b_2}{{a_2}^2+{b_2}^2}$. Therefore,

$$\begin{aligned}
Q(\overline{BC}) &= \left(b_1 - \frac{a_1a_2+b_1b_2}{{a_2}^2+{b_2}^2}\cdot b_2\right)^2 + \left(\frac{a_1a_2+b_1b_2}{{a_2}^2+{b_2}^2}\cdot a_2 - a_1\right)^2 \\
&= \left(\frac{b_1({a_2}^2+{b_2}^2)-(a_1a_2+b_1b_2)b_2}{{a_2}^2+{b_2}^2}\right)^2 + \left(\frac{(a_1a_2+b_1b_2)a_2 - a_1({a_2}^2+{b_2}^2)}{{a_2}^2+{b_2}^2}\right)^2 \\
&= \frac{(b_1{a_2}^2+b_1{b_2}^2-a_1a_2b_2-b_1{b_2}^2)^2+(a_1{a_2}^2+b_1b_2a_2-a_1{a_2}^2-a_1{b_2}^2)^2}{({a_2}^2+{b_2}^2)^2} \\
&= \frac{(b_1{a_2}^2-a_1a_2b_2)^2+(b_1b_2a_2-a_1{b_2}^2)^2}{({a_2}^2+{b_2}^2)^2} \\
&= \frac{(a_2(b_1a_2-a_1b_2))^2+(b_2(b_1a_2-a_1b_2))^2}{({a_2}^2+{b_2}^2)^2} \\
&= \frac{{a_2}^2(b_1a_2-a_1b_2)^2+{b_2}^2(b_1a_2-a_1b_2)^2}{({a_2}^2+{b_2}^2)^2} \\
&= \frac{\left(({a_2}^2+{b_2}^2)^2\right)(b_1a_2-a_1b_2)^2}{({a_2}^2+{b_2}^2)^2} \\
&= \frac{(b_1a_2-a_1b_2)^2}{{a_2}^2+{b_2}^2}.
\end{aligned}$$

Therefore, $s(l_1,l_2) = \dfrac{Q(\overline{BC})}{{a_1}^2+{b_1}^2} = \dfrac{(b_1a_2-a_1b_2)^2}{({a_1}^2+{b_1}^2)({a_2}^2+{b_2}^2)}$.

Partitions

The Background

Starting with any positive integer, a *partition* is simply a list of positive integers, given in decreasing order, that adds up to the starting integer. For example, some partitions of 12 are $(5, 3, 2, 1, 1)$, $(4, 4, 4)$, and (12). The individual numbers that make up the partition are called the *parts*. There are no restrictions on the parts other than they have to be in decreasing order.

The only partition of 1 is (1). For 2 there are two: (2) and $(1, 1)$. There are three partitions of 3 and five of 4: (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, and $(1, 1, 1, 1)$. Let $p(n)$ represent the number of partitions of the number n. Thus we have just seen that $p(4) = 5$.

In the problems below, the instruction *list* means to write a list of all the partitions asked for in the problem, while the instruction *determine the number* asks for just a number, with no proof necessary. In particular, when asked to determine the number of partitions of a particular type, you do not have to list all the partitions in your answer (even though you may have solved the problem by listing them all out and counting them). Additionally, you may feel free to use the "floor" notation: $\lfloor x \rfloor$ is the largest integer less than or equal to x. So $\lfloor 3.5 \rfloor = 3$, $\lfloor -3.5 \rfloor = -4$, and $\lfloor 5 \rfloor = 5$.

There are 40 points available on this contest.

The Problems

1. a. It is a fact that $p(6) = 11$. List the eleven partitions of 6.

   b. Determine $p(10)$, the number of different partitions for 10 (just give the answer—not the list!).

2. a. List the partitions of 11 whose parts consist only of odd numbers.

   b. List the partitions of 11 whose parts are all different.

   c. Determine the number of partitions of 12 whose parts consist only of odd numbers.

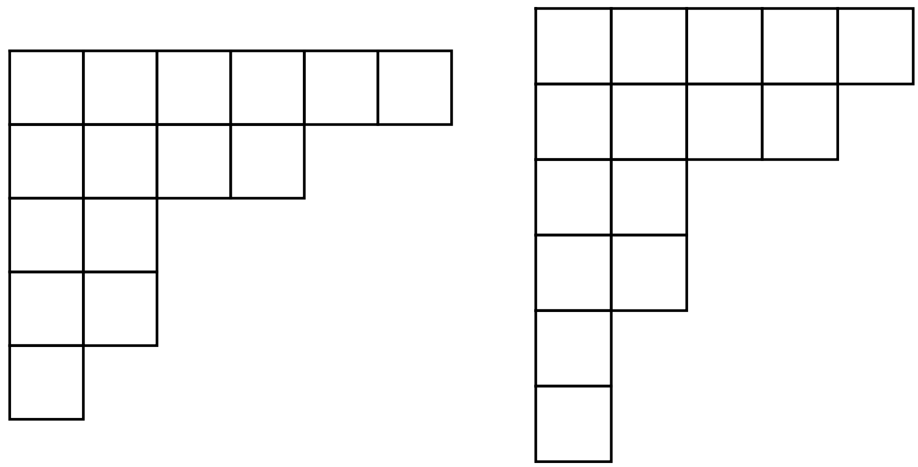
   d. Determine the number of partitions of 12 whose parts are all different.

Sometimes we manipulate a partition to turn it into another partition. For example, if a partition has any even part we can break that part into two halves, and repeat until there are no even parts left. For example, $(12, 7, 4, 4, 3, 1) \longrightarrow (7, 6, 6, 3, 2, 2, 2, 2, 1) \longrightarrow (7, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.

3. a. Describe the reverse of this transformation.

   b. If the reverse transformation was carried out on $(7, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ (the final result of the example above) as far as possible, what is the resulting partition?

Partitions can be represented visually by means of a *Young diagram*. This is simply a bunch of rows of boxes, aligned on their left hand sides, with the parts telling the number of boxes in each row:



The two Young diagrams shown above are the diagrams corresponding to the partitions $(6, 4, 2, 2, 1)$ and $(5, 4, 2, 2, 1, 1)$ respectively. Notice how these two diagrams are "flips" of each other. More precisely, if we were to use the parts of the partition as the number of boxes in a column instead of a row, $(6, 4, 2, 2, 1)$ would give the second diagram and $(5, 4, 2, 2, 1, 1)$ would give the first. Such pairs of diagrams are called *conjugates* of each other, and the corresponding partitions are also called conjugates.

4. Draw the Young diagram for $(7, 5, 4, 4, 3, 2, 2, 2, 1)$. Then draw the conjugate diagram and find the conjugate partition.

A partition that is its own conjugate is called *self-conjugate*. These have diagrams that are symmetric across their NW-SE diagonal. For instance, the partition $(3, 1, 1)$ of 5 is self-conjugate. There are two self-conjugate partitions of 8, namely $(4, 2, 1, 1)$ and $(3, 3, 2)$.

5. Find the smallest positive integer that has at least three different self-conjugate partitions. List these partitions.

Manipulating partitions or their Young diagrams can often lead to quick proofs of deep relationships, sometimes even relationships between partitions that seem to have no relation between each other. Conjugation often figures into either the relationship or the manipulation.

6. a. Find, with explanation, the number of partitions of a general number n into exactly two parts.

   b. Using conjugation, prove that the number of partitions of any number n into exactly two parts is equal to the number of partitions of n into parts where the largest part is two.

   c. Prove that the number of partitions of a number n into at most five parts is the same as the number of partitions of n into parts none of which is larger than 5.

7. Prove that the number of self-conjugate partitions of a number is equal to the number of partitions of that number into different odd parts.

8. Prove that the number of partitions of a number n where none of the parts are one is $p(n) - p(n-1)$.

9. Prove that the number of partitions of a number into a list of (different) consecutive integers is equal to the number of odd factors of that number. For example, 15 has four odd factors, (1, 3, 5, and 15) and four partitions into consecutive integers: $(1, 2, 3, 4, 5)$, $(4, 5, 6)$, $(7, 8)$, and (15).

10. Prove that the number of partitions of a number into all odd parts equals the number of partitions of that number into all different parts.

11. Explain why the number of partitions of n into all different parts is given by the coefficient of x^n in the expansion of the product $(1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^n)$.

The Solutions

1. a. The eleven partitions of 6 are (6), $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, and $(1,1,1,1,1,1)$.

   b. There are 42 of them. Since we're just getting started, the easiest way to do this is to exhaustively list them all. So here we go:

   $$\begin{array}{l}
   (10),(9,1),(8,2),(8,1,1),(7,3),(7,2,1),(7,1,1,1)\\
   (6,4),(6,3,1),(6,2,2),(6,2,1,1),(6,1,1,1,1)\\
   (5,5),(5,4,1),(5,3,2),(5,3,1,1),(5,2,2,1),(5,2,1,1,1),(5,1,1,1,1,1)\\
   (4,4,2),(4,4,1,1),(4,3,3),(4,3,2,1),(4,3,1,1,1),(4,2,2,2),(4,2,2,1,1)\\
   (4,2,1,1,1,1),(4,1,1,1,1,1,1)\\
   (3,3,3,1),(3,3,2,2),(3,3,2,1,1),(3,3,1,1,1,1),(3,2,2,2,1),(3,2,2,1,1,1)\\
   (3,2,1,1,1,1,1),(3,1,1,1,1,1,1,1)\\
   (2,2,2,2,2),(2,2,2,2,1,1),(2,2,2,1,1,1,1),(2,2,1,1,1,1,1,1)\\
   (2,1,1,1,1,1,1,1,1),(1,1,1,1,1,1,1,1,1,1)
   \end{array}$$

   There are better ways to count them, such as totaling by the largest part, but this will do for now.

