Counting Configurations

Spring 2026 ARML Power Contest

The Solutions

1. (a) Since there are four vertices of the square, there are $\binom{4}{2} = 6$ pairs of points to measure the distance between. Four are sides, which are equal, and two are diagonals which equal each other and are longer than the sides. Thus, to make the shortest distance equal to 1, set the sides to length one, making the diagonals have length $\sqrt{2}$. This leads to the signature $\boxed{(1, 1, 1, 1, \sqrt{2}, \sqrt{2})}$ or its shorthand version $\boxed{(1_{(4)}, \sqrt{2}_{(2)})}$.

   (b) Again, four sides have equal length. The choice of vertex angles makes the shorter diagonal also have this same length (it splits the rhombus into two equilateral triangles). The remaining diagonal is longer than the sides. So make the sides of length 1 and then the long diagonal will have length $\sqrt{3}$. The signature is thus $\boxed{(1, 1, 1, 1, 1, \sqrt{3})}$ or $\boxed{(1_{(5)}, \sqrt{3}_{(1)})}$.

   (c) All diagonals of a regular pentagon have the same length. For mathletes, it is worthwhile to know that this length is the golden ration, $\frac{1+\sqrt{5}}{2}$. This can be found by examining the many similar triangles formed by a regular pentagon and its diagonals. The signature of the figure is thus $\boxed{(1_{(5)}, \frac{1+\sqrt{5}}{2}{}_{(5)})}$.

   (d) A regular hexagon has six "short" diagonals and three "long" diagonals that pass through the center of the hexagon. By assembling six equilateral triangles with sides of length 1 around a central point, it can then be seen that the signature is $\boxed{(1_{(6)}, \sqrt{3}_{(6)}, 2_3)}$.

2. Counting the numbers of point and different distances in each of the parts of the previous problem, the types are (a) $(4, 2)$, (b) $(4, 2)$, (c) $(5, 2)$, and (d) $(6, 3)$.

3. Each two points in the configuration have a distance between them. There are $\binom{n}{2} = n(n-1)/2$ pairs of points. Alternately, each of the n points has a distance between it and each of the other $n-1$ points, for a total of $n(n-1)$ distances, but each pair of points is counted twice in this arrangement, so the result must be divided by two.

4. (a) Since m counts the number of different distances within the configuration, and the total number of distances (by the previous problem) is $n(n-1)/2$, the number of different distances certainly cannot exceed that amount!

   (b) One way to do this is to add new points one at a time, each one so far away from all previous points so that the sum of distances of the previous points is less than the shortest distance of the new point from any of the old ones. A simple, though of course not the only, way to do this is to place the points at $x = 1, 2, 4, 8, \ldots, 2^{n-1}$.

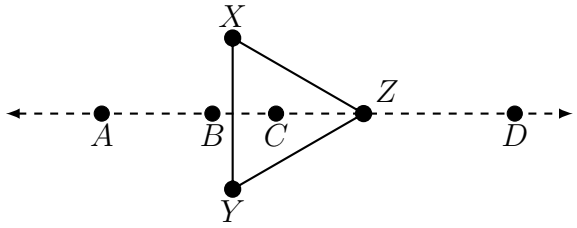
5. Type $(3, 2)$ means there are three points in the configuration with exactly two different distances between them. Since there are three pairs of points but only two different distances, one of the distances is used twice—the configuration is an isosceles (but not equilateral!) triangle. So an infinite collection of different configurations is the collection of sets of vertices of 1-x-x isosceles triangles with $x > 1$.

6. A configuration of type $(4, 1)$ would mean that there are four points, each equidistance from each other. Start with one point. Each other point is equidistant from it, so all three other points lie on a circle of radius 1 about the first point. But they are equidistant from each other as well, so form an equilateral triangle. The equilateral triangles whose vertices lie on a circle of radius 1 have sides of length $\sqrt{3}$, not 1, so the desired configuration is not possible.

7. As hinted at in the background material, any non-square rectangle will have three different distances—the short sides, the long sides, and the diagonals. So the vertices of any rectangle with sides of length 1 and x, for $x > 1$, will be a configuration of type $(4, 3)$ and no two different choices of x lead to similar rectangles so these are all different.

   There are other possible configurations. For instance, choose three vertices of an equilateral triangle with a fourth point almost anywhere on the line between a vertex and the midpoint of the opposite side ("almost" because some choices of position for this point lead to fewer distances, such as choosing the center of the triangle—see the next problem where all configurations of type $(4, 2)$ are determined).

