Condensing Conundrums

Spring 2023 ARML Power Contest

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth. Be sure to read the problems carefully and answer the question that is being asked!

Notation: In this problem, different number bases will be indicated by subscripts in parentheses. For example, $785_{(13)}$ should be interpreted in base 13, so equals 1,292. Numbers without subscripts will always be in base 10. The base will always be rendered in base 10. Single digits' values never change regardless of the base being used, so for convenience will never receive a subscript.

Consider the number 785. One might split the number into two parts and add them: $78 + 5 = 83$. Repeating, $8 + 3 = 11$ and finally $1 + 1 = 2$. The split-and-add process stops once a single digit has been reached, because no further changes occur.

Instead, the number could have been split differently: $7 + 85 = 92$. But then $9 + 2 = 11$ and the final result is the same. The original number even could have been split-and-added as $7 + 8 + 5 = 20$, but $2 + 0 = 2$ once again ends at the same single digit.

The process of splitting a number and adding the pieces repeatedly until a single digit is reached will be called *condensing* a number.

The condensing process can also be applied in bases other than 10. For example, if 785 is viewed as a number in base 13, the condensing process might proceed as $7+8+5 = 17_{(13)}$ and then $1 + 7 = 8$. Or $78_{(13)} + 5 = 80_{(13)}$, and then $8 + 0 = 8$. Or $7 + 85_{(13)} = 8\mathsf{C}_{(13)}$ (C being a digit representing 12), and then $8 + \mathsf{C} = 17_{(13)}$, ending with $1 + 7 = 8$. Note that once again, the pathway to condensing the number didn't alter the outcome; the final result of condensing a number always yielded the same single digit.

With larger numbers, there are many more options for choosing the steps of the condensing process. For example, 123456 might be condensed as $123 + 456 = 579$ and then $5 + 7 + 9 = 21$ and $2 + 1 = 3$. Or the process could have begun $12 + 34 + 56 = 102$, and then $1+0+2 = 3$. No matter the choices made, the resulting single digit will always be the same, depending only on the starting number and the base, but not the particular choices made during the condensing process. This will be proven later.

<u>The Problems</u>

1. Condense each of the following numbers:

   (a) 9876. [1 pt.]

   (b) 888 888 888 888 888 (there are fifteen 8's). [1 pt.]

   (c) The number 777...777 consisting of 2023 7's. [1 pt.]

2. Condense each of the following base-15 numbers:

(a) $9876_{(15)}$. [1 pt.]

(b) $888\,888\,888\,888\,888_{(15)}$ (there are fifteen 8's). [1 pt.]

(c) The number $777\ldots777_{(15)}$ consisting of 2023 7's. [2 pts.]

3. When working in base 10, show that if A is any positive integer, and B is obtained from A by one single step of the condensing process, then $A - B$ is divisible by 9. For example, for $A = 138293156$ and $B = 1382 + 931 + 56 = 2369$, their difference is $A - B = 138290787$ can be checked to be divisible by 9. [3 pts.]

When working in a base b other than base 10, the corresponding result is that the difference of A and any number obtained from A by one step of the condensation process is divisible by $b - 1$.

4. Prove that, no matter the steps chosen in the condensation process, the resulting single digit will always be the same for any given starting number. [2 pts.]

5. Show that, given any starting number, the smallest possible number that can be achieved by one step of the condensation process is the result of adding the digits of the number. That is, no way of splitting the number into parts and adding the parts results in a smaller number than if each part is a single digit. [2 pts.]

In base 10, the numbers 1–9 are already condensed. The numbers 10–18 take only one step to condense, but 19 requires two steps: $1 + 9 = 10$; $1 + 0 = 1$.

6. When each condensing step consists of simply adding the digits, compute the least positive integer for which the condensation process requires:

   (a) three steps. [1 pt.]

   (b) four steps. [1 pt.]

In light of problem 5 it might seem that adding the digits at each condensation step requires the fewest steps. But that is not the case! For example, to condense 982, one might add the digits to obtain $9 + 8 + 2 = 19$, and then $1 + 9 = 10$, finishing with the third step $1 + 0 = 1$. But instead, one could start the process with $98 + 2 = 100$ and then finish in a total of two steps with $1 + 0 + 0 = 1$. Being free to choose how to split the number before adding the pieces can reduce the number of steps needed to completely condense a number.

7. Compute the least positive integer for which the condensation process requires at least three steps, regardless of how the number is split into pieces at each step. [2 pts.]

How dramatic can the reduction in the number of steps actually be? Believe it or not, with the correct choices in how to split the number at each step:

- *Every* positive integer, written in base 4 or larger can be completely condensed in at most four steps.

- *Every* positive integer, written in base 3 can be completely condensed in at most three steps, and only 11 numbers require three steps; all others can be condensed in at most two steps.

- *Every* positive integer, written in base 2 can be completely condensed in at most two steps.