2. a. (11), $(9,1,1)$, $(7,3,1)$, $(7,1,1,1,1)$, $(5,5,1)$, $(5,3,3)$, $(5,3,1,1,1)$, $(5,1,1,1,1,1,1)$, $(3,3,3,1,1)$, $(3,3,1,1,1,1,1)$, $(3,1,1,1,1,1,1,1,1)$, and $(1,1,1,1,1,1,1,1,1,1,1)$. There are twelve of them.

   b. (11), $(10,1)$, $(9,2)$, $(8,3)$, $(7,4)$, $(6,5)$, $(8,2,1)$, $(7,3,1)$, $(6,4,1)$, $(6,3,2)$, $(5,4,2)$, and $(5,3,2,1)$. There are twelve of these, also.

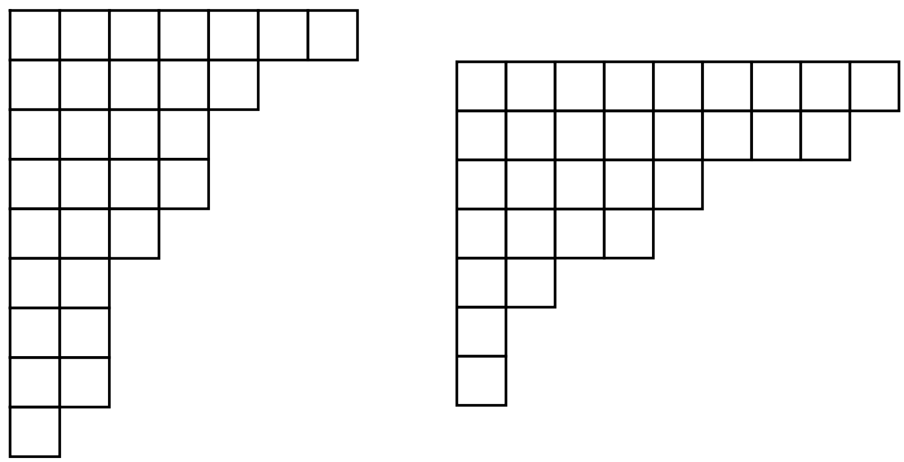
   c. 15. (Each of the answers in part (a) could have a one added as a last part, and you can additionally have $(9,3)$, $(7,5)$, and $(3,3,3,3)$.)

   d. 15. (Each of the answers in part (b) could have its largest part increased by one, and you can additionally have $(6,5,1)$, $(5,4,3)$, and $(5,4,2,1)$. It's no coincidence that (a) and (b) have the same answer, as do (c) and (d)—see problem 10.)

3. a. Since the transformation takes an even part and breaks it into two equal smaller parts, the reverse will take two equal small parts and combine them into a single part that is double their size.

   b. $(7,3,3,3,3,3,1,1,1,1,1,1,1,1,1) \longrightarrow (7,6,6,3,2,2,2,2,1) \longrightarrow (12,7,4,4,3,1) \longrightarrow (12,8,7,3,1)$. Notice how the transformation, if carried out as long as possible, always leads to a partition with all odd parts, because any even part must be split in two. The reverse transformation will always lead to a partition with all different parts, because if any parts are the same they must be combined into a single part that is twice as large.

4. The Young diagram for $(7, 5, 4, 4, 3, 2, 2, 2, 1)$ is as shown below. Taking the mirror image along the diagonal gives the conjugate diagram, which represents the partition $(9, 8, 5, 4, 2, 1, 1)$.



5. The answer is 12. It is easy to see that 9 has two self-conjugate partitions ($(3, 3, 3)$ and $(5, 1, 1, 1, 1)$), 10 also has two ($(4, 3, 2, 1)$ and $(5, 2, 1, 1, 1)$), as does 11 ($(4, 3, 3, 1)$ and $(6, 1, 1, 1, 1, 1)$). But 12 has three different self-conjugate partitions: $(4, 4, 2, 2)$, $(5, 3, 2, 1, 1)$, and $(6, 2, 1, 1, 1, 1)$.

6. a. To break a number n into two parts, think about how large the smaller part can be. We can make this smaller part $1, 2, 3, \dots$ up to $n/2$ if n is even, but only up to $(n-1)/2$ is n is odd. A quick way to write this is $\lfloor n/2 \rfloor$.

   b. If you draw the Young diagram for a partition with exactly two parts there are exactly two rows of boxes in the diagram. When you flip this, there is at least one row with two boxes in it, and no row has more than two boxes. So the conjugate of a partition with exactly two parts is a partition whose largest part is two. Thus conjugation gives a one-to-one correspondence between partitions into exactly two parts and partitions with largest part equal to two, so there are the same number of each.

   c. The reasoning is the same as in the previous paragraph, except for we must deal with the "at most" phrase rather than the "exact" phrase. So the diagram for a partition with at most five parts has at most five rows of boxes. When flipped, no row can have more than five boxes, so conjugation gives a one-to-one correspondence between partitions into at most five parts and partitions with largest part at most five.

7. Consider the Young diagram for a self-conjugate transformation. There are as many boxes in the first row as there are in the first column, and the top-left box is counted in both. So the total number of boxes in the top row and first column is odd. Now peel off the top row and column and consider what's left. Because the original diagram was symmetric across the diagonal, so is the remainder, so what's left after peeling off the first row and column is a Young diagram for a self-conjugate partition of some other (smaller!) number. So its

top row and first column also form an odd number of boxes, and so forth. So by peeling apart the Young diagram by layers, we see that a self-conjugate partition has a corresponding partition into all odd parts. The correspondence for the three self-conjugate partitions of 12 are $(4,4,2,2) \longleftrightarrow (7,5)$, $(5,3,2,1,1) \longleftrightarrow (9,3)$, and $(6,2,1,1,1,1) \longleftrightarrow (11,1)$.

But why are all the odd numbers *different*? Well consider the first and second rows in a Young diagram. The second row has *at most* as many boxes as the first, and may have fewer. And when we peel off the first column, that leaves the second row with *strictly fewer* boxes than the original first row. That means each odd number we peel off is strictly smaller than the previous one. Thus, each self-conjugate partition corresponds to a partition into different odd numbers.

We can also go backwards, by taking each odd number and "bending" it in the middle. For instance, 13 would "bend" into a row of 7 and a column of 7, overlapping in the top-left square of course. If we have all different odd numbers, then we can bend each in the middle and nestle them inside each other in order of decreasing size to obtain a self-conjugate partition.

In this way, there is a one-to-one correspondence between self-conjugate partitions and partitions into different odd numbers, so there must be the same number of each.

8. Keep in mind that there are $p(n)$ partitions of n altogether. We can split these into two types: those that have a part of one and those that don't. We are trying to count the latter. But it is much easier to count the former! For if you have any partition of n and at least one of the parts is one, if you delete that part you have a partition of $n-1$. So there are $p(n-1)$ of these, and therefore $p(n) - p(n-1)$ partitions that do not have a part of size one.

9. Say we are breaking n into k consecutive parts, and let those parts be a, $a+1$, $a+2$, $\dots$, $a+(k-1)$. The number k is the number of parts and can be any positive integer (keep in mind that we are counting a single part as a list of (one) consecutive integer!). So $n = a + (a+1) + \cdots + (a+(k-1))$. The right-hand side is an arithmetic series, so we apply the familiar formula to obtain $n = ka + k(k-1)/2$, or $2n = k(2a+k-1)$. Now if k is odd, $2a+k-1$ is even and vice-versa (we say they have *opposite parity*).

   Now if xy is a way of factoring n, $(2x)y$ is a way of factoring $2n$. So each odd factor of n is also an odd factor of $2n$ and vice versa. And since a will always be at least one, $2a+k-1$ is larger than k. So given a way to factor $2n$ into an odd number and an even number, set k to be the smaller factor and $2a+k-1$ to be the larger factor and we have the list of consecutive numbers that add up to n.

   For example, when $n = 15$, we have $2n = 30$ which factors as:

   - 1×30, giving $k = 1$ and $2a = 30$ so $a = 15$ and the partition is (15);
   - 2×15, giving $k = 2$ and $2a + 1 = 15$ so $a = 7$ and the partition is $(7,8)$;
   - 3×10, giving $k = 3$ and $2a + 2 = 10$ so $a = 4$ and the partition is $(4,5,6)$;
   - 5×6, giving $k = 5$ and $2a + 4 = 6$ so $a = 1$ and the partition is $(1,2,3,4,5)$.

10. The solution to this problem lies in the transformation described prior to problem 3. As we noticed when investigating this transformation, if we continue to apply it as many times as

possible to a partition, the result is a partition with no even parts because any even part must be split in half. If we apply the reverse transformation as many times as possible, the result is a partition with no two equal parts, because those parts could be combined. So the idea is that given an partition with all odd parts, we apply the reverse transformation until we obtain a partition with all different parts, and given a partition with all different parts we apply the transformation until we obtain a partition with all odd parts. In this way, for each partition with all odd parts we can find a related one with all different parts, and vice versa. We will be done if we can show that this correspondence is one-to-one.

To show this, keep in mind that every positive integer can be expressed uniquely as $2^a b$ where a is a non-negative integer and b is an odd positive integer. If $a > 0$ then the integer is even, and the transformation would take it to two copies of $2^{a-1}b$. If you have multiple copies of the same number $2^a b$ then the reverse transformation takes two of those copies and gives you one copy of $2^{a+1}b$. The important thing to notice is that the transformation and its reverse *never change* b. And since the transformation ends with all odd parts, that part must be b.

