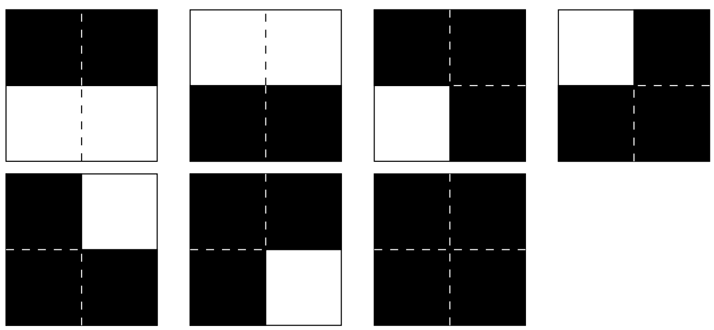
Paths in a Grid
Fall 2022 ARML Power Contest

The Solutions

1. In an $m \times n$ grid there are mn squares, each of which may be painted either black or white, so the multiplication principle dictates that there will be $\boxed{2^{mn}}$ ways to paint the grid.

2. Any of the ways to paint the grid where either 3 or 4 squares have been painted black has an end-to-end path. There cannot be an end-to-end path with 0 or 1 black squares, because there must be at least one black square in each column. When there are only 2 squares painted black, they much be touching horizontally to create an end-to-end path. So the 7 possible paintings are:



3. For there to exist an end-to-end path, there certainly must be a path from the first column to the second. Thus, the first two columns of a painted 2×3 grid with an end-to-end path will be painted in one of the seven ways that are solutions to the previous problem.

   For each of the four solutions to the previous problem that have just one black square in the second column, there are two ways to paint the squares in the third column to create an end-to-end path: either both squares in the third column are painted black, or just the square adjacent to the black square in the second column is black. This leads to $4 \cdot 2 = 8$ possibilities. For each of the three solutions to the previous problem where both squares in the second column are black, there are three ways to paint the third column to create an end-to-end path—as long as at least one of the squares is black a path is created. This contributes $3 \cdot 3 = 9$ more possiblities. Thus, there are $\boxed{17}$ ways to paint a 2×3 grid that yield an end-to-end path.

4. The values for $E_2(0)$ and $E_2(1)$ are known to be correct from information given in the problems. The recursion remains to be verified. To that end, compare with the previous problem. Let $A_2(n)$ be the number of ways to paint a $2 \times n$ grid that have end-to-end paths in which there is only one painted square in column n, and $B_2(n)$ be the number of ways to paint a $2 \times n$ grid that have end-to-end paths where both squares in column n are painted black. So $E_2(n) = A_2(n) + B_2(n)$.

   Now, as was explained in the previous solution, if just one of the squares in column $n-1$ is painted black, there are two ways to paint the squares in column n to extend the end-to-end path. If both squares in column n are painted black, though, there are three ways to paint the squares in column n to obtain an end-to-end path. So

$$E_2(n) = 2A_2(n-1) + 3B_2(n-1).$$

Rearranging this a little leads to

$$\begin{aligned} E_2(n) &= 2A_2(n-1) + 2B_2(n-1) + B_2(n-1) \\ &= 2(A_2(n-1) + B_2(n-1)) + B_2(n-1) \\ &= 2E_2(n-1) + B_2(n-1). \end{aligned}$$

But every end-to-end path of length $n-2$ can be extended by painting both squares in column $n-1$ black, and each end-to-end path with both squares in column $n-1$ painted black is an end-to-end path of length $n-2$ if that last column is simply erased. That is, $B_2(n-1) = E_2(n-2)$. Substituting in the previous formula gives the desired result.

5. Given such a number, write each digit of the number above one column of the grid. If the digit is 1, paint the top square black; 2 paint the bottom square black; 3 paint both squares black. This painting will create an end-to-end path, for since the digits 1 and 2 are never consecutive, there is a never a pair of columns where the path can't get from one to the other. Clearly, an end-to-end path gives rise to such a number by reversing this process. Thus the paths and the numbers are in one-to-one correspondence, so a formula that counts one will naturally count the other as well.