8. (a) There are four such configurations, as indicated by the diagram below:



   Let the sides of the triangle have length 1, the vertex through which the line passes be denoted Z, and the two vertices on the side through which the line passes be called X and Y. As the fourth point lies on the perpendicular bisector of $\overline{XY}$, it is always equidistant from those two vertices, while its distance to Z is a different distance (unless the fourth point is at the circumcenter of the triangle). Since only two distances are allowed in the configuration, one of these two distances must be 1 (again, unless the fourth point is at the circumcenter).

   If the distance of the fourth point to X and Y is 1 (point A) then the figure is the rhombus with 60- and 120-degree angles. As the point moves toward the triangle, there is a point (labeled B) that is one unit aways from Z on the opposite side of $\overline{XY}$ from it as Z. This configuration is actually not standard, as the shortest distance $BX = BY$ is less than one. The point then crosses into the triangle, and

its distance from all three vertices is less than one, but it does reach point C, the circumcenter of the triangle. The only other point that is distance 1 from both X and Y is already occupied by point Z, so the traveling point now moves out of the triangle past Z until it reaches D, at distance 1 from Z on the same side of $\overline{XY}$ as Z. This accounts for all possibilities; there are four such.

(b) Let the triangle be XYZ and the fourth point be W. Since only two distances are allowed in the configuration, at least two of WX, WY, and WZ must be equal by the pigeonhole principle. If, say, $WX = WY$ then W is on the perpendicular bisector of $\overline{XY}$, which passes through Z and the midpoint of $\overline{XY}$. Thus if the configuration contains the vertices of an equilateral triangle, the fourth point must be on one of the lines through a vertex and the midpoint of the opposite side.

(c) All cases where three of the points form the vertices of an equilateral triangle have been considered. Of the $\binom{4}{2} = 6$ distances between the points, if five of them are equal (say $WX = WY = WZ = XY = XZ$ with YZ a different distance) then WXY is an equilateral triangle. So these possibilities have already been considered.

If four of the six distances are equal, then to avoid an equilateral triangle there must be a cycle: $WX = XY = YZ = ZW$. Thus the four points form a rhombus. The other two distances WY and XZ must be equal, and a rhombus with equal diagonals is a square.

Finally, there could be three each of two different distances. Thus $WX = XY = YZ$ and $WY = XZ = WZ$. This can only occur with the *golden trapezoid* where one base and the two sides have length 1 and the other base and diagonals have length $\frac{1+\sqrt{5}}{2}$. This is the configuration that results from taking the five vertices of the regular pentagon and eliminating one.

9. Label the vertices V_0, V_1, $\ldots V_{2j}$. By symmetry, the distance between vertices V_a and V_b is the same as that between V_c and V_d if and only if $b - a \equiv \pm(d - c) \pmod{n}$. Now, modulo n, these differences take on the values ± 1, ± 2, $\ldots$, $\pm j$, so there are k possible distances between the points.

10. The three easiest configurations are modifications of the previous problem.

- First, let $k = j + 1$. Then the vertices of the $2j + 1$-gon have j distance between them. Add in the point at the center of the polygon, equidistant from all vertices (with that distance only equalling the distance between points on the regular polygon if that polygon has a multiple of 6 sides, which is not possible because $2j + 1$ is odd). That is, the $(n - 1)$-gon together with its center is of type (n, k).
- The regular n-gon itself is also the correct type, as an argument similar to the previous problem shows.
- A cute trick is to use the $(2k + 1)$-gon which, by the previous problem, is of type $(2k + 1, k)$. Now simply delete one of the vertices.

There are certainly other configurations that will work (there were six configurations

found of type $(4, 2)$ in problem 8!). Certainly any other correct descriptions will be accepted.

11. First, let $k = 2j + 1$ be odd. Then there are j different distance among the points in the inner polygon, and j more among the point in the outer polygon, while there are $j + 1$ different distances from a vertex of outer polygon to vertex of the inner, totaling $3j + 1$. Thus, this configuration is of type $(4j + 2, 3j + 1)$

    Second, let $k = 2j$ be even. Then there are j different distance among the points in the inner polygon, another j distances among the point in the outer polygon, and a further $j + 1$ distances from a vertex of the outer to a vertex of the inner, totaling $3j + 1$. So in this case, the type is $(4j, 3j + 1)$.

    Both cases can be written using the formula $(n, 3\lfloor \frac{n}{4} \rfloor + 1)$.

12. It is already known that the configuration of the vertices of a regular pentagon has type $(5, 2)$. We demonstrate no other configuration has this type.

    If one of the points is deleted from a configuration of type $(5, 2)$ the remaining points would form a configuration of type $(4, 2)$ or $(4, 1)$. Problem 6 showed there are no configurations of the latter type. So consider the five configurations of type $(4, 2)$ found in problem 8 other than the golden trapezoid.