So now we can easily see why the transformation and its reverse describe a one-to-one correspondence. Starting with the partition with all different parts, collect those parts that have the same b. In fact, those parts will be different powers of 2 times b. Let's say the total of those power of 2 is k. As we apply the transformation, these parts will dissolve into a lot of copies of b. In fact, into k copies. Now when we apply the reverse transformation, copies of b will be turned into copies of $2b$, then $4b$, and so on until there are no two copies of the same number. Thus, we will have turned all copies of b into powers of 2 times b. Since there is a unique way of writing any number k as a sum of powers of 2 (think binary notation!) we have shown that starting with a partition into different parts, applying the transformation to get a partition into odd parts, and then applying the reverse transformation returns to the original partition. So the transformation and its reverse must be one-to-one.

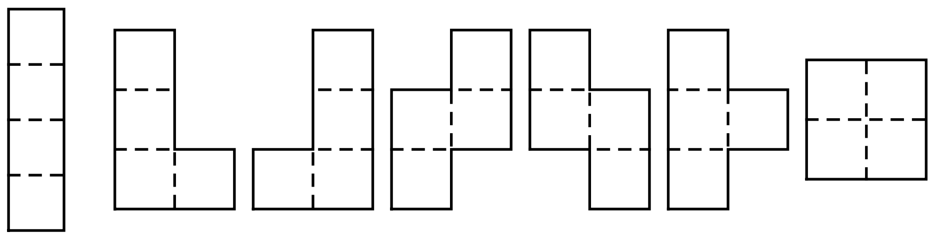
For example, here's a partition with different parts: $(52, 48, 46, 39, 30, 24, 21, 16, 15, 12, 10, 5, 2, 1)$. We could write this as $(2^2\cdot 13, 2^4\cdot 3, 2\cdot 23, 1\cdot 39, 2\cdot 15, 2^3\cdot 3, 1\cdot 21, 2^4\cdot 1, 1\cdot 15, 2^2\cdot 3, 2\cdot 5, 2\cdot 1, 1\cdot 1)$. If we collect the parts whose b is, say, 3, we have the 48, the 24, and the 12—a total of 28 3's. Then when we use the reverse transformation, we first obtain 14 6's, then 7 12's. One of these is left behind and we continue combining the other six to get 3 24's and a 12, and two of those 24's are combined to end with 48, 24, and 12, exactly the multiples of 3 we started with. None of the other parts interferes in any way with our 3's

11. Let's say we multiplied out the expression $(1+x)(1+x^2)\cdots(1+x^n)$ but did *not* collect like terms. Each term in the resulting expression then comes from multiplying a bunch of *different* powers of x with a bunch of ones. Since multiplying powers of x causes their exponents to add, each term with total exponent k comes from some partition of k into different parts (different because no number occurs twice as an exponent in $(1+x)(1+x^2)\cdots(1+x^n)$). So when we do combine like terms, the coefficient of x^k represents the number of ways to write k as a sum of different numbers less than or equal to n. In particular, since a partition of n itself never needs any parts larger than n, the number of partitions of n into different parts will be the coefficient of x^n in our expanded product.

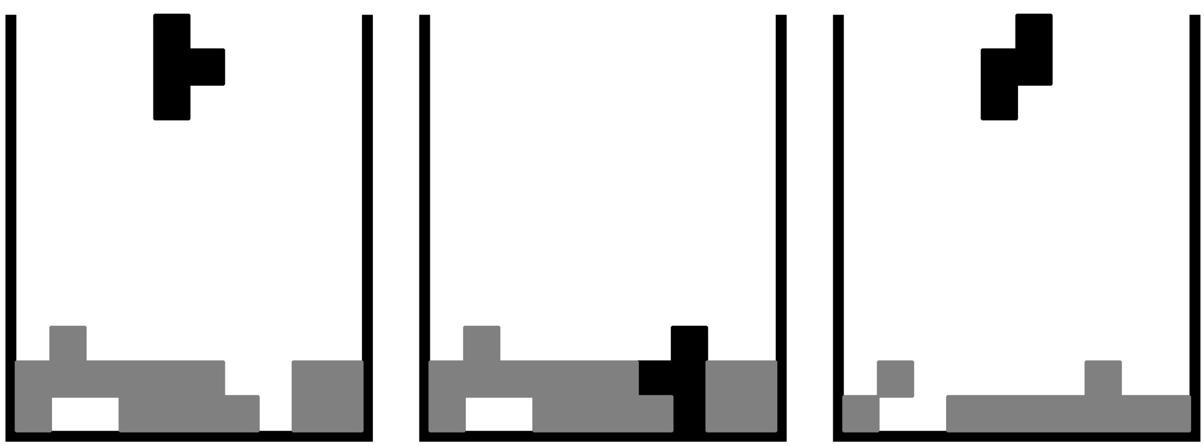
Falling Blocks

The Background

This year marks the thirtieth anniversary of the computer game Tetris®. In this game, there is an area ten squares wide and a number of squares tall. Pieces chosen randomly from among the seven "tetrominoes" made up of four squares glued together, as shown below, fall from the top of the screen.



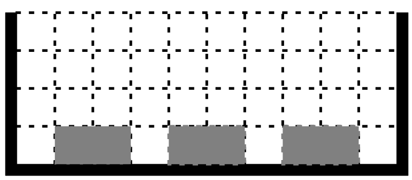
As the pieces fall, the player may rotate them or slide them left or right, but once they touch a piece below them they stick in place. If the player is able to fit the pieces together so as to leave no gaps in a row, that row disappears and all the blocks above fall to leave more room for new blocks. Otherwise the screen fills up with blocks and the game ends. In the example below, the player has been playing for a little while; the gray area representing earlier pieces that have become stuck, and the black T-shaped block has just started to fall. The player rotates it and slides it over to fit into the gaps in the bottom rows, and in the second diagram the T has fallen into the gap. Note that the bottom row still has holes in it, but the second row has now been filled in completely. So the second row will disappear and all blocks above it will drop by one row—even if it means defying gravity a little—leaving the grid in the third diagram, where another piece is about to start falling.

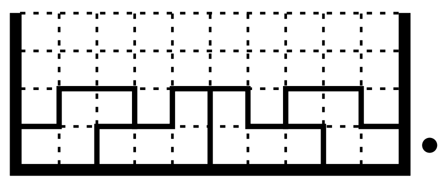


This problem is about the kinds of patterns that can be formed on a Tetris® screen, and similar related puzzles. Obviously, if the pieces are chosen at random, the player has little control over the patterns that form. But what if the player could choose the piece to come next? Can any pattern be left on the screen? The answer to this is easily seen to be *no*. For instance, there can never be just a single square—or any odd number of squares—because the pieces all are made of four squares, and squares disappear from the screen only in complete rows of ten squares. Since both four and ten are even numbers, the player can never have an odd number of squares remaining. But what *is* possible?

In the problems that follow, you will be asked to find a way to end up with the given pattern. The answer will be to draw the outlines of the pieces where they end up, drawing a small circle next to rows that disappear due to being complete. An example appears below. In your solutions, you may use as many lines as necessary—only a couple lines will be shown for the patterns to save space, not as an indication of how tall the solution should be. That is, your solution may need many lines to disappear before you can obtain the desired pattern, but unless otherwise instructed any solution will be acceptable regardless of how tall it ends up being compared to the original problem. Other questions may ask you to prove that certain patterns are impossible. Be careful to check the width of each puzzle, because some of the puzzles are not the standard ten columns wide.

Example: Create the given pattern. A solution is on the right; there are other solutions (such as turning the given one upside-down). Note the small circle at the right to indicate that the bottom row will disappear, allowing the desired pattern to fall into place along the bottom of the screen.

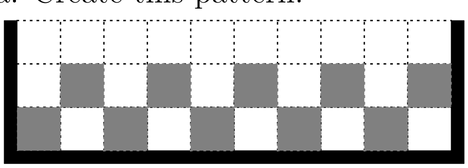




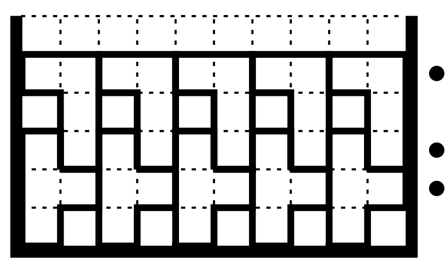
<u>**The Problems**</u>

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. a. Create this pattern:

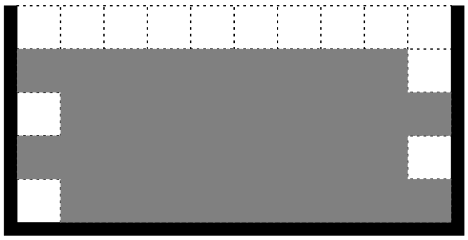


b. Faced with having to create this pattern, suppose that a careless ARML team submitted the answer shown to the right. The team would not have received any points, because this proposed solution doesn't actually create the pattern desired. Explain why this proposed solution does not work.

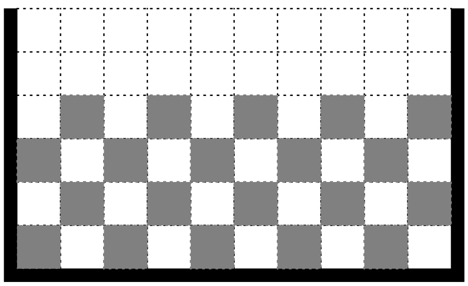


2. Create each of the following patterns:

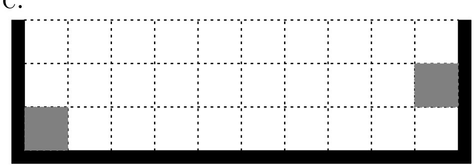
a.