6. (a) Proceed by induction. When $n = 1$ the relationships read

$$E_2(1) = E_2(0) + 2F_2(0) = 1 + 2 = 3$$

and

$$F_2(1) = E_2(0) + F_2(0) = 1 + 1 = 2$$

which are both true. Similarly, when $n = 2$ the formulas read

$$E_2(2) = E_2(1) + 2F_2(1) = 3 + 4 = 7$$

and

$$F_2(2) = E_2(1) + F_2(1) = 3 + 2 = 5.$$

Since the definition of $F_2(2) = 2F_2(1)+F_2(0) = 4+1 = 5$, both of these statements are true also. So both statements are true in the base cases of $n = 1$ and $n = 2$. For the induction step, assume both statements are true for the case of $n-1$. By the recursion for $E_2(n)$ proven in question 4, $E_2(n) = 2E_2(n-1) + E_2(n-2)$. Then by the inductive assumption on $E_2(n-1)$:

$$\begin{aligned} E_2(n) &= 2E_2(n-1) + E_2(n-2) \\ &= E_2(n-1) + E_2(n-1) + E_2(n-2) \\ &= E_2(n-1) + E_2(n-2) + 2F_2(n-2) + E_2(n-2) \\ &= E_2(n-1) + 2(E_2(n-2) + F_2(n-2)) \\ &= E_2(n-1) + 2F_2(n-1) \end{aligned}$$

the last step by the inductive assumption on $F_2(n-1)$. Similarly, using the recursion in the definition of $F_2(n)$ and the inductive assumptions for both $E_2(n-1)$ and $F_2(n-1)$,

$$\begin{aligned}
F_2(n) &= 2F_2(n-1) + F_2(n-2) \\
&= F_2(n-1) + F_2(n-1) + F_2(n-2) \\
&= F_2(n-1) + E_2(n-2) + F_2(n-2) + F_2(n-2) \\
&= F_2(n-1) + (E_2(n-2) + 2F_2(n-2)) \\
&= F_2(n-1) + E_2(n-1)
\end{aligned}$$

which is the desired result.

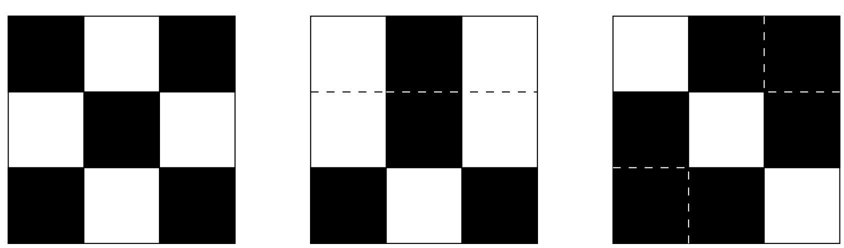
(b) Again, proceed by induction. The statement is clearly true when $n = 0$, as $E_2(0)^2 = 1^1 = 1 = 2 \cdot 1 - 1 = 2F_2(0)^2 - (-1)^0$. So assume the statement to be true for $n-1$. Then, using the relationship proven in the first part of this problem,

$$\begin{aligned}
E_2(n)^2 - 2F_2(n)^2 &= (E_2(n-1) + 2F_2(n-1))^2 - 2(E_2(n-1) + F_2(n-1))^2 \\
&= E_2(n-1)^2 + 4E_2(n-1)F_2(n-1) + 4F_2(n-1)^2 \\
&\quad - 2E_2(n-1)^2 - 4E_2(n-1)F_2(n-1) - 2F_2(n-1)^2 \\
&= -E_2(n-1)^2 + 2F_2(n-1)^2 \\
&= -(E_2(n-1)^2 - 2F_2(n-1)^2) \\
&= (-1)^{n-1} = -(-1)^n
\end{aligned}$$

where the last step is the inductive assumption.

7. The formula is $\boxed{m(n-1) = mn - m}$. For if there is a row that doesn't have at least one white square, that row is an end-to-end path by itself. So at least m squares are painted white, and at most $mn - m$ painted black. But this bound is easily achieved by simply painting all the squares in any single column white, leaving a grid of $mn-m$ black squares with no end-to-end path.

8. There are many such examples. Three are shown below.



9. This problem is a lot more subtle than it may appear at first. It is very easy to make true statements that seem obvious, but that aren't actually proven, or are logically equivalent to the original question. In a large grid there may be patterns of white and black squares that spiral around each other in complicated ways so that it might seem

almost possible for one to "get past" the other and create both end-to-end and top-to-bottom paths. A correct proof must address all these possibilities. The subtlety comes from the relationship of this problem to the *Jordan Curve Theorem* whose complete proof is very advanced mathematics indeed!

A clever trick can be used in this problem by exploiting the finite size of the grid. The general continuous case is a much harder problem.