    For the four configurations that consist of an equilateral triangle plus a point on one of the medians, first note that the four possible locations of that fourth point each lead to a different distance being created in addition to the length of the triangle sides. If the original triangle has sides of length 1, then the point A is at distance 1 from X and Y, and distance $\sqrt{(}3)$ from Z. B is distance 1 from Z and $\frac{\sqrt{6}-\sqrt{2}}{2}$ from X and Y. C is distance $\sqrt{3}/3$ from all of X, Y, and Z, while D is distance 1 from Z and $\frac{\sqrt{6}+\sqrt{2}}{2}$ from X and Y. Since pairs of distances are all different, the fifth point, when added back, would have to be at the symmetric location on a different median of the triangle. It is routine to check that these two points would then be at a third distance from each other.

    Next, consider the square $ABCD$ with $AB = 1$ and $AC = \sqrt{2}$. There are two points located at distance 1 from both A and B, but neither of these is at distance 1 or $\sqrt{2}$ from C or D. The same holds for the points that are $\sqrt{2}$ from both A and B, or that are $\sqrt{(}2)$ form A and distance 1 from B. So there is no type $(5, 2)$ configuration that contains a square.

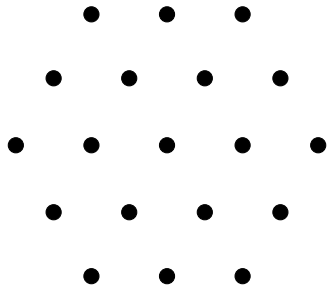
    Trying a similar analysis for the golden trapezoid does yield a configuration that is of type $(5, 2)$—the regular pentagon.

13. See Appendix A, the table of configurations at the end of these solutions. At present, the author has not confirmed that this list is comprehensive. Any other configurations found will be added to this table; a proof that all configurations have been found would be most welcome!

14. Continuing the argument from problem 12, if one point were removed the remaining configuration would be of type $(5, 2)$. The vertices of a pentagon are the unique con-

figuration of this type. Similar to the argument from that problem, take circles of radius 1 and $\frac{1+\sqrt{5}}{2}$ about two of the points. The possible intersections of these points are either already points in the configuration or would add at least one more distance, so no configuration of more points can be found that have only two distances.

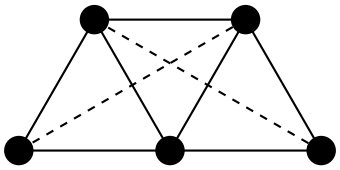
15. Take the points in triangular lattice that form a hexagon with three points on each side. It can be counted, with care, that there are exactly 8 different distances among points in this collection.



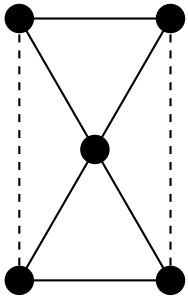
Appendix A: Catalog of Type $(5, 3)$ Configurations

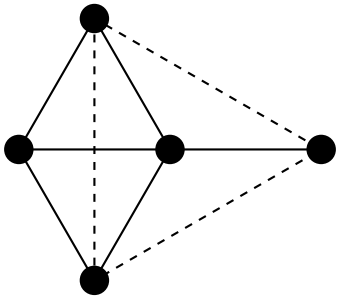
There are a few type $(5, 3)$ configurations where one of the three distances occurs at least six times. No distance can occur 8 times, for by the pigeonhole principle that would require one point to be at that distance from all four other points—all four would be a on a circle about the central point. Casework then shows that it is impossible to arrange four points on a circle so that four of the six distances between them are equal to the radius.

If one of the three distances occur between 7 pairs of points, casework quickly shows that there must be one point at this distance from each of the four others. In this case, those points lie on a circle about the central point, and since the radius must occur three more times as a distance between these four points, it can be fairly quickly concluded that the only configuration possible is "half a hexagon"—four consecutive vertices of a regular hexagon plus the center point. The length of the short diagonal ($\sqrt{3}$ if the hexagon has side length 1) occurs twice, and the length of the long diagonal—twice the radius—occurs once.

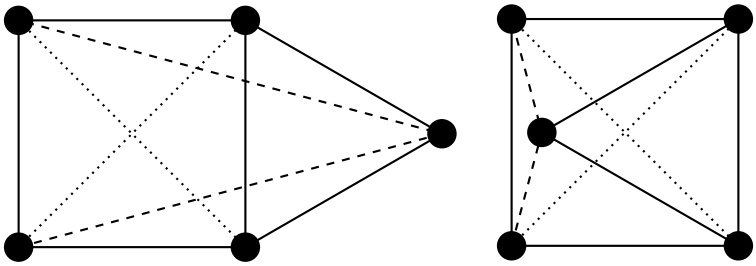


There are six configurations where one distance occurs six times. Two of them are the regular hexagon with two non-consecutive vertices deleted.