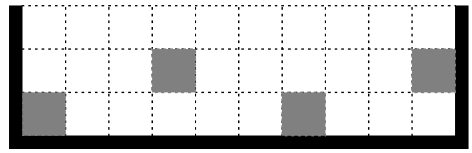
b.



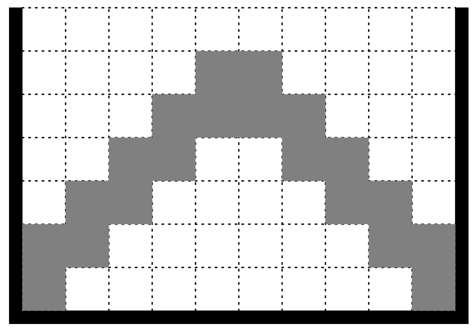
c.



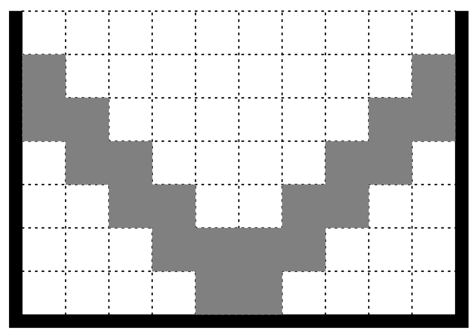
d.



e.



f.



3. a. If the puzzle is ten spaces wide and the pattern to be obtained consists of exactly two squares (that is, after complete rows disappear, only two squares are left that didn't disappear), explain why the number of complete rows that disappear in the solution must be odd.

   b. If the puzzle is ten spaces wide, there are $\binom{10}{2} = 45$ different patterns where two squares remain on the bottom line and all other squares have disappeared. Some of these patterns can be achieved with just one row disappearing, while others require at least three rows to disappear. (None require more than three.) How many of each kind are there? (You do not have to provide all 45 solutions, just the numbers.)

4. If you try to repeat part (b) of the previous problem but without using the "T"-shaped block you will discover that some patterns still require only one line to disappear, others require three lines to disappear, but now some are impossible altogether. Again, none of the possible patterns require more than three disappearing lines.

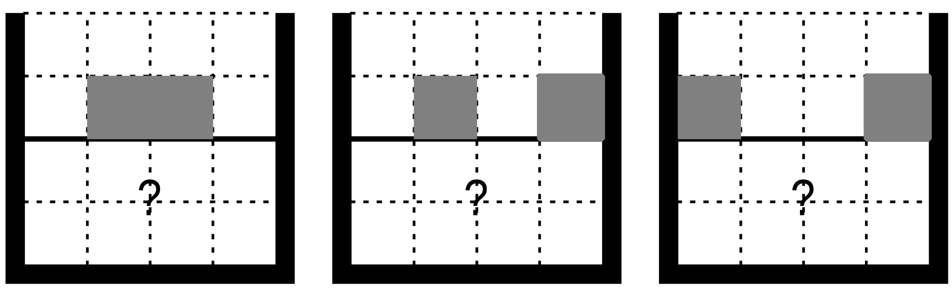
   a. Determine how many of the 45 patterns still require only one line, how many now require three, and how many are now impossible.

   b. Give a simple explanation why some patterns are impossible without the "T"-block.

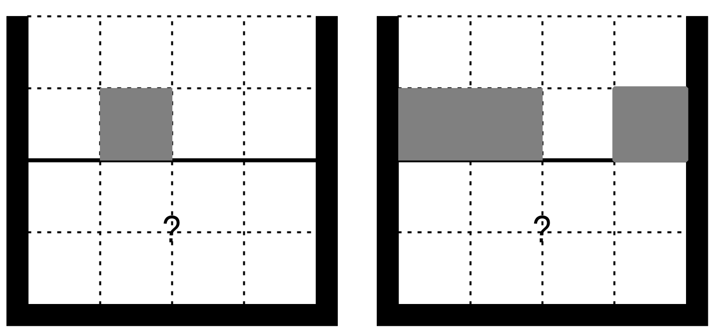
It turns out that *every* pattern with an even number of squares can be created by dropping appropriate blocks in an appropriate order, and that every such pattern can be "cleared"—by dropping the right blocks in the right places every square will eventually be part of a complete row and disappear, leaving the entire screen empty. The proof of these facts is not any more difficult than the rest of these problems, but has too many little parts that need verification to include the whole solution. But we can model the proof for a simplified case.

5. Consider if the game were played on a screen only four squares wide, and the blocks that fell were "triominoes"—three squares glued together along their edges. There are only two different triominoes. One is the straight triomino, three squares glued together with their centers in a straight line. The other is the "L"-triomino, three squares glued together with their centers in a right triangle. We will prove that any pattern on the screen can be cleared. Start by showing that no matter what the top line of the pattern is, by dropping the correct blocks it can be cleared, leaving a pattern with fewer uncleared lines:

   a. Show how to drop blocks to clear the top line if the top line is one of these (or their left-right mirror image) (the question mark indicates that we don't care what is below the top line, only that we can clear the top line):



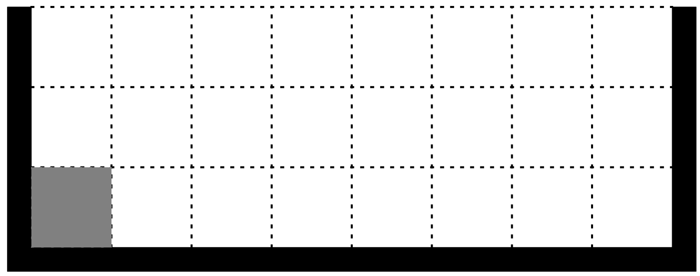
   b. Show how to clear the top line of these patterns (or their mirror images):



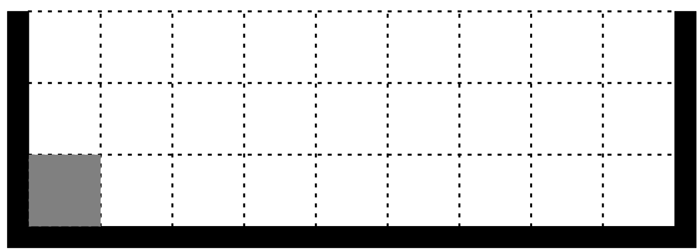
   c. Show how to clear any remaining possible top-line patterns.

   d. Complete the proof that any pattern in a four-square wide screen can be completely cleared.

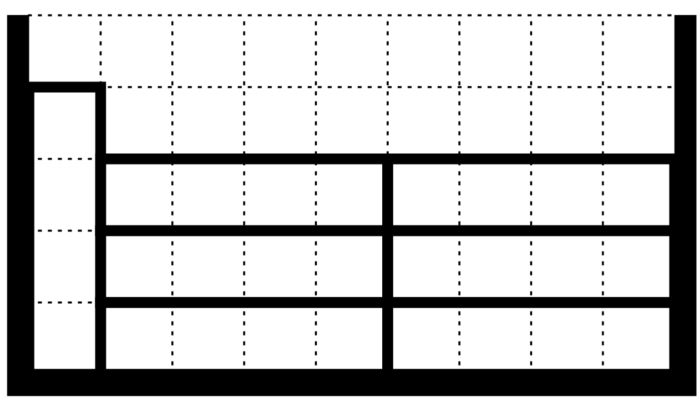
6. If you are playing on a screen that is eight squares wide but the only block you can use is the straight triomino, prove that it is impossible to either create or clear the pattern with just a single square in the corner.



7. This time, we are playing on a *nine*-square-wide screen, with the regular tetrominoes. Let's say we want to create the pattern with just a single square in the lower left-hand corner:



If we allow ourselves only one of the seven possible blocks, the straight tetromino can certainly solve the following puzzle:

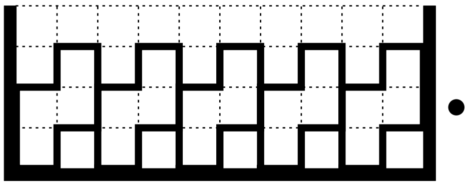


It turns out that no other tetromino can create this pattern by itself. But several pairs of the other blocks can create the pattern. Find *three* solutions to the pattern, each using only two different tetromino blocks. Each solution should use a different pair of blocks. Don't forget that the mirror image of a non-symmetric block is a *different* block!

The Solutions

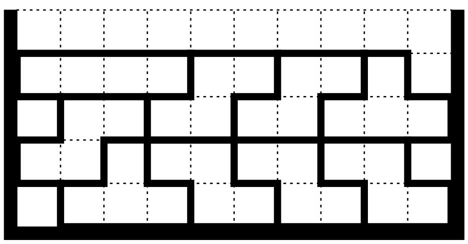
Note: Many, if not all, of the pattern-finding problems have multiple solutions. Only one will be given, but any correct solution will score full points.

1. a.