The remainder of this problem will be devoted to proving the statement about numbers expressed in base 2.

8. Note that in base 2, the only single-digit number is $1_{(2)}$. Use this fact to explain why a positive integer, written in base 2, can be condensed in one step if and only if it is a power of two. [2 pts.]

Thus, to fulfill the claim that every number, in base 2, can be condensed in at most two steps it must be shown that every number that is not already a power of two can be condensed in one step to a power of two! To facilitate the argument, let the starting number be denoted a, let m be the number of 1's in its base-2 expansion, and let k be such that $2^k < m \leq 2^{k+1}$. For example, for $a = 45 = 101101_{(2)}$, $m = 4$ and $k = 1$. Note that, since m is a power of two, this number can be condensed to a power of two simply by adding its digits.

When $m = 1$ the number is a power of two, and can be condensed in one step by adding its digits. When $m = 2$ or $m = 4$ adding the digits takes one step, and results in a power of two that can be condensed in a second step by adding the digits.

9. When $m = 3$ the number has exactly three 1's, and may have any number of 0's in its binary expansion. Construct a strategy for splitting any such number into pieces so that the sum of the pieces will result in a power of two. Make sure to demonstrate that the strategy will work for *any* positive integer that has exactly $m = 3$ 1's in its binary expansion. [3 pts.]

10. Create and prove a strategy that will take any number with $m = 5$ 1's in its binary expansion and condense it to a power of two in one step. [3 pts.]

11. Create and prove a strategy that will work when $m = 6$ or $m = 7$. [2 pts.]

Of course, when $m = 8$ simply adding the digits produces a power of two.

For larger m, observe the following. Given any number written in binary, if the individual digits are simply added, the resulting number will be m. But what if we create some two- or three-digit pieces? If, instead of $\cdots + 1 + 0 + \cdots$ the digits are grouped $\cdots + 10 + \cdots$ then those two digits will contribute 2 instead of 1 to the sum of the pieces. If $\cdots + 1 + 1 + \cdots$ is regrouped as $\cdots + 11 + \cdots$ then those two digits will contribute 3 instead of 2 to the sum. For example, for $38 = 100110_{(2)}$, when the pieces are all one-digit long, the total is $1 + 0 + 0 + 1 + 1 + 0 = 3$. If the pieces are $10_{(2)} + 0 + 1 + 1 + 0$ the total increases to 4, as it does when the pieces are $1 + 0 + 0 + 11_{(2)} + 0$ or $1 + 0 + 0 + 1 + 10_{(2)}$. A total of 5 could be reached by making more than one two-digit piece.

12. As just seen, the sum of the pieces increases by one when two digits are grouped together. Determine the possible effects on the sum when a three-digit piece is created. [2 pts.]

Note that when $1+1+1$ is replaced with the group $111_{(2)}$ these three digits contribute $7/3$ times as much to the new sum as they did when standing as single digits. $1+1+0$ replaced by $110_{(2)}$ multiplies the contribution of these three digits by 3. $1+0+1$ to $101_{(2)}$ is a multiplier of $5/2$, and $1+0+0$ to $100_{(2)}$ is a multiplier of 4.

13. Prove that, starting at the left end of the number and creating as many three-digit pieces as possible, the sum of these pieces is at least $2m$. [3 pts.]

Note that $m \leq 2^{k+1} < 2m$, so that somewhere in the process of creating three-digit pieces the sum of the pieces must become greater than 2^{k+1}. Stop creating three-digit pieces at this point, and instead create two-digit pieces as long as the sum remains under 2^{k+1}. For example, consider $110\,110\,111\,010\,011_{(2)}$ with $m = 10$. The strategy produces:

$$
\begin{aligned}
1+1+0+1+1+0+1+1+1+0+1+0+0+1+1 &= 10 \\
110_{(2)}+1+1+0+1+1+1+0+1+0+0+0+1 &= 13 \\
110_{(2)}+110_{(2)}+1+1+1+0+1+0+0+0+1 &= 17 \text{ (too big!)} \\
110_{(2)}+11_{(2)}+0+1+1+1+0+1+0+0+0+1 &= 14 \\
110_{(2)}+11_{(2)}+0+11_{(2)}+1+0+1+0+0+0+1 &= 15 \\
110_{(2)}+11_{(2)}+0+11_{(2)}+10_{(2)}+1+0+0+0+1 &= 16
\end{aligned}
$$

When the tripling-up resulted in a sum larger than the next power of two, that step was undone and the digits were only grouped into two-digit pieces after that.

14. Prove that if $m \geq 10$ then this strategy will always be able to produce a power of two. [4 pts.]

15. Create and prove a strategy that will create a power of two with one condensation step when $m = 9$. [3 pts.]

Taken together, these problems show that, for any positive integer written in base 2, it is possible to choose a way to split the number into pieces so that the sum of these pieces is a power of two, and then a second condensation step consisting of just adding the digits will result in a single digit, thus proving the claim.