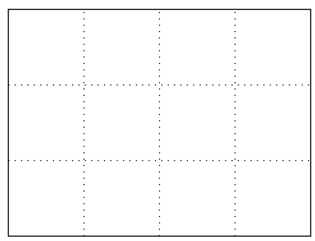
Paths In A Grid

Fall 2022 ARML Power Contest

The Background

The Fall 2022 ARML Power Contest concerns patterns of black and white squares within a rectangular grid, particularly, patterns in which there is a path of black squares that connects one side of the grid to the other. The goal will be to obtain and prove formulas for the numbers of such patterns depending on the size of the grid.

For the purposes of this contest, an $m \times n$ grid will specifically mean a grid of squares in which there are m rows and n columns. The diagram below shows a 3×4 grid:

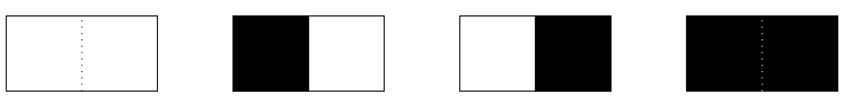


Some squares within the grid may be painted black, while the others will be painted white. The problems will not concern the edges of the squares, so their color should not be considered when solving the problems.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth. These are also indicated on the cover sheet, which should be the front page of your solution packet. Please make sure to write on only one side of the answer sheets.

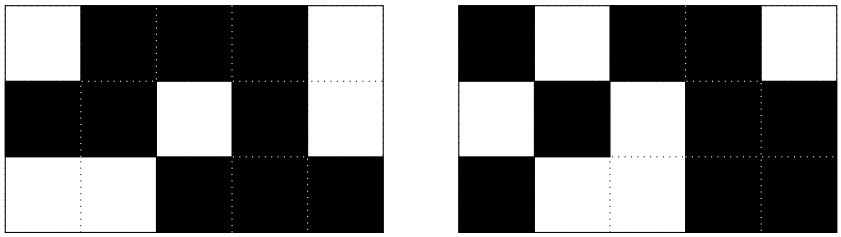
For a 1×1 grid—a single square—there are two ways to paint the grid. The square could be painted white or it could be painted black. For either a 1×2 or a 2×1 grid, there are two squares, and there are four ways to paint the grid, as shown below for the 1×2 grid:



1. Determine a formula for the number of ways to paint the squares of an $m \times n$ grid either black or white. Justify your formula. [2 pts.]

A *path* between two squares (call these squares A and B) in a grid is a sequence of squares $s_1, s_2, \ldots, s_n$ where $s_1 = A$, $s_n = B$, and for each i with $1 \leq i \leq n-1$ the squares s_i and s_{i+1} have an edge in common. The path may pass through any given square multiple times and may double back on itself. We are only concerned about whether the squares are connected by a path, not on the particulars of how the path gets from one to the other.

We will say that a painted grid has an *end-to-end path* if there is a path of squares where A is in the left-most column and B is in the right-most column, and every square in the path has been painted black. The grid on the left below has an end-to-end path, the one on the right below does not.



It should be obvious that a $1 \times n$ grid has an end-to-end path if and only if every square is painted black. It should also be clear that, of the four ways to paint a 2×1 grid, three of the ways of painting it have an end-to-end path—any of the three ways that have at least one black square.

2. Of the 16 ways to paint a 2×2 grid, exactly seven have an end-to-end path. Draw these seven painted grids. [3 pts.]

3. Compute the number of ways to paint a 2×3 grid so that it has an end-to-end path. [2 pts.]

Let $E_2(n)$ be the number of ways to paint a $2 \times n$ grid so that it has an end-to-end path. Thus, $E_2(1) = 3$, $E_2(2) = 7$, and $E_2(3)$ is the number you computed in the previous problem (assuming you computed it correctly!). For simplicity, you may assume that $E_2(0) = 1$, since there is only one way to paint a grid that doesn't have any squares (don't paint anything!) and every "path" in this grid has all of its squares painted black.