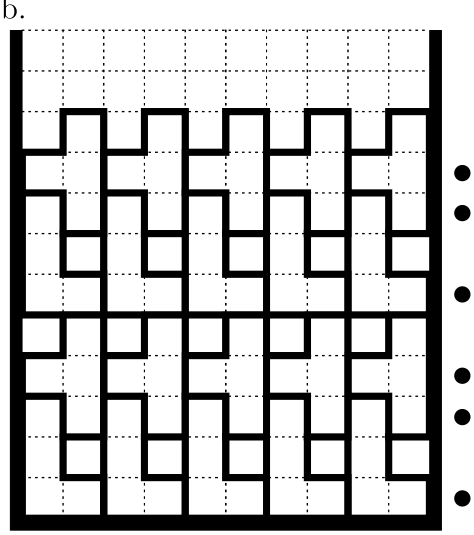


b. As the blocks are dropped into place to form the proposed solution, the final "T"-shaped block must be placed before the "L"-shaped block that sits on top of it. But as soon as that "T"-block is placed, the row indicated by the bottom-most circle will disappear. Then, when the "L"-shaped block that rests on top of it comes down, the protrusion from the "T" upon which it is supposed to come to rest won't be there, and it will fall too far to leave the desired pattern.

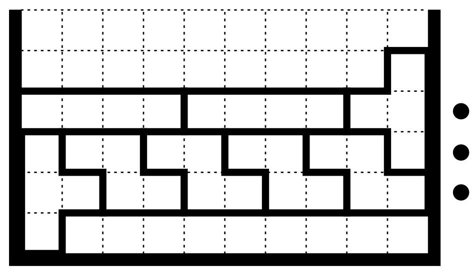
2. a.



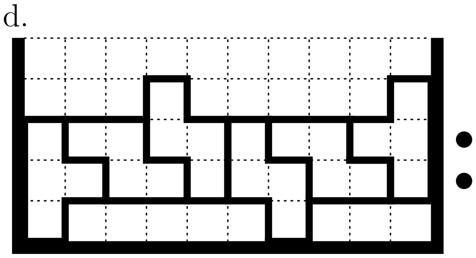
b.



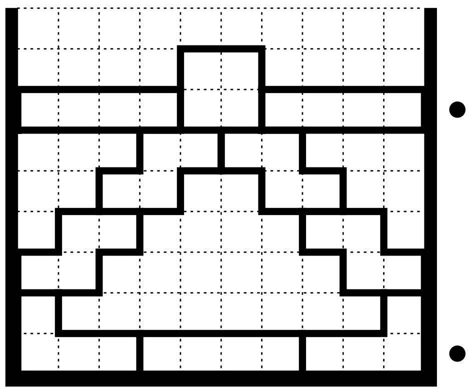
c.



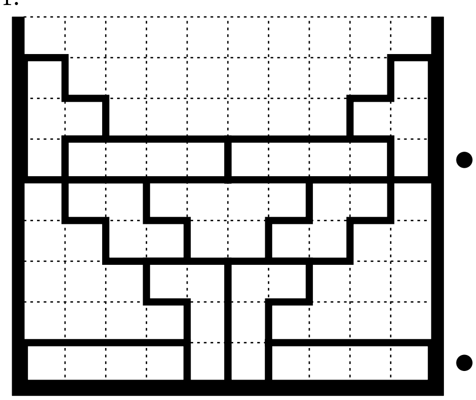
d.



e.

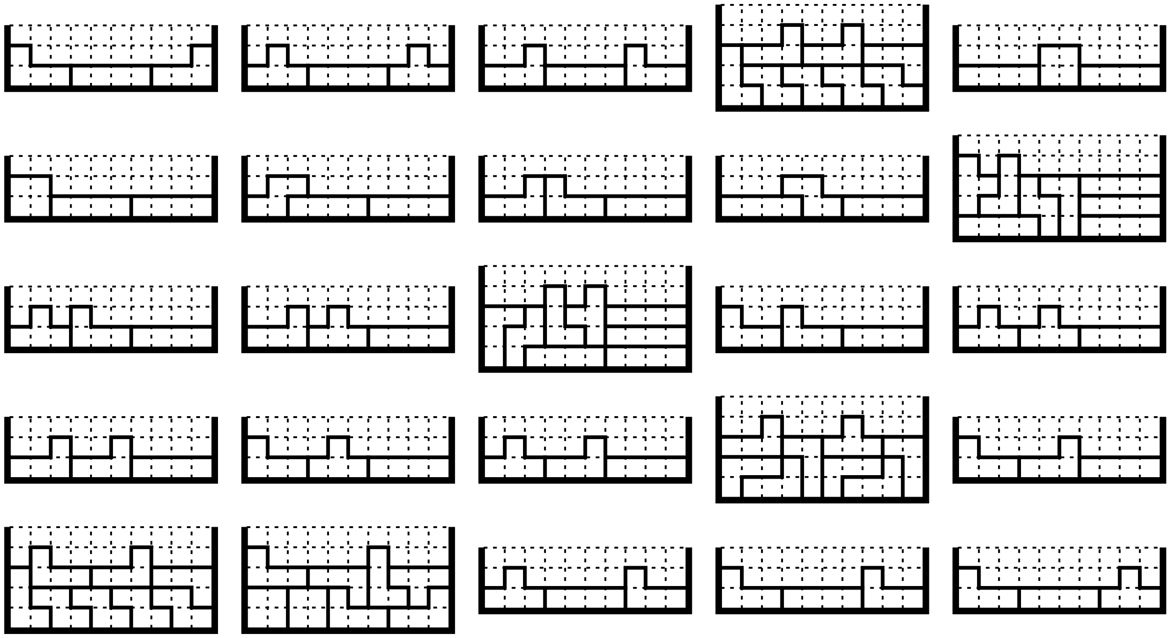


f.

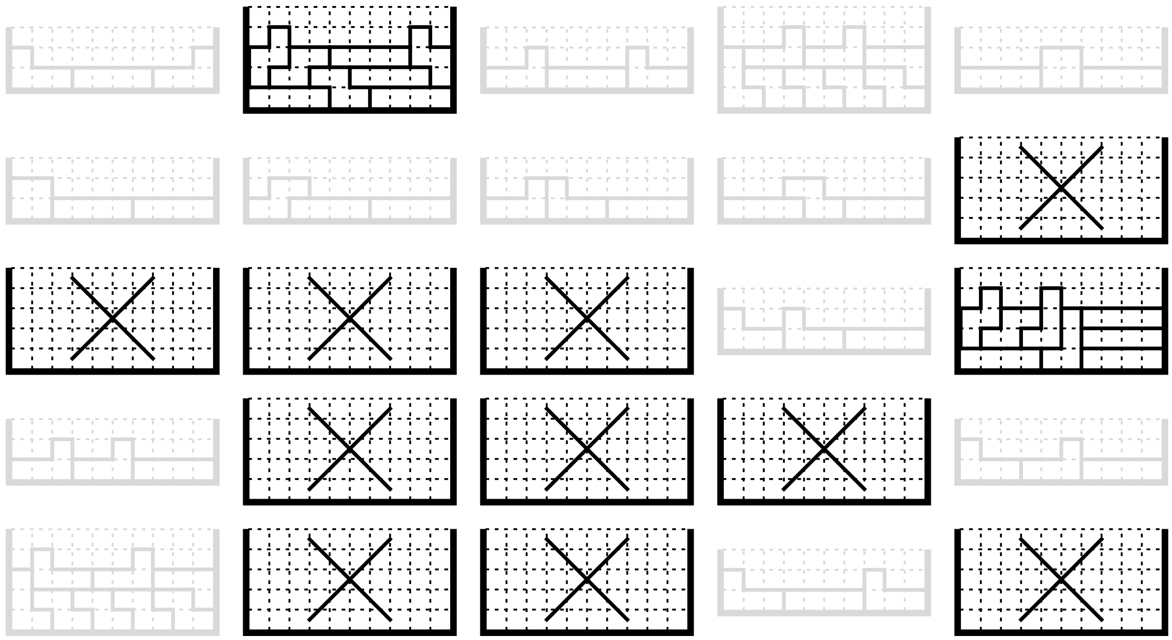


3. a. All of the different pieces have four squares in them, so the total number of squares in all blocks that have fallen is a multiple of four. If there are to be two squares left after all the complete rows have disappeared, the total number of squares that have fallen is $10n+2$ where n is the number of lines that disappear. If n is even, then $10n+2$ is not divisible by four.

   b. The diagram below shows how to achieve each of the patterns with two squares remaining on the bottom line (in each, all but the topmost row disappear, so the circles have been omitted). The first line of patterns are left-right symmetric, while the remaining patterns can each be reversed left-for-right to obtain the 20 patterns that are not explicitly shown. So clearly there are 34 patterns that can be created with just one row disappearing, while the remaining 11 require three rows



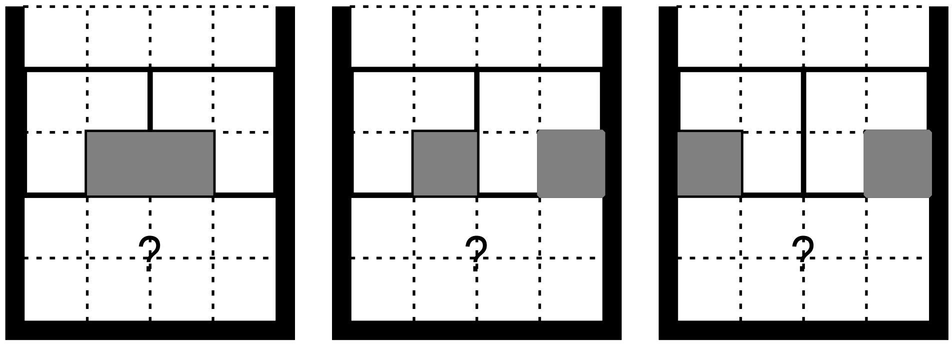
4. a.