As was pointed out in the text preceding question 2, a $1 \times n$ grid can only have an end-to-end path if it is entirely painted black, precluding the existence of a top-to-bottom path. Symmetrically, $m \times 1$ grids with a top-to-bottom path are entirely white, and have no end-to-end paths. So any grid that has both an end-to-end path and a top-to-bottom path must have at least two rows and at least two columns. This also serves as the base cases for an induction on both the number of rows and columns. That is, it may be assumed that no grid of dimensions $(m-1) \times n$ or $m \times (n-1)$ possesses both end-to-end and top-to-bottom paths.

Now for the sake of contradiction, consider an $m \times n$ grid with both end-to-end and top-to-bottom paths. Further, let the total length of these two paths be as short as possible. That is, if there are k black squares in the end-to-end path, and l white squares in the top-to-bottom path, then there is no painted grid where there is both an end-to-end path and a top-to-bottom path where the total number of squares in those paths is less than $k + l$. If there are any painted grids with both kinds of paths, then there is certainly such a minimum because any end-to-end path must be at least n squares and any top-to-bottom path must be at least m squares, and there are finitely many painted grids so if necessary they could all be tested to determine which ones have both top-to-bottom and end-to-end paths, and then to pick one with shortest total length. Because of this minimality, the top-to-bottom path has only its first square in the top row. For any part of the path preceding the last time it leaves the top row can be eliminated and the remaining path is still a top-to-bottom path.

Now focus on the end-to-end path. If this path never reaches the top row, then *eliminate* the top row. This doesn't affect the end-to-end path since it never reached the top row, and the remaining part of the top-to-bottom path is a top-to-bottom path in the smaller grid since the top-to-bottom path in the original grid never returned to the top row after leaving its first square. Thus, the grid so produced is $(m-1) \times n$, and possesses both end-to-end and top-to-bottom paths. This contradicts the inductive assumption. Similarly, the end-to-end path much reach the bottom row. That is, contained within the end-to-end path there is a top-to-bottom path (of black squares). By a symmetric argument, the top-to-bottom path contains an end-to-end path of white squares.

The end-to-end path must start and end in corners of the grid. Assume it does not. Delete the part of the path preceding the first time it reaches either the top or the bottom row. This shorter path is still a top-to-bottom path (of black squares), while the other path hasn't been altered and is still both a top-to-bottom and an end-to-end path of white squares. By swapping the colors of the two paths the top-to-bottom path is now (correctly) colored white while the now-black path is end-to-end. Thus a grid has been created with both an end-to-end path and a top-to-bottom path, whose total lengths are less than $k + l$. Since that is the minimum possible total length for such pairs of paths, the assumption is incorrect and the end-to-end path starts in a corner of the grid. Similarly it ends in a corner, and the top-to-bottom path also starts and ends at corners of the grid. (In fact, each of the two paths must start and end at diagonally opposite corners. For if, say, the end-to-end path both started and ended at the two corners in the top row of the grid, then it must, at some time, visit the bottom row (because it is known to be a top-to-bottom path also). Then this portion of the path can be deleted, leaving a top-to-bottom path while the other path is still also an end-to-end path, and (changing the colors of the paths if necessary) a grid with end-to-end and top-to-bottom paths of shorter total length has been created. This fact is not necessary, though.)

Now consider the path that starts at the top-left square of the grid. Its first step must either be down to the leftmost square in the second row, or right to the top square in the second column. But both of these are impossible! For if the first step is down one square, the first square could be eliminated from the path and the path remains end-to-end, while the other path remains top-to-bottom. Swapping the colors of the paths if necessary once again contradicts the shortest total length of the two paths rule! If instead the first step were to the right, the first square could be eliminated and the path remains top-to-bottom! Of course, the other path remains end-to-end, and once again shorter paths have been discovered.

To summarize, the existence of an end-to-end path and top-to-bottom path of minimal total length in an $m \times n$ grid either implies the existence of end-to-end and top-to-bottom paths of shorter total length (a self-contradiction) or the existence of such paths in a smaller grid (contradicting the induction hypothesis). So these paths cannot exist, and the proof is complete.

10. $E_3(2) = \boxed{37}$. There are many possible explanation for this; here is a simple one. Another will be given in the solution to the next problem.

    Obviously a grid with and end-to-end path must have at least one black square in each column. There are $2^3 - 1 = 7$ ways to paint the squares of a column where at least one is black.