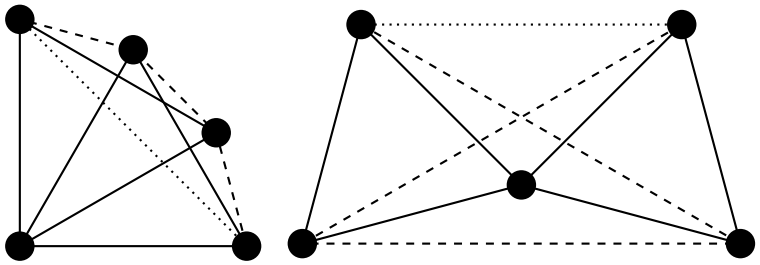




Two others two are squares with an equilateral triangle glued to one side—either interior or exterior to the square.



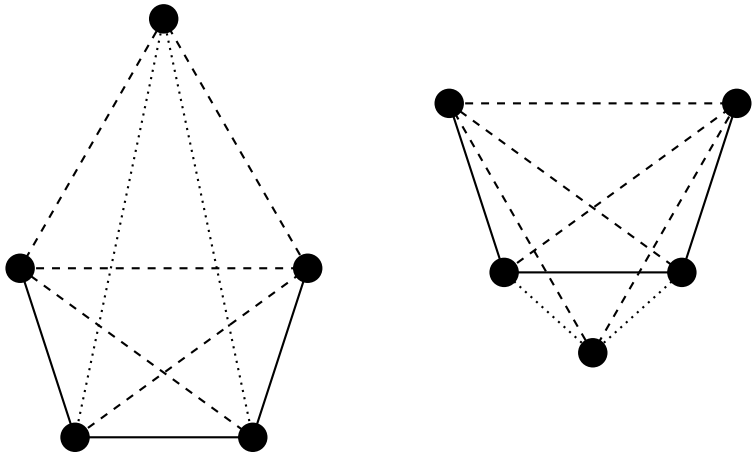
The final two cases where a distance occurs six times are configurations consisting of the center of a circle and four points lying on the circle, but unlike the first two cases (which also have four points lying on a circle about the fifth) the four points on the circle to not lie at vertices of a regular hexagon. One has the points spaced along the circle at 30° intervals, the other at 150° intervals.



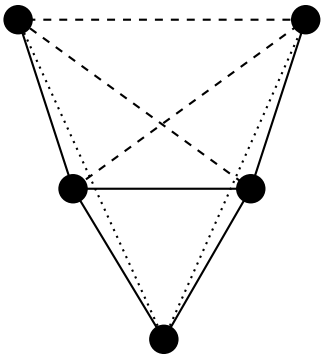
The cases where the most common distance occurs 5 or 4 times are much more plentiful and difficult to categorize. (Note that since there are only three distances, the most common one must occur at least 4 times.)

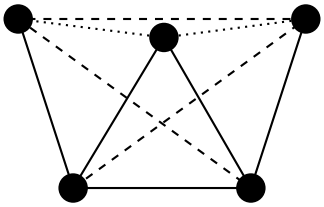
One case to consider is where four of the five points are four of the vertices of a regular pentagon. These form a trapezoid with three sides of one length, and one side and both diagonals of a second length, which is $\frac{1+\sqrt{5}}{2}$ times as long. With considerable casework, it can be shown that the only possible locations for a fifth point that does not introduce at least two new distances is along the line of symmetry of the figure. Of course, one of these locations is the fifth point of the pentagon, which introduces no new distances; problem 2 showed the type is $(5, 2)$. The other locations as as follows:

Two configurations occur when an equilateral triangle is attached to the long side of the trapezoid. It could either point outward or inward.

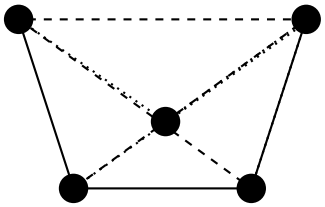


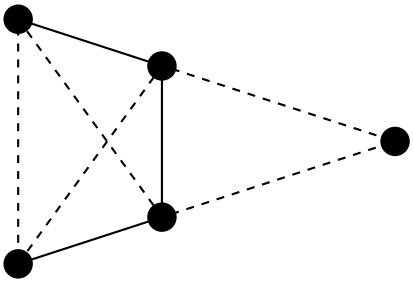
Similarly, the trapezoid could have an equilateral triangle attached to the shorter base, again pointed either outward or inward.