4. Prove that $E_2(n)$ satisfies the recursive formula [3 pts.]

$$E_2(n) = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 2E_2(n-1) + E_2(n-2) & n \geq 2 \end{cases}.$$

5. Prove that $E_2(n)$ also counts the number of n-digit numbers whose digits are all 1, 2, or 3, where the digits 1 and 2 are never consecutive. For instance, $E_2(2) = 7$ because there are seven such 2-digit numbers: 11, 13, 22, 23, 31, 32, 33 (the combinations 12 and 21 both have a 1 and 2 consecutive). [2 pts.]

Let $F_2(n)$ be defined by

$$F_2(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2F_2(n-1) + F_2(n-2) & n \geq 2 \end{cases}.$$

That is, $F_2(n)$ is the same recursion as $E_2(n)$ with a different starting value when $n = 1$.

6. Prove the following facts about the relationship between $E_2(n)$ and $F_2(n)$:

   (a) For $n \geq 1$, $E_2(n) = E_2(n-1) + 2F_2(n-1)$ while $F_2(n) = E_2(n-1) + F_2(n-1)$. [3 pts.]

   (b) $E_2(n)^2 = 2F_2(n)^2 - (-1)^n$. [3 pts.]

Let's change the direction of our investigation a little:

7. Find, with proof, a formula involving m and n that computes the maximum number of squares in an $m \times n$ grid that can be painted black without creating an end-to-end path. [3 pts.]

A *top-to-bottom path* is defined similarly to an end-to-end path, except that it must start and end on the top and bottom rows of the grid and consist entirely of squares that are painted *white*.

8. Draw a painted 3×3 grid that has *neither* and end-to-end path nor a top-to-bottom path. [2 pts.]

9. Prove that no painted grid can have *both* an end-to-end path and a top-to-bottom path. [4 pts.]

10. Let $E_3(n)$ be the number of ways to paint a $3 \times n$ grid so that there is an end-to-end path. Compute, with explanation, $E_3(2)$. [2 pts.]

11. Following the established pattern, $E_k(2)$ is the number of ways to paint a $k \times 2$ grid so that there is an end-to-end path. In other words, there is at least one row in which both squares are painted black. We know from our work so far that $E_1(2) = 1$, $E_2(2) = 7$ and $E_3(2)$ is the answer to the previous question. Show that $E_k(2)$ satisfies both the recurrences $E_k(2) = 4E_{k-1}(2) + 3^{k-1}$ and $E_k(2) = 3E_{k-1}(2) + 4^{k-1}$. [4 pts.]

12. Compute, with explanation, $E_3(3)$. [3 pts.]

13. Two players play a game using a 6×6 grid. On each turn, player 1 paints one square black, while player 2 paints one square white. Player 1 goes first, and the players alternate after that. Player 1 wins is she can create an end-to-end path. Player 2 wins by preventing the creation of an end-to-end path. Describe a strategy that player 2 can follow to ensure a victory. Prove your strategy works. [4 pts.]

    (To describe a strategy, explain which square player 2 will paint white given player 1's previous move(s). For an example, a strategy could be "always paint the square directly above the square which player 1 just painted; if that square is already painted then paint any available square." Which is a losing strategy, as player 1 could just paint each of the squares in the bottom row one right after another and win after six moves.)

The sequences $E_2(n)$ and $F_2(n)$ were known by Theon of Smyrna (flourished *circa* 100 CE), and in fact the sequence of ratios $\frac{1}{1}, \frac{3}{1}, \frac{7}{5}, \frac{17}{12}, \cdots$ are known as his *diagonal numbers*. This sequence converges rapidly to $\sqrt{2}$ and is, in fact, the sequence of convergents of the continued fraction for $\sqrt{2}$. If anyone can explain why the number of end-to-end paths should be related to $\sqrt{2}$ I'd love to hear about it!

The game in the last problem is a very simplified version of the game of *Hex*. Instead of a square grid, Hex is played on a hexagonal grid. player 1 tries to form an end-to-end path while player 2 has to make a top-to-bottom path. Because the hexagonal grid has more connections it is harder for player 2 to block player 1, and in fact player 1 theoretically can always win. But no one knows how!