    - If the first column has just one black square—and there are three choices of the square to paint—the second column has to have a black square in the same row for there to be an end-to-end path. The other two squares in the second column could be painted in any way, so each of the three ways to paint one square in the first column has $2^2 = 4$ ways to paint the second column to obtain an end-to-end path. This accounts for $3 \cdot 4 = 12$ ways to paint the grid.
    - If the first column has two black-painted squares (there are three choices of how to do this), then only one of the seven ways of painting the second column does not lead to an end-to-end path. So this accounts for $3 \cdot (7 - 1) = 18$ end-to-end paths.

- If all the squares in the first column are painted black, then all seven ways of painting the second column lead to end to end paths.

Thus the total number of ways to paint a 3×2 grid that yield an end-to-end path is $12 + 18 + 7 = 37$.

11. If the bottom row does not have both squares painted black, then for there to be an end-to-end path it must exist completely in the first $k-1$ rows. But if the bottom row has both squares painted black, then there is an end-to-end path regardless of how the other $2(k-1)$ squares are painted. Thus $E_k(2) = 3E_{k-1}(2) + 2^{2(k-1)} = 3E_{k-1} + 4^{k-1}$.

    Alternately, if there is an end-to-end path entirely within the first $k-1$ rows, then any of the four ways to paint the bottom row will still leave the grid with an end-to-end path. There are also 3^{k-1} ways to paint the first $k-1$ rows so as not to create an end-to-end path, for a grid with two columns has an end-to-end path if and only if both squares in some row are painted black. If there is no end-to-end path in the top $k-1$ rows, then the bottom row must have both its squares painted black. This leads to $4E_{k-1}(2) + 3^{k-1}$ ways to paint the grid to obtain an end-to-end path.

    Using the logic of the previous paragraph, the exact number of paintings can be seen to be the total number of ways to paint the grid minus the number of ways to paint it without an end-to-end path. This will work out to $4^k - 3^k$, which naturally satisfies both recurrences.

12. Mimic the technique for question 10. Concentrate on the middle column.

    - If the middle column has only one square painted black (there are three choices for which square to paint black) then the adjacent square in each of the first and third columns must also be black, and any other squares in those columns may also be black. There are four choices of ways to paint the first column, and four choices for how to paint the third column that will create an end-to-end path. This leads to a total of $3 \cdot 4 \cdot 4 = 48$ paths.
    - There are two ways to choose two neighboring squares in the middle column to paint black. As seen in question 10, each of these leads to 6 ways to paint the first and 6 ways to paint the third column that will result in an end-to-end path. This adds another $2 \cdot 6 \cdot 6 = 72$ paths.
    - If the top and bottom squares in the middle column are painted there are two possibilities:
        - either the top or bottom, but not both, squares in the first column are painted. Whether or not to paint the middle square is a separate choice. In this case, only one of the two rows (top or bottom) is the beginning of an end-to-end path, and there are four ways to paint the third column to allow it to become end-to-end. So this accounts for $2 \cdot 2 \cdot 4 = 16$ end-to-end paths.
        - both the top and bottom squares in the first column are painted. The middle square's color is a separate choice. Then, as in question 10, there are six ways to choose to paint the third column to obtain an end-to-end path. Thus this option accounts for $2 \cdot 6 = 12$ end-to-end paths.

- Finally, if all three squares of the middle column are black, there are 7 choices of ways to paint the first column so that at least one square is black, and 7 choices for the third column as well. The total here is $1 \cdot 7 \cdot 7 = 49$.

Altogether, these cases result in a total of $E_3(3) = 48 + 72 + 28 + 49 = \boxed{197}$ ways to obtain an end-to-end path in a 3×3 grid.

13. There are many strategies that will work. Here is a simple one. Any time that player 1 paints a square in the first column black, player 2 paints the square in the same row in the second column white; any time that player 1 paints a square in the second column black then player 2 will paint the square in the same row in the first column white. Otherwise, player 2 chooses any unpainted square in the last four columns to paint.

    This strategy never tells player 2 to paint a square that has already been painted. In the first two columns, as soon as one square is painted the square in the same row in the other column will be painted immediately, so that player 1 never has a choice of painting a square whose neighbor has already been painted. For the rest of the columns, there are an even number of squares so if player 2 paints one of them player 2 will always have at least one to choose from. So player 2 always has a legal "move" in this strategy. The strategy ensures that player 2 will prevent player 1 from creating an end-to-end path because black squares in the first column will never be adjacent to a black square in the second column. So no path can ever start in the first column and make it to, much less past, the second column. Thus, no end-to-end path can be created.