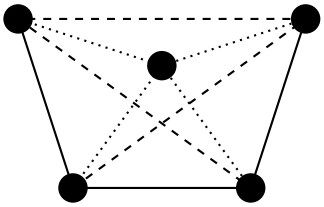


There is a case where a vertex of the regular pentagon is reflected over the diagonal of the pentagon, and another where that vertex is reflected over the opposite side of the pentagon (shown sideways to save space).

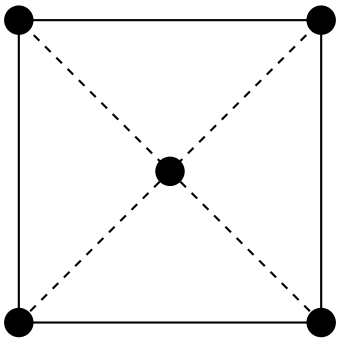




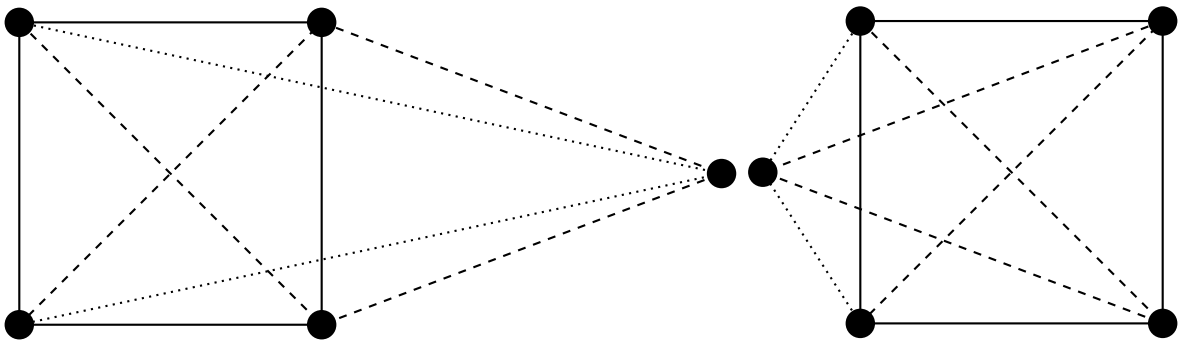
Finally, there is the trapezoid together with its circumcenter.



Next, consider the cases where four of the points lie at the vertices of a square. If the square has sides of length 1, then its diagonals have length $\sqrt{2}$. The fifth point can add at most one more distance. Thus, of the four distances from this new point to the vertices of the square, at least two must be equal. If three are equal, the fifth point is the center of the square (and all four will be equal).



Otherwise, the fifth point will lie either on the diagonal (possibly extended) or the common bisector of opposite sides. Quick case work shows that any point on the diagonal will add at least two new distances, so there are no such cases. For the cases where the fifth point is on the common bisector of opposite sides (say, the left and right sides), if it is not the center of the square then its distances from the left two vertices and from the right two are different, so one of those distances must be either 1 or $\sqrt{2}$ or else two new distances will be introduced. So there are four cases: a 1-1-1 triangle pointing outward, a 1-1-1 triangle pointing inward, a 1-$\sqrt{2}$-$\sqrt{2}$ triangle pointing outward, and pointing inward. The first two of those were already cataloged in the initial section because they have six equal distances.



The next cases to consider are where four of the points are the vertices of a non-square rectangle. There are already three distances in such a figure—the short sides of the rectangle, the long sides, and the diagonals. Thus, any fifth point must be at one of these distances from each of the four vertices of the original rectangle. In fact, by the pigeonhole principle, this fifth point must be at one of those distances from at least *two* of the four original points. Being equidistant from those two points places the fifth point on the perpendicular bisector of the segment determined by those two points. If the fifth point is equidistant from at least *three* of the rectangle's vertices, it is the center of the rectangle, and the only possibility is that the original rectangle is four of the points from a regular hexagon, a case which was already covered when six equal distances were considered.

There are no cases where the fifth point lies on the perpendicular bisector of a diagonal of the rectangle. Let the rectangle be $ABCD$, and the fifth point, E, lie on the perpendicular bisector of $\overline{AC}$, on the same side of this diagonal as point B. Then E is closer to B then any other vertex of the rectangle, so EB is the length of the shortest side of the rectangle. Similarly, E is farther from D than from A or C, so ED must be the length of the diagonal, while $EA = EC$ must be the longer side of the rectangle. Assume $AB = CD$ is the long side of the rectangle. Then there are only two locations for E where ED equals the length of the diagonal and AE equals the length of the longer side. One of them is at point B, the other is B's mirror image over segment $\overline{AD}$, whose distance from B is twice the long side of the rectangle, certainly not equal to the short side!