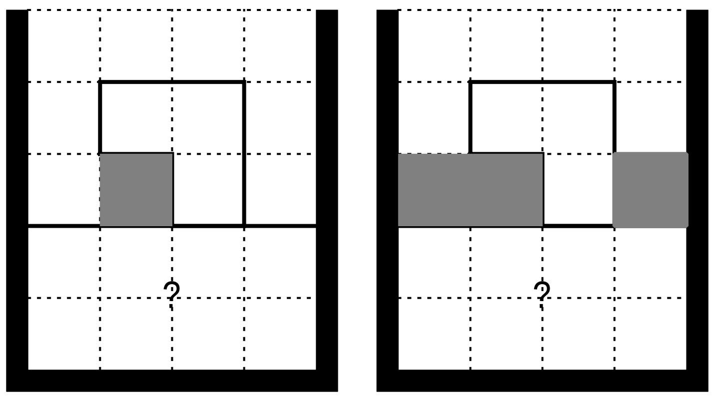
The diagram above shows the situation. The entries in light gray haven't changed since the previous problem—the pattern was already solved without the "T" in as few lines as possible. There are two entries that used to have one line and now have three, and there are twelve entries marked with an "X" which represent patterns that are impossible without the "T." So there are now 19 of the 45 patterns that are possible with one line disappearing, six that are possible with three lines disappearing, and 20 that are impossible without the "T."

b. Paint the squares that make up the falling blocks alternately black and white, like a chessboard. Notice that each block except the "T" always has the same number of black and white squares. And when a row of squares is completed and disappears, the number of white and black squares that disappear is the same. Now that doesn't mean that the non-"T" pieces can't create a pattern with different numbers of black and white squares—problem 1(a), for example —because when a line disappears and squares above it fall, every one of those squares changes colors. In 1(a) squares above the disappearing line changed color to match squares below it which didn't change color, resulting in more of one color than another. But for this problem, the remaining squares are all on one line, so there are no squares below the disappearing lines to match!

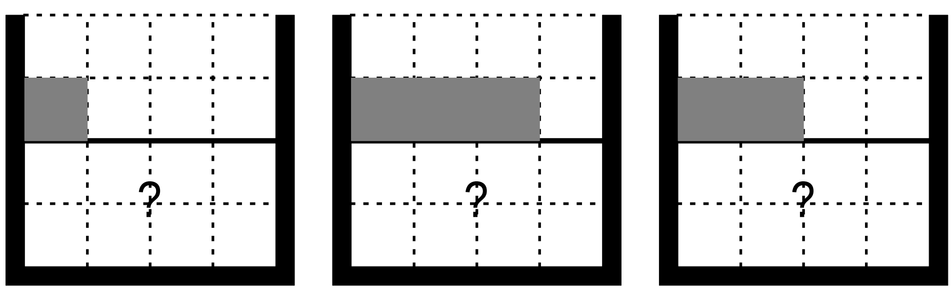
5. a. Drop two of the "L"-triominoes as shown. The first one will not complete the line to be eliminated, so the second one will be supported as shown, and then both the squares to be eliminated and all the new squares added by dropping these blocks will be part of complete lines and vanish.



b. Drop one "L"-triomino as shown. In the first case, we are left with two lines that can be cleared like the first diagram in part (a); in the second case we are left with one such line.



c. The remaining cases are those shown below (and their mirror images):



We know these are all possible patterns because (counting mirror images) we have accounted for 14 different possibilities of the top row. Together with the empty row and the full row (which aren't problems!) this takes care of all $2^4 = 16$ possible rows of squares.

Now in the first diagram above with just one square in the row to be eliminated, the solution is to drop a copy of the straight triomino in its horizontal position into the wide gap. This will eliminate the top row right away (unless there is also a wide gap in the next row down, in which case that row will vanish, and so forth for as many rows have this wide gap. Whatever the row the new triomino lands in must have a single square in it holding up the rows above it—there are no empty rows in the middle of a pattern!). No matter which row vanishes, the resulting pattern has one fewer row than the original.

In the second diagram, drop the straight triomino into the narrow gap. Keep dropping them until one reaches the height of the row to be eliminated or higher. At that point, the row will be eliminated, though there may be some single squares above that row that remain if the straight triomino ends up reaching above the row. No matter; we learned in the previous paragraph how to eliminate rows with a single square along the edge.

In the final diagram, drop the "L"-triomino, oriented so the it has two squares on the bottom and one above, into the gap. Keep dropping them until one reaches the height of the row to be eliminated or higher. If it only reaches the row to be eliminated, that row now has three squares in it, and previous work shows how to make it vanish. If it reaches the row above, then the bottom two squares of the new triomino filled the gap in the row to be eliminated, causing it to disappear. The top square of the new triomino now is all alone on the row to be eliminated, but we already know how to eliminate a row with only one square in it.

d. A short induction proof is all that is needed now. For the base case, if the pattern has zero rows of squares to eliminate, the whole pattern has been cleared and we are done. Now assume that all patterns of height n can be cleared. The previous steps show how to reduce any pattern to one with one fewer row, so a pattern of height $n+1$ can be reduced to one of height n. This can be cleared, so by induction any pattern can be cleared.

6. The original solution I had to this problem turned out to be incorrect, and will be discussed below. Here is a relatively simple explanation (a composite of some of the actual teams' solutions—it's always humbling when the students are smarter than the writer!).

   Let's say that triominoes could be dropped to leave just a single square in the lower left corner. Consider the number of complete rows that would end up disappearing to leave just the single square. Since eight squares disappear at a time, if k is the number of rows that disappear we have the total number of squares that drop is $8k+1$. Since the squares drop in groups of three, $8k+1$ is a multiple of three. That requires k itself to be one larger than a multiple of three (in other words, $k \equiv 1 \pmod 3$). So $k = 3l+1$. So $3l+1$ squares drop into every column except the leftmost, which sees a total of $3l+2$ squares.

   Now the triominoes land either horizontally or vertically. Number the columns left-to-right from 1 to 8, and let h_n be the number of triominoes that land horizontally with their left-most block in column n, and v_n be the number of triominoes that land vertically in column n. Clearly $h_7 = h_8 = 0$ because a triomino landing horizontally with its left-most block in the seventh or eighth column would stick out to the right past the edge of the screen, which is not allowed.

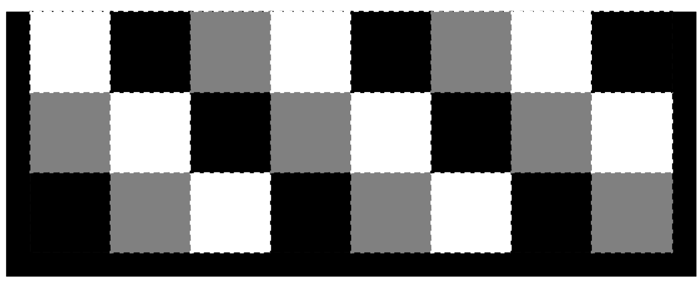
   Count the number of squares that end up in the first column. This is clearly $3v_1 + h_1$, so this must equal $3l+2$. This tells us that $h_1 - 2$ is a multiple of three ($h_1 \equiv 2 \pmod 3$).

Now any triomino that lands horizontally with its left-most square in the first column crosses into the second and third columns. So the total number of squares in the second column is $3v_2 + h_2 + h_1$ and this must equal $3l + 1$. Since h_1 is two more than a multiple of three, this relationship requires h_2 to also be two more than a multiple of three. Continuing to the right, $3v_3 + h_3 + h_2 + h_1 = 3l + 1$, so h_3 is a multiple of three. Then $3v_4 + h_4 + h_3 + h_2 = 3l + 1$ so h_4 is two more than a multiple of three. Then h_5 will be two more than a multiple of three, and h_6 will be a multiple of three. Next we get to the seventh column, where we find that $3v_7 + h_7 + h_6 + h_5 = 3l + 1$. Since h_6 is a multiple of three and h_5 is two more than a multiple of three, we find that h_7 must be two more than a multiple of three. But we already know that $h_7 = 0$, so it is impossible for it to be two more than a multiple of three. This contradiction shows that our original assumption that the triominoes could be dropped to leave a single square in the corner must have been incorrect.

This same idea will show that you can't leave a single square in columns 2, 4, 5, 7, or 8. The argument doesn't apply in columns 3 or 6, and in fact it *is* possible to leave a single square in those columns—I'll let the reader figure out the solution to that puzzle.

Now here is the original, incorrect proof for this problem and a brief discussion of what went wrong.

