Counting Configurations

Spring 2026 ARML Power Contest

The Background

This set of problems is concerned with counting configurations of finite sets of points in the plane with restrictions on the number of different distances that are allowed between the points. In this problem, configurations of points that are similar, in the usual Euclidean geometry sense of the word, will be considered the same. For example, all sets of vertices of an equilateral triangles are the same configuration, as are all sets of vertices of squares, though there are infinitely many configurations of four points that are the vertices of a rectangle because rectangles with different ratios of length to height are not similar. A 3-4-5 triangle and a 30-40-50 triangle are also the same configuration. A finite set of points is a *standard configuration* if it has been scaled so that the least distance between any two points in the set is 1. The vertices of a 3-4-5 triangle are not a standard configuration, but those of the 1-$\frac{4}{3}$-$\frac{5}{3}$ triangle are.

The *signature* of a configuration is defined as follows. If the configuration is standard, its signature is the non-decreasing sequence of distances between the points, written as an ordered tuple. If it is not standard, first select a similar configuration that is standard and then take its signature. For example, the vertices of a 3×4 rectangle are not a standard configuration, so first select the similar $1 \times \frac{4}{3}$ and take its signature: $(1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3})$. If the same number appears several times in the signature, the signature can be written shorthand by writing each distance only once, with a parenthesized subscript indicating the number of times that distance occurs. The previous signature could have been written $(1_{(2)}, \frac{4}{3}{}_{(2)}, \frac{5}{3}{}_{(2)})$. As another example, the three vertices of any equilateral triangle have the signature $(1_{(3)})$.

The *type* of a configuration is the ordered pair (n, m) where n is the number of points in the configuration and m is the number of different distances. The rectangle in the previous paragraph has type $(4, 3)$ while the equilateral triangle has type $(3, 1)$. As you will show in the problems, n has a strong influence on the range of possible m. If there are very few points in the configuration, there can't be too many different distances between them! More interestingly, for a given number of points, there is a minimum number of distances that must exist in any configuration of n points.

In the problems, you will be asked to *describe* configurations of points. Any unambiguous description will be acceptable. A list of ordered pairs locating points in the plane will suffice, for example. But it may be easier to use geometric descriptions, such as "the vertices of a square together with its center point." When asked to describe infinitely many configurations, it is sufficient to include a parameter that can be varied. For example, "the set of points $(0, 0)$, $(1, 0)$, and $(0, y)$ for $y > 1$" describes infinitely may configurations of type $(3, 3)$ because the signature of such a set is $(1, y, \sqrt{y^2 + 1})$ which is a different triple of real numbers for every different $y > 1$.

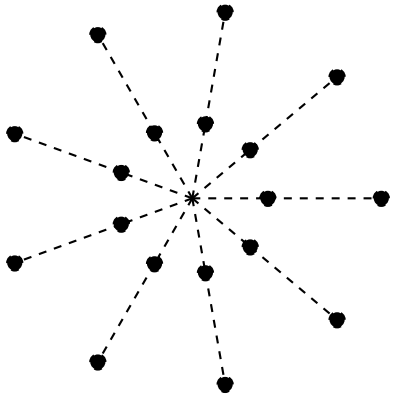
The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth. Be sure to read the problems carefully and answer the question that is being asked!