So the fifth point must lie on the common perpendicular bisector of a pair of opposite sides of the rectangle. To further the analysis, let the rectangle have vertices at $(a, 1)$, $(-a, 1)$, $(a, -1)$, and $(-1, -1)$ with $a > 0$. The sides and diagonals of this rectangle have length 2, $2a$, and $2\sqrt{a^2 + 1}$ respectively. The fifth point can be assumed to lie on the positive y-axis, having coordinates $(0, y)$. We look for all possible choices of a and y so that no new distances are created. (Note that $y > 0$, as $y = 0$ puts the fifth point at the center of the rectangle, which is half the diagonal, which can only be equal to one of the other distances we already have in our diagram if the diagonals have length 2 or $2a$, and in either of those cases the original rectangle is four vertices of a regular hexagon, and we already found this configuration—it is the second example found overall.)

By the Pythagorean theorem, the squares of the distances of $(0, y)$ from $(a, 1)$ and $(a, -1)$ are $a^2 + (y - 1)^2$ and $a^2 + (y + 1)^2$ respectively. Since y is positive these will necessarily be different, with the first smaller than the second. So these must be squares of the lengths of

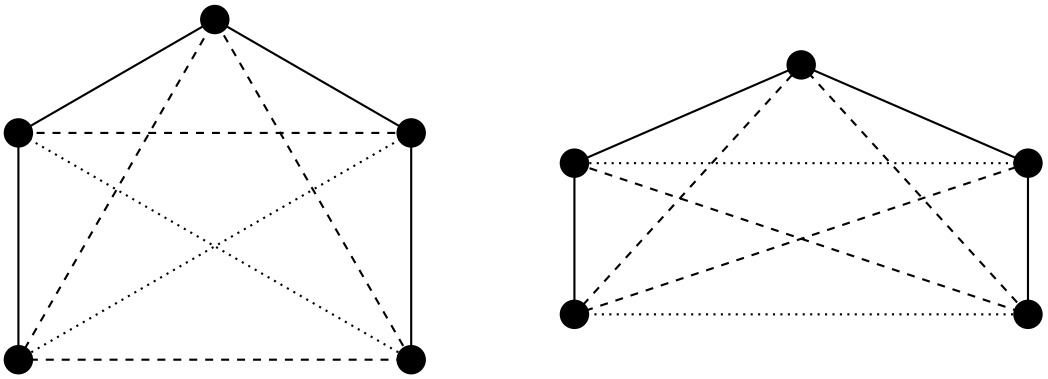
the sides or diagonals. That is, one of the following three systems must hold:

$$\begin{array}{lll} a^2+(y-1)^2=4 & a^2+(y-1)^2=4 & a^2+(y-1)^2=4a^2 \\ a^2+(y+1)^2=4a^2 & a^2+(y+1)^2=4a^2+4 & a^2+(y+1)^2=4a^2+4 \end{array}$$

The first system can be solved by subtracting the first equation from the second, leavine $y=a^2-1$. Substituting that back into the second equation then yields $a=\sqrt{3}$ and then $y=2$. The resulting configuration is five of the vertices of a regular hexagon.

The second system can be solved the same way, yielding $a=\frac{1+\sqrt{13}}{2}$ and $y=\frac{7+\sqrt{13}}{2}$, yielding a wider shape than the previous case.

The last system has solution $a=0$ and $y=1$, which is impossible since a must be greater than zero.

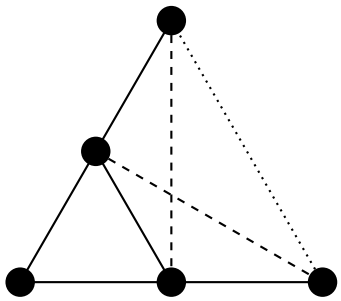


The next cases we will consider are there there are three equally-spaced collinear points. This has already occurred in the first three configurations cataloged as well as the square with its center, so we will avoid those configurations. For simplicity, start with the collinear points being $(-1,0)$, $(0,0)$, and $(1,0)$. These points are either at distance 1 or at distance 2 from each other. The fourth and fifth points cannot create more than one new distance.

The only points on the x-axis that create at most one new distance are $(-2,0)$ and $(2,0)$, each creating four collinear equally spaced points. But there is no location possible for the fifth point that doesn't create a new distance. So the remaining points cannot be on the x-axis. Without loss of generality, assume the y-coordinate of the fourth point is positive.