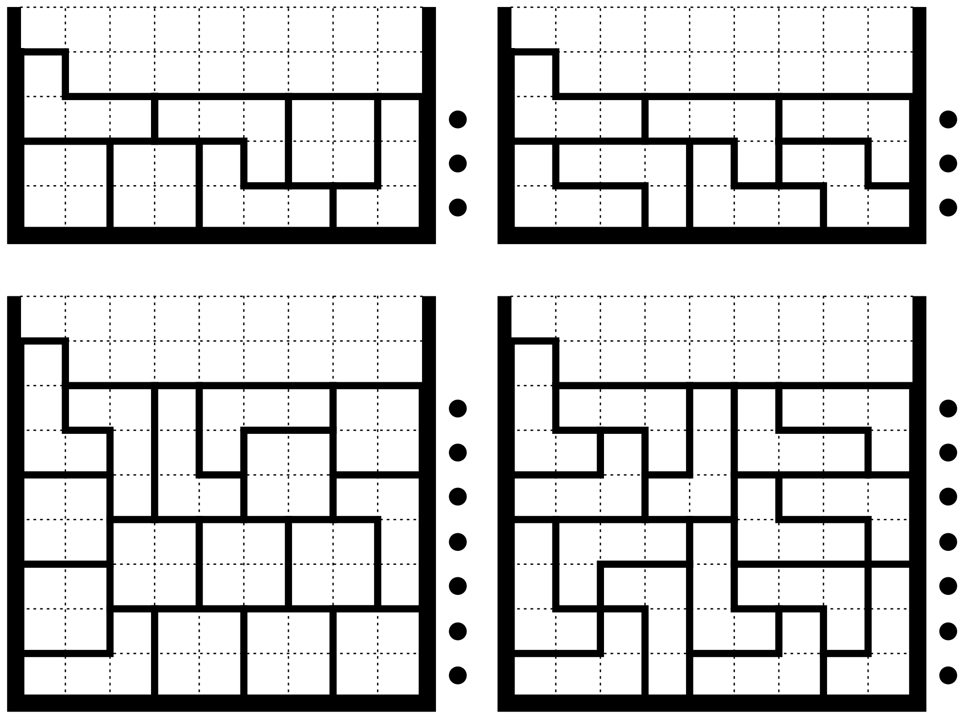
Start by coloring the squares of the grid with three colors, as shown:



Now every straight triomino will fall into one square of each color, so that (before complete rows vanish) there are always exactly the same number of each color square that are covered by triominoes. Unfortunately, no matter how many rows are involved, creating the pattern with just a single square (or clearing it) requires more of one color square than some other color. For example, if we try to create the pattern with a single square in the lower left, and the diagram is colored as shown, then any number of complete rows with a single square on top, where the total number of squares is divisible by three, will require one more gray square than there are black squares, and one fewer white square than black.

So what is wrong? The problem is that as entire lines disappear, the colors of the squares above that line change, and vertical blocks need no longer cover one square of each color. Skillful timing of when each row disappears will now allow the player to leave patterns that initially have uneven numbers of each color.

7. Here are some sample solutions; obviously there are many others.



Please note that only solutions that did not use the straight tetromino could earn credit, as the question asked for “pairs of the other blocks.”

Pancake Sorting

The Background

In 1975, a mathematician named Jacob Goodman posed a problem in the *American Mathematical Monthly* which went like this:

> The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function $f(n)$ of n) that I will ever have to use to rearrange them? (*American Mathematical Monthly* v. 82 (1975), p. 1010)

Unfortunately, no one knows exactly how to compute the function $f(n)$. Small values of n are relatively easy, though. For instance, if there are just two pancakes, either they are already in order, or the larger might be on top of the smaller, so the stack would have to be flipped. Thus $f(2) = 1$. If there are three pancakes, they might be in the order 231, which means the middle-sized pancake is on top, then the largest, then the smallest on the bottom of the stack. If this is the case, we have to flip the top two (resulting in the stack 321) and then flip the whole stack (to get 123) so that they are in the correct order. This does *not* mean that $f(3) = 2$, however, because there is an ordering of three pancakes that takes more than two flips (see problem 1).

Notation: As just indicated, a stack of pancakes will be denoted by listing the relative sizes of the pancakes, from top to bottom. To indicate a sequence of flips, underline the part you wish to flip—it will always start at the left (representing the top of the stack) and include at least two numbers (because flipping just one pancake doesn't help re-order the stack). Then write an arrow, followed by the result of that flip (the part that you previously underlined will be reversed). You can string several flips in a row. For instance, the flips that change 231 to the proper order 123 would be written as

$$\underline{23}1 \longrightarrow \underline{321} \longrightarrow 123.$$

The Problems

1. There are six arrangements of three pancakes: 123, 132, 213, 231, 312, and 321. Obviously, if you start with the pancakes in the correct order 123 no flips are required to put them in correct order. The order 231 was demonstrated above. For each of the other four orders, find a shortest sequence of flips that puts the stack in order.

2. From your information in problem 1, determine $f(3)$.

3. For one of the stacks in problem 1 there is a different, equally short sequence of flips that will put it in order. Show this alternate set of flips.

4. It is known that $f(4) = 4$ and that there are three different stacks that require a sequence of four flips. Find those stacks (you don't need to show the sequence of flips needed to put them in order, just the stack).

5. It is known that $f(5) = 5$. Find any stack of five pancakes that requires five flips to put in order.

In any stack of pancakes, you can arrange for any single pancake to end up on top of the stack with at most one flip—if it's not already on top, then put your spatula just beneath it an flip. Similarly, you can arrange for the top pancake to end up in any position with one flip. If you want it to become the k^{th} pancake, just flip the top k pancakes.

6. Use these ideas to show that $f(n+1) \leq f(n) + 2$.

7. Prove, for all $n \geq 2$, that $f(n) \leq 2n - 3$.

We have discovered an *upper bound* on the number of flips necessary to order the stack. In reality, $2n - 3$ is much larger than needed, and one of the main results of Bill Gates' paper is the improved upper bound of $\dfrac{5n+5}{3}$. A more recent trick was able to lower the slope from 5/3 to 18/11. Since we don't know the function exactly, it is good to have as small an overestimate as possible.

On the other hand, we'd also like to find an underestimate. In a stack of pancakes, define an *adjacency* to be a pancake where the pancake below it is immediately larger or immediately smaller in size (that is, these two pancakes belong together in the final sorted stack), *or* when the largest pancake is on the bottom of the stack (we can pretend we are treating the plate as a sort of unflippable largest pancake).

8. Count the number of adjacencies in each of the following stacks.
(a) 28713645 (b) 54312768

9. Explain why a single flip cannot change the number of adjacencies by more than one.

10. For every $n \geq 4$ explain how to construct a stack of n pancakes that has zero adjacencies.

11. Use the previous two results to explain why $f(n) \geq n$ for $n \geq 3$.

12. Find a stack of pancakes for which no flip increases the number of adjacencies.

13. Find a stack of six pancakes which requires *seven* flips to put in order. Hint: After three flips you reach a stack for which no flip increases its adjacencies.

Another result in Gates' paper is an improved lower bound, by finding stacks of $16k$ pancakes, where k is any positive integer, that take at least $17k$ flips to put in order. Thus the slope of the underestimate is raised from 1 to 17/16 (others have since raised the slope to 15/14).

Cohen worked on a variation of the problem that deals with "burnt" pancakes. That is, each pancake has a good side and a burnt side, and when you serve the stack to the customer not only should they be in order from smallest to largest, but each pancake should have its burnt side down so that at least it looks pretty. We will let $g(n)$ represent the maximum number of flips necessary to properly order a stack of n burnt pancakes.

Notation: The notation will stay the same as before, and a pancake that is burnt-side-up will be surrounded by parentheses. You can put a single set of parentheses around a bunch of pancakes if every one of them is upside down. For example, 43(5)2(1) is a stack where the smallest pancake is burnt-side-up on the bottom of the stack, the second-smallest pancake is right-side-up on top of it, then the upside-down largest pancake, and so forth. The stack (123) is an easier way to write (1)(2)(3) which has all the pancakes in order, but burnt-side-up.

Notice that a flip not only reverses the order of the numbers that are underlined, but also makes right-side-up pancakes become upside-down and vice versa. Also, with burnt pancakes it may be necessary to flip just the top pancake. One way of fixing the example stack above is

$$\begin{aligned} \underline{43(5)}2(1) \quad &\longrightarrow \quad \underline{5}(34)2(1) \longrightarrow \underline{(534)2(1)} \longrightarrow \underline{1(2)}435 \longrightarrow \underline{(4)2(1)3}5 \\ &\longrightarrow \quad \underline{(3)1(2)}45 \longrightarrow \underline{2}(1)345 \longrightarrow \underline{(21)}345 \longrightarrow 12345. \end{aligned}$$

14. If there are n burnt pancakes, derive an expression for the number of possible stacks, counting both order and sided-ness.

15. Use the ideas from problems 6 and 7 and the paragraph before them to show that $g(n) \leq 3n - 2$.

This upper bound of $3n-2$ can be improved very easily to $2n$ as we shall see. First, let's define an adjacency for burnt pancakes to also require that the burnt side of the smaller pancake is touching the unburnt side of the larger. The largest pancake is only adjacent to the plate when it is right-side-up. For example, if pancakes 3 and 4 are to be deemed adjacent, the stack must look like $\cdots 34 \cdots$ or $\cdots (4)(3) \cdots$. As in the unburnt case, a stack of n pancakes is sorted if and only if there are exactly n adjacencies.

16. Assume we have a stack of n burnt pancakes. Show with proof, in each case below, a sequence of two flips that will create one more adjacency without ruining any other adjacency.

(a) Pancake n is right-side-up somewhere in the stack. Get it to the bottom of the stack, right-side-up, in two or fewer flips.

(b) Pancake p is the largest right-side-up pancake, $p < n$, and pancake $p+1$ is upside-down and higher up in the stack than p. That is, the stack looks like $\cdots (p+1) \cdots p \cdots$.

(c) Pancake p is the largest right-side-up pancake, $p < n$, and pancake $p+1$ is upside-down and lower in the stack than p. That is, the stack looks like $\cdots p \cdots (p+1) \cdots$.

(d) All pancakes are upside-down, but some pancake (p) has $(p+1)$ somewhere higher up in the stack. That is, the stack looks like $\cdots (p+1) \cdots (p) \cdots$.

The only stack not covered by one of the cases above is where the pancakes are in the correct order—no pancake is ever below a larger pancake—but every pancake is burnt-side-up. That is, the stack $(123\cdots n)$.

17. Starting with this stack, flip the entire stack, then flip all but the bottom, and repeat this pair of flips k times. Prove that the resulting stack is $(k+1)(k+2)\cdots(n)12\cdots k$.