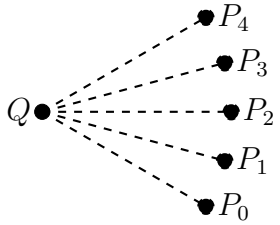
1. Determine the signature of each of the following configurations of points:
   (a) The vertices of a square. [1 pt.]
   (b) The vertices of a rhombus whose angles are $60°$ and $120°$. [1 pt.]
   (c) The vertices of a regular pentagon. [1 pt.]
   (d) The vertices of a regular hexagon. [1 pt.]
2. Determine the type of each configuration from problem 1. [2 pts.]
3. Given a configuration of n points, show that the number of distances in its signature (which is equal to the sum of the subscripts in the shorthand signature) is equal to $n(n-1)/2$. [2 pts.]
4. (a) Given a configuration of n points, show that it is impossible for its type (n, m) to have $m > n(n-1)/2$. [1 pt.]
   (b) Explicitly construct a set of n collinear points of type $(n, n(n-1)/2)$. (You can do this by showing how to construct a set of numbers x_1, x_2, ..., x_n so that the points $(x_1, 0)$, $(x_2, 0)$, ..., $(x_n, 0)$ on the x-axis have exactly $n(n-1)/2$ different distances between them.) [2 pts.]
5. There is only one configuration of type $(3, 1)$, the vertices of an equilateral triangle. Describe infinitely many different configurations of type $(3, 2)$. [2 pts.]
6. Show that there are no configurations of type $(4, 1)$. [2 pts.]
7. Describe infinitely many different configurations of type $(4, 3)$. [2 pts.]
8. There are a finite number of different configurations of type $(4, 2)$. Explore them!
   (a) Describe all configurations of type $(4, 2)$ consisting of the vertices of an equilateral triangle, with the fourth point located somewhere on the line through a vertex and the midpoint of the opposite side of this triangle. Show that you have found all such configurations. [3 pts.]
   (b) Show that there are no configurations of type $(4, 2)$ consisting of the vertices of an equilateral triangle where the fourth point does *not* lie on one of the median lines. [1 pt.]
   (c) Describe all other configurations of type $(4, 2)$. Show that your collection of such configurations is complete. [2 pts.]
9. Let $n = 2j + 1$ with $j \geq 1$. Show that the vertices of the regular n-gon are a configuration of type (n, j). [1 pt.]

10. Let $n = 2k$ with $k \geq 2$. Describe three different configurations of type (n, k). (One point for each configuration found.) [3 pts.]

11. Let $n = 2k$ be even. Let G be the configuration consisting of two concentric regular k-gons whose vertices are centrally aligned. That is, the line containing a vertex from the outer k-gon and the corresponding vertex from the inner k-gon passes through the shared center of the two k-gons. See the diagram below for the case of $k = 9$. Assume there is no special relationship between the sizes of the k-gons (so that, for instance, the length of a side or diagonal of the outer k-gon is not equal to a distance from a vertex of the outer k-gon to a vertex of the inner k-gon). Determine the type of the configuration G [2 pts.]



Consider points P_0, P_1, ..., P_n arranged in a fan-shaped configuration about point Q so that $P_0Q = P_1Q = \cdots P_nQ$ and $\angle P_0QP_1 = \angle P_1QP_2 = \cdots = \angle P_{n-1}QP_n$. The case of $n = 4$ is shown below:



Note that there are n distinct distances between the different P_i and (except for certain special choices of angle) one more distance from each P_i to Q. Thus, by varying the angle between the P's there are infinitely many different configurations of type $(n + 2, n + 1)$. So, for large enough m there are infinitely many configurations of type (n, m). In fact, by varying the relative sizes of the k-gons in problem 11, there are infinitely many configurations when m is as small as roughly $\frac{3}{4}n$. On the other hand, if m is too small, there isn't enough freedom to arrange the points to make a configuration of n points. You have already shown there are no configurations of type $(4, 1)$, for example. Somewhere in the middle there are combinations of m and n with finitely many combinations. We will look at a few more special cases.

Let's specifically explore the case of $n = 5$. From the fan-shaped example it can be deduced that there are infinitely many configurations of type $(5, 4)$. One could also use the four vertices and center of rectangles of various height-to-width ratios to obtain the same result. On the other hand...

12. Prove that there is exactly one configuration of type $(5, 2)$. [3 pts.]

13. Draw as many different configurations as you can find of type $(5, 3)$. [4 pts.]

(The scoring for this last problem will be one point for each six different configurations found. There will be a bonus if you are able to prove you have found all possible configurations of type $(5, 3)$.)

A couple more interesting cases:

14. Show there are no configurations of type $(6, 2)$. [2 pts.]

15. From problem 9, the regular 19-gon has type $(19, 9)$. Describe a configuration of type $(19, 8)$. [2 pts.]