- If the fourth point is on the positive y-axis, with coordinates $(0,y)$ then its distances to the points $(-1,0)$ and $(1,0)$ is $\sqrt{y^2+1}$ and its distance to $(0,0)$ is y.
  - If $\sqrt{y^2+1}=1$ then $y=0$. This is ruled out since $y>0$.
  - If $\sqrt{y^2+1}=2$ then $y=\sqrt{3}$. This point, together with $(-1,0)$ and $(1,0)$ form an equilateral triangle, and $(0,0)$ is the midpoint of one side. Casework (the fifth point now must be at a distance of 1, 2, or $\sqrt{3}$ from one of these first four points) leads to the conclusion that the fifth point must be the midpoint between $(\pm 1,0)$ and $(0,\sqrt{3})$.
  - If $\sqrt{y^2+1}$ is some other number, then y must be 1 or 2. If $y=1$ then the only possible configuration is the square with center point which has already been cataloged, while if $y=2$ there are no configurations that work.

– So the fourth point being on the y-axis led to one more configuration:

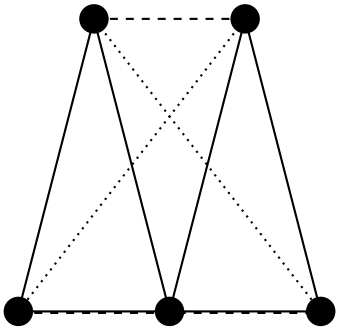


- Otherwise, we may take the fourth point to be in the first quadrant. Split into three further cases:
  - If the fourth point has x-coordinate less than $1/2$, then it has three different distances to the points on the x-axis already, so the point must be distance 1 from one of thos points and distance 2 from another. It is closest to the origin and farthest from $(-1, 0)$:
    - The only point that is distance 1 from the origin and distance 2 from $(-1, 0)$ is $(1, 0)$, which is not in the first quadrant.
    - The only point that is distance 1 from the origin and distance 2 from $(1, 0)$ is $(-1, 0)$ which is not in the first quadrant either.
    - The point in the first quadrant that is distance 1 from $(1, 0)$ and distance 2 from $(-1, 0)$ has x-coordinate larger than $1/2$.

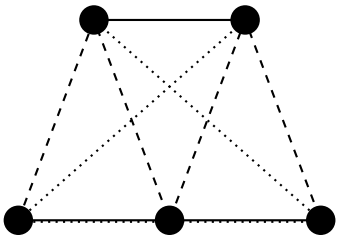
    So there are no possible configurations in this case.
  - If the fourth point has x-coordinate larger than $1/2$, then it has three different distances to the points on the x-axis already, so the point must be distance 1 from one of those points and distance 2 from another. It is closest to $(1, 0)$ and farthest from $(-1, 0)$:
    - The only point that is distance 1 from the origin and distance 2 from $(-1, 0)$ is $(1, 0)$ which is not in the first quadrant.
    - The only point that is distance 1 from $(1, 0)$ and distance 2 from the origin is $(2, 0)$ which is not in the first quadrant either.
    - The point in the first quadrant that is distance 1 from $(1, 0)$ and distance 2 from $(-1, 0)$ is the point $(\frac{3}{4}, \frac{\sqrt{15}}{4})$, which has distance $\sqrt{6}/2$ from the origin. Casework now shows there is no location for a fifth point so that the distances to these first four points are among the numbers 1, $\sqrt{6}/2$, and 2.

    So there are no configurations in this case, either.
  - The remaining case is where the fourth point lies on the vertical line $x = 1/2$. Now it is equidistant from the origin and $(1, 0)$, leaving a little more flexibility.
    - If this point is located at a distance of 1 from $(0, 0)$ and $(1, 0)$ then these four initial point are three of the vertices and the center of a regular hexagon. All such configurations have already been cataloged—the fifth point must lie at one of the other three vertices of the hexagon.

* If this point is located at a distance of 2 from $(0,0)$ and $(1,0)$ then its coordinates are $(1/2, \sqrt{15}/2)$. The fifth point must be located symmetrically at $(-1/2, \sqrt{15}/2)$ forming a "mountain range":