Thus, every stack of n burnt pancakes can be sorted in at most $2n$ flips. Use the steps in problem 16 to create as many adjacencies as possible. At most, two flips are used per adjacency. When no more adjacencies can be created by those techniques, we have reached a stack of the form $(12\cdots k)\cdots n$. Pancakes $k+1$ through n are safely in proper order on the plate, and we have created $n-k$ adjacencies using at most $2(n-k)$ flips. Then use the technique of problem 17 to get the top k pancakes right-side-up, using $2k$ additional steps. So $g(n) \leq 2n$.

Gates' and Cohen's papers show that a lower bound for $g(n)$ is $3n/2$.

The problem becomes at once simpler and more complicated if you allow more kinds of flips. Let's return to the unburnt pancakes, but now allow *any* part of the stack to be flipped. That is, you may lift a few pancakes from the top of the stack, then flip the next bunch, then set the top few back down. In notation, the underline indicating the flip is now allowed to start and end anywhere in the list, not just starting at the first pancake. It is now much easier to sort stacks, since it will require fewer moves, but more complicated to analyze since there are far more options.

18. Prove that any stack of n (unburnt) pancakes can be sorted in no more than $n-1$ moves if you are allowed to flip any part of the stack.

Since reversal of part of a strand of DNA is one of the most common types of mutations in genetics, it is important to know the fastest way to sort a string of DNA using these kinds of flips. One genetic test of the "distance" between two species is the number of flips it would take to get from the genome of one to the genome of the other.

The Solutions

1. The shortest sequences of flips that put each stack in order are as follows.

$\underline{321} \longrightarrow 123$

$\underline{21}3 \longrightarrow 123$

$\underline{312} \longrightarrow \underline{21}3 \longrightarrow 123$

$\underline{132} \longrightarrow \underline{23}1 \longrightarrow \underline{321} \longrightarrow 123$ or $\underline{13}2 \longrightarrow \underline{312} \longrightarrow \underline{21}3 \longrightarrow 123$

2. Of all the stacks of three pancakes, there is one (132) that requires three flips, and none require more than three flips, so $f(3) = 3$.

3. The stack is 132, and both flip sequences are shown in the answer to problem 1.

4. The three stacks are $2413, 3142$, and 4231. Though inelegant, probably the best way to see this is to start with 1234 and catalog all the stacks you can get from various flips.

5. There are 20 such stacks. Using the idea of adjacencies introduced in later problems, this problem can be easily solved by looking for stacks with no adjacencies. A few examples are 53142, 52413, 42513, and 25314. Of course, any of the 20 correct solutions is acceptable for full credit.

6. Given any stack of $n + 1$ pancakes, it takes at most two flips to get the biggest pancake on the bottom of the stack (if it's not already on the bottom, then (if it's not already on top) flip the largest to the top, then flip the whole stack). Then the smallest n pancakes can be sorted in at most $f(n)$ flips without disturbing the largest pancake, so the whole process takes at most $f(n)+2$ flips.

7. This is a straightforward induction starting with a base case of $n = 2$. We know that a two-pancake stack takes at most one flip, so $f(2) = 1 = 2 \cdot 2 - 3$. so the formula holds for $n = 2$. Now if $f(n) \leq 2n - 3$, then using the result of the previous problem, we see that $f(n + 1) \leq f(n) + 2 \leq 2n - 3 + 2 = 2(n + 1) - 3$.

8.
(a) There are only two adjacencies: the 87 and the 45.
(b) There are four adjacencies: $54, 43, 12$, and 76.

9. During the flip, the number of adjacencies among the pancakes that are flipped can't change, nor can the number of adjacencies among the pancakes that are left on the plate. The only place that you can create or destroy an adjacency is the one place where you put the spatula!

10. There are many ways to do this. A simple one would be to make a stack with all the even-sized pancakes (in order) first, followed by the odd-sized pancakes (in order). No two even numbers nor any two odd numbers are adjacent, and the place where you switch from even to odd can't be an adjacency because you go from the largest even-size pancake (at least four) to the smallest odd pancake (size 1). For four pancakes, this creates the stack 2413, which is one of the stacks we discovered in problem 4 to require the maximum number of flips.

11. For $n = 3$, we already know $f(3) = 3$ so it is true in this case. For $n \geq 4$, by problem 10 we can find a stack with zero adjacencies. A completely sorted stack has n adjacencies—12, 23, 34, ..., until you count the adjacency of the largest pancake lying on the plate. But by problem 9 we can create *at most* one new adjacency with each flip, so we need at least n flips to create the n adjacencies in the sorted stack. Thus $f(n) \geq n$.

12. There was a typo in the problem—it was supposed to ask for a stack *other than the fully sorted stack* for which no flip would increase the number of adjacencies, because the fully sorted stack is a trivial example. In fact, every flip possible for the sorted stack will *decrease* the number of adjacencies. In the proper form, the question would require an answer something like 2143. The possible flips are 1243 (no adjacencies are affected), 4123 (one adjacency, 23 is created, but another, 34 is destroyed) and 4312 (no adjacencies affected).

13. There are two such stacks—462513 and 536142. It is clear that both of these start with zero adjacencies, so would require at least six flips to put into order. But trying the various possibilities, each of these always leads to a position where no available flip will increase the number of adjacencies, so each of these stacks takes at least a seventh flip to order.

14. In terms of pancake size, there are $n!$ orders. But each pancake can be right-side-up or upside-down, so there are 2^n ways to choose which side is up for all the pancakes. Since the order and the upsides can be chosen independently, we multiply: there are $2^n n!$ possible stacks of n burnt pancakes.

15. A stack of one pancake might need to be flipped (if the pancake is burnt-side-up). So $g(1) = 1 = 3 \cdot 1 - 2$ to start the induction. Now to order and right-side $n + 1$ pancakes, we can put the biggest pancake right-side-up on the bottom of the stack in at most three flips. For if it is not already in that position, we can bring it to the top (using one flip, unless it is already there), then flip it so that the burnt side is up (using one flip, if it isn't that way already), and then flip the whole stack to that it ends up good-side-up on the bottom. Then the remaining pancakes can be positioned using at most $g(n)$ moves. So we learn $g(n + 1) \leq g(n) + 3$. If we assume by induction at $g(n) \leq 3n - 2$ then we find $g(n + 1) \leq g(n) + 3 \leq 3n - 2 + 3 = 3(n + 1) - 2$.

16. Let $q = p + 1$ in the explanations below.

(a) If pancake n is right-side-up, it cannot be adjacent to the pancake just below it unless it is on the plate. So flip the top of the stack down to, and including n, and then flip the whole stack. In other words, perform $\underline{\cdots n}\cdots \longrightarrow \underline{(n)\cdots} \longrightarrow \cdots n$. Any adjacencies in the top part of the stack (above n) are preserved because that part of the stack is never changed, and the same goes for the adjacencies below the original location of n. Since, as already mentioned, pancake n was not adjacent to pancake initially below it, no adjacencies were destroyed, and pancake n is now right-side-up on the plate, so one more adjacency has been created.

(b) Perform the flips $\underline{\cdots (q)\cdots p}\cdots \longrightarrow \underline{(p)\cdots}q\cdots \longrightarrow \cdots pq\cdots$. In other words, flip the top down to and including p. Since p is the largest right-side-up pancake, it is not adjacent to the pancake below it (which would have to be a right-side-up $q = p + 1$). So this flip doesn't ruin any adjacencies. Then flipping the stack down to, but *not* including the right-side-up q puts these two together properly.

(c) Perform the flips $\underline{\cdots p\cdots (q)}\cdots \longrightarrow \underline{q\cdots}(p)\cdots \longrightarrow \cdots (qp)\cdots$. No adjacencies are ruined, for the same reasons as the previous part. And again p and q are now adjacent (though upside-down this time).

(d) This time, the flips are $\underline{\cdots (q)}\cdots (p)\cdots \longrightarrow \underline{q\cdots}(p)\cdots \longrightarrow \cdots (qp)\cdots$. Since, once again, the place where the spatula goes cannot be an adjacency, we never ruin any adjacencies and always create one.

17. Looks like a job for induction! When $k = 1$, we get $\underline{(1)(2)\cdots(n)} \longrightarrow \underline{n\cdots 2}1 \longrightarrow (2)\cdots(n)1$, which demonstrates the base case.

Then, if $2(k-1)$ flips takes $(1)(2)\cdots(n)$ to $(k)(k+1)\cdots(n)12\cdots k-1$, two more flips produce

$$\underline{(k)(k+1)\cdots(n)12\cdots k-1} \longrightarrow \underline{(k-1)(k-2)\cdots(2)(1)n\cdots k+1}k \longrightarrow (k+1)\cdots(n)12\cdots k,$$

as desired.

18. A simple solution, though certainly not the only one, is to sort the pancakes into order by using each move to put the next pancake that is not in order into its proper position from top to bottom. That is, after k flips, assume the top (at least) k pancakes are in order on top of the stack. At step $k+1$, lift these k pancakes, flip the bunch down to and including $k+1$, and set the top k back down. In notation, we do the flip $12\cdots k\underline{a\cdots b\,k+1}\cdots \longrightarrow 12\cdots k\,k+1\,b\cdots a\cdots$. We can be a little more proper with the induction. The base case is $k = 0$ and certainly after zero flips, there are at least zero pancakes in proper order. If, after k flips the top k pancakes are in order, the $(k+1)^{\text{st}}$ flip puts pancake $k+1$ in its proper position. After at most $n-1$ flips the top $n-1$ pancakes are in order, and by the pigeon-hole principle, pancake n must be on the bottom, so the whole stack is in order.