# Condensing Conumdrums

Spring 2023 ARML Power Contest

**The Solutions**

1. Since the order in which condensation proceeds is being assumed not to matter, each number will be condensed by adding all the digits together, repeating as many times as needed.

   (a) $9+8+7+6=30$, and then $3+0=\boxed{3}$.

   (b) The fifteen 8's add to 120, and $1+2+0=\boxed{3}$.

   (c) The 2023 7's add to $2023 \cdot 7 = 14161$. Then $1+4+1+6+1=13$, and $1+3=\boxed{4}$.

2. Again, condensation will proceed by adding the digits of the number, repeating as necessary.

   (a) $9+8+7+6=30=20_{\text{(fifteen)}}$ and then $2+0=\boxed{2}$.

   (b) The fifteen 8's add to 120, which is $80_{\text{(fifteen)}}$, and $8+0=\boxed{8}$.

   (c) The 2023 7's add to $14161 = 42\mathsf{E}1_{\text{(fifteen)}}$. Since $\mathsf{E}$ represents 14, these digits add to 21, or $16_{\text{(fifteen)}}$, and finally $1+6=\boxed{7}$

3. Let $A = d_n d_{n-1} \ldots d_2 d_1 d_{0\text{(ten)}}$. This can be expanded to $A = d_n 10^n + d_{n-1}10^{n-1} + \cdots + d_2 10^2 + d_1 10^1 + d_0$. When the number is split into pieces, each piece is a base-10 representation of some number, so has the same coefficients $d_i$ with possibly different powers of 10. For example, of one of the pieces is $d_{k+3}d_{k+2}d_{k+1}d_k$ then instead of being attached to $10^{k+3}$, $d_{k+3}$ will simply be attached to $10^3$. So $B$ is just $A$ with some of the exponents on the power of 10 changed. But the difference $A-B$ will then be a sum of terms like $d(10^a - 10^b)$. This can be factored as $d10^b(10^{a-b}-1)$. The last factor in that is always divisible by 9 ($10^{a-b}-1 = (10-1)(10^{a-b-1} + 10^{a-b-2} + \cdots + 10^2 + 10^1 + 10^0$ by factoring), so the difference between $A$ and $B$ is a sum of numbers each of which is divisible by 9, so their sum is also.

4. From the previous problem, any two ways of condensing a number will results in single digits that differ by a multiple of 9. The only single digits that differ by 9 are 0 and 9, and 0 cannot be obtained by condensing, so the final digits obtained from the condensing process will be the same no matter what route is taken.

5. Following the proof in problem 3, when a number is condensed the power of 10 that each digit is multiplied by is reduced. The maximum reduction possible is then the power is reduced to zero. That is, each digit is multiplied by $10^0 = 1$, so must appear as a single digit being summed in the condensation process.

6. (a) If the digits of a number sum to something less than 19 then condensation will complete in one more step. So the sum of the digits must be at least 19. The smallest number for which this occurs is $\boxed{199}$. Note that 199 can be condensed

in two steps via $199 \to 1 + 99 = 100 \to 1 + 0 + 0 = 1$, but the question asked specifically for the case of simply adding the digits without creating any multiple-digit blocks.

(b) This time the digits need to add to at least 199. The smallest number for which this happens is $1999 \cdots 999$ in which there are exactly twenty-two 9's. That is, the number is $\boxed{19\,999\,999\,999\,999\,999\,999\,999}$.

7. The digits of the required number should add to 19, though as seen above the number 199 can actually be condensed in two steps. The next candidate would be $\boxed{289}$. This can be checked to work. If broken into pieces as $28 + 9$ the result is $289 \to 28 + 9 = 37 \to 3 + 7 = 10 \to 1 + 0 = 0$ taking three steps. The other ways of breaking 289 into pieces are $289 \to 2 + 89 = 91 \to 9 + 1 = 10 \to 1 + 0 = 1$ or $289 \to 2 + 8 + 9 = 19 \to 1 + 9 = 10 \to 1 + 0 = 1$.

8. In base 2 there is only one 1-digit number, namely 1. Thus the digits of the number to be condensed must add to 1, so there must be one digit of 1 and all the rest of the digits are zeroes. So the number, in base 2, is $100 \cdots 00_{(\text{two})}$ where there can be any nonnegative integer number of zero digits. Such a number represents $2^n$ where $n$ is the number of zero digits. Conversely, $2^n$, written in base 2, consists of 1 followed by $n$ zeroes, so condensing by adding the digits gives a result of 1.

9. If the number is $111_{(\text{two})}$ then condense as follows: $111_{(\text{two})} \to 1 + 11_{(\text{two})} = 100_{(\text{two})} \to 1 + 0 + 0 = 1$. Otherwise the number has 0 for at least one of its digits. Take the first 0 that occurs after a 1 and group them into one piece, while leaving all other digits as single pieces. For example, $1100010_{(\text{two})}$ should be grouped as $1 + 10 + 0 + 0 + 1 + 0$. In this grouping, the single 0's can be ignored, while the two singleton 1's add to two and the remaining $10_{(\text{two})}$ represents 2, so the sum of the digits is 4, a power of two.