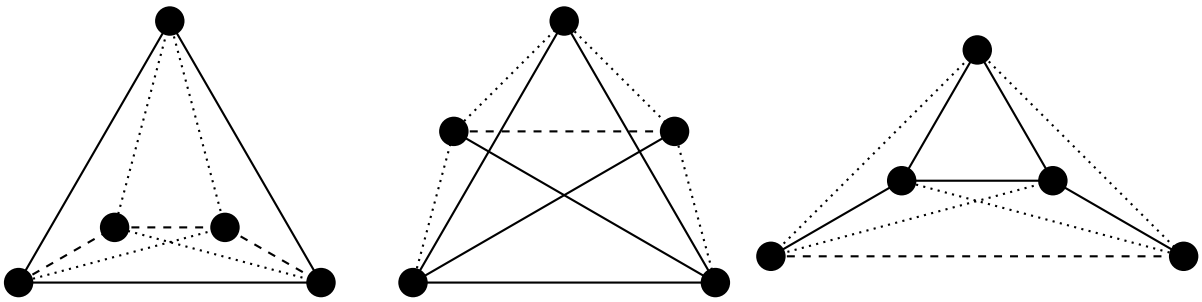


* There is another mountain range where it is the distance from the peaks to the base of the opposite peak is 2:



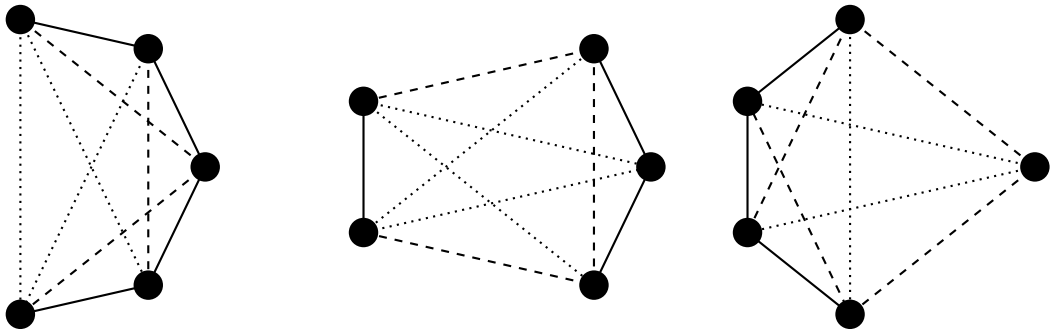
There are only two configurations where there are three collinear points that are not even spaced! These three points already have three distances between them, so the remaining two points must be one of these three distances from each other and each of the three points. The only such configurations are where a vertex of a regular pentagon is reflected over a side or a diagonal, and these were cataloged among the configurations containing four points of a regular pentagon.

There are a few configurations consisting of an equilateral triangle with two more points on the angle bisectors (either internal or external!) of the triangle. Internally, these bisectors could be extended until they have length equal to the distance between their endpoints, or until their lengths equal the sides of the triangle. The other possibilities have already been cataloged (equilateral triangle with midpoints of two of its sides, and five vertices of the regular hexagon). Externally, the bisectors need to have length equal to the side of the triangle to create such a configuration.

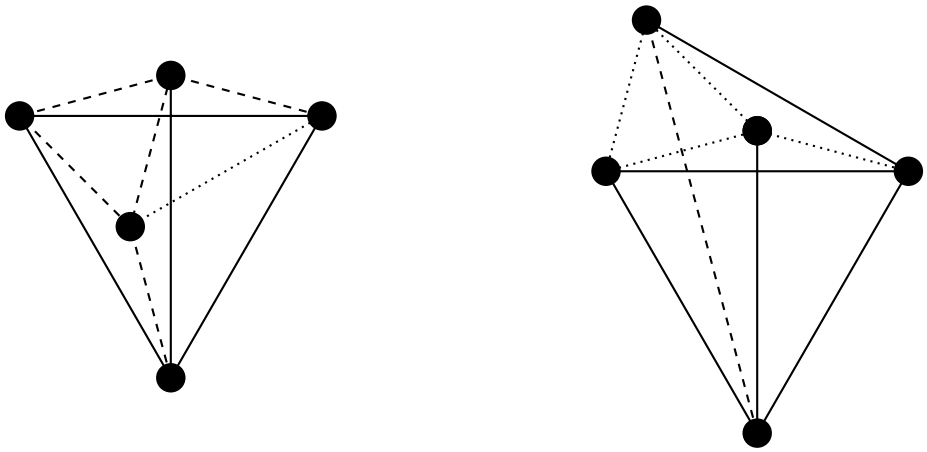


The vertices of a regular heptagon have three distances between them: the side, the short diagonal, and the long diagonal. Thus, removing any two vertices will regular in a type-$(5,3)$

configuration. The two points removed could be connected by a side, a short diagonal, or a long diagonal.



Finally, there are two configurations that possess no line of symmetry! The author was quite surprised when his colleague discovered these two. They consists of two equilateral triangles of different sizes attached at a vertex, with the smaller one tilted 15° relative to the larger.



The author believes that this catalog of 30 configurations is complete, but would be delighted to hear of others. A proof of completeness would be even better!