10. Time for some casework. The following show how to obtain a power of two after one condensation step for any base-2 number with exactly $m = 5$ 1 digits:

Case 1: $11111_{(\text{two})} \to 1111_{(\text{two})} + 1 = 10000_{(\text{two})}$.

Case 2: $111110_{(\text{two})} \to 1111_{(\text{two})} + 1 + 0 = 10000_{(\text{two})}$.

Case 3: All other numbers with a single 0 in them can be split, for example, as $1 + 101_{(\text{two})} + 1 + 1 = 1000_{(\text{two})}$.

Case 4: Any number with two consecutive 0 digits can be split as $100_{(\text{two})}$ plus four single 1's, totaling 8.

Case 5: Any number with two or more zeroes, none of which are consecutive must have a 101 group somewhere in it, so can be split as in case 3 above.

11. In the case of $m = 7$, simply group the two leading digits into one piece. This will add one to the digital sum. For $m = 6$, group the leading two digits into one piece. There are at least three 1's left, so group the left-most remaining 1 with the digit following it, and now the digital sum has been increased by two. For example, $11000110101_{(\text{two})} \to 11 + 0 + 0 + 0 + 11_{(\text{two})} + 0 + 1 + 0 + 1 = 8 = 1000_{(\text{two})}$.

12. There are four cases:

    (a) $+1+0+0+$ is replaced with $+100+$, changing the contribution from 1 to 4.
    (b) $+1+0+1+$ is replaced with $+101+$, changing the contribution from 2 to 5.
    (c) $+1+1+0+$ is replaced with $+110+$, changing the contribution from 2 to 6.
    (d) $+1+1+1+$ is replaced with $+111+$, changing the contribution from 3 to 7.

    So the sum increases by either 3 or 4.

13. Assume that the process of creating three-digit pieces collects every 1 into one of the pieces. Then since each piece contributes at least $7/3$ as much to the sum as the individual digits did, the sum will be at least $7m/3 > 2m$. If the three-digit pieces leave out a single 1, then the sum will be at least $7(m-1)/3+1 = 7m/3-4/3$. If two 1's are left out, the sum will be at least $7(m-2)/3+2 = 7m/3-8/3$. Using the fact that $m \geq 8$, note that $7m/3 = 2m + m/3 \geq 2m + 8/3$, so in all cases the new sum is at least $2m$.

14. Let $q$ be the number of 1's that have been collected into three-digit pieces. At the start, $q = 0$ and the sum of the pieces is $m$. As more 1's are collected into three-digit pieces, the sum at any given point will be at least $\frac{7}{3}q + (m-q)$. By the previous problem, if enough 1's are collected into three-digit pieces the sum will be at least $2m$ at some point. So at some point as the sum changes from $m$ to $2m$ it must either equal or become larger than a power of 2. In fact, if $m$ is not already a power of 2 (in which case the process stops immediately!) then $2m \geq 2^k + 2$. For $m$ between 10 and 15, $2m \geq 2^k + 4$. Otherwise, $m$ will be at least 17.

    If $10 \leq m \leq 15$ then 1's are collected into three-digit groups as long as $\frac{7}{3}q + (m-q) < 2m-4$ or $\frac{4}{3}q < m-4$ and thus $q < m-5$. On the other hand, is $m \geq 17$ then $\frac{7}{3}q + (m-q) < 2m-2$ so $\frac{4}{3}q < m-2$ and once again $q < m-5$. So in either case, the process must stop creating three-digit groups while there are still *at least* six 1's left.

    But at this point, the current sum must be at least $2^k - 3$ because creating a 3-digit group can only increase the sum by 3 or 4. Since there are at least six 1's remaining, at least 3 2-digit groups can be formed, raising the sum by 3. That is, any deficit in creating a sum of $2^k$ by forming 3-digit groups can be made up by forming as many 2-digit groups as needed from the remaining 1's. Thus any number with $m \geq 10$ can be condensed to a power of 2 in one step.

15. The 3-digit strategy will work in the case of $m = 9$ as well. Care needs to be taken to make sure of all the details.

    When the process begins, there are $m = 9$ 1's, and the current sum is 9.

    - If the first 3-digit group created is 111 or 110, then the current sum is 13, and there are six 1's remaining, which can form at least 3 2-digit groups which will achieve a sum of 16.
    - if the first 3-digit group created is 101 or 100, then the current sum is 12. Consider the next 3-digit group created.

- If it is 111 or 110 then a sum of 16 has been achieved.
- If it is 100 or 101 then a sum of 15 has been achieved, and there are still at least 5 1's so that a 2-digit group can be created to achieve a sum of 